Homework set 1

In the following $\mathcal{L}_d^1 \simeq \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ denotes the space of unimodular lattices in $\mathbb{R}^d$.

(1) Recall that we equipped $\mathcal{L}_d^1$ is the factor space topology. Namely, a subset $U \subset \mathcal{L}_d^1$ is open if $\pi^{-1}(U)$ is open in $\mathrm{SL}(d,\mathbb{R})$ where $\pi : \mathrm{SL}(d,\mathbb{R}) \to \mathcal{L}_d^1$ is the factor map.
  (a) Prove that for $L_n, L \in \mathcal{L}_d^1$, we have $L_n \to L$ as $n \to \infty$ if and only if there exist bases $\{e_i^{(n)}\}$ and $\{e_i\}$ of L_n and L respectively such that $e_i^{(n)} \to e_i$ as $n \to \infty$.
  (b) Show that the space $\mathcal{L}_d^1$ is not compact.

(2) A subgroup Γ of $G = \mathrm{SL}(d,\mathbb{R})$ is called a *lattice* if it is discrete and there exists a (measurable) set $F \subset G$ such that $\mathrm{vol}(F) < \infty$ and $G = F\Gamma$ (for example, $\Gamma = \mathrm{SL}(d,\mathbb{Z})$ is a lattice in G).
  (a) Show that if $\Gamma_1 \subset \Gamma_2$ are lattices in G, then Γ_1 has finite index in Γ_2.
  (b) Let Γ be a lattice in G. Prove that the space G/Γ is compact if and only if e is not an accumulation point of $\{g\gamma g^{-1} : g \in G, \gamma \in \Gamma\}$.

(3) (a) Give a formula for left and right invariant measures for the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+,\, b \in \mathbb{R} \right\}$$

  (b) Show that the group G has no lattice subgroups.

(4*) Let $G = \mathrm{SO}(1,n)(\mathbb{R})$ be the orthogonal group.
  (a) Construct the Iwasawa decomposition for G.
  (b) Construct Siegel sets of $G = \mathrm{SO}(1,n)(\mathbb{R})$ and $\Gamma = \mathrm{SO}(1,n)(\mathbb{Z})$.
  (c) Show that Γ is a lattice in G.

(4) Prove that for every $v > 0$ there exists a number α which is v-approximable.

(5) Prove that every quadratic irrational is badly approximable.

(6) Prove that if in the Minkowski theorem the domain B is closed, then the condition "$\mathrm{vol}(B) \geq 2^d \mathrm{vol}(\mathbb{R}^d/\Lambda)$" is sufficient.

(7) Use the Minkowski theorem to prove the Lagrange theorem: every positive integer is a sum of four squares. You may wish to follow the following steps:
  (a) Prove that if integers m and n are sums of four squares that so is $m \cdot n$ (hint: introduce a "norm" on the field of quaternions).

(b) Show that for a lattice

$$\Lambda = \begin{pmatrix} p & 0 & a & b \\ 0 & p & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^4,$$

there exists $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ such that $0 < x_1^2 + x_2^2 + x_3^2 + x_4^2 < 2p$.

(c) Prove the Lagrange theorem for prime number (hint: arrange that for $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ one have $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \mod p$.

(8) Let $d \in \mathbb{N}$ and $(x, y) \in \mathbb{Z}^2$ be a solution of the Pell equation $x^2 - dy^2 = 1$. Show that every such solution gives rise to a periodic orbit of the geodesic flow on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ with period $2\cosh^{-1}(x)$.

(9) An element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is called *primitive* if it cannot be written as $\gamma = \gamma_0^m$ for some $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ and $m \in \mathbb{N}$. Show that every element in $\mathrm{SL}_2(\mathbb{Z})$ is a power of primitive element.

(10) Prove that there is a one-to-one correspondence between periodic orbits of the geodesic flow on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ and conjugacy classes of primitive elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{Tr}(\gamma) > 2$. Show that this correspondence the periods of the orbits are given by $2\cosh^{-1}(\mathrm{Tr}(\gamma)/2)$.

Let for $\alpha \in \mathbb{R}^d$ and $t \in \mathbb{R}$

$$\Lambda_\alpha = \begin{pmatrix} id & \alpha \\ 0 & 1 \end{pmatrix} \langle e_1, \ldots, e_{d+1} \rangle \in \mathcal{L}_{d+1}^1,$$

$$g_t = \mathrm{diag}(e^{dt}, e^{-t}, \ldots, e^{-t}) \in \mathrm{SL}_{d+1}(\mathbb{R}).$$

(11) Prove that a vector $\alpha \in \mathbb{R}^d$ is well approximable if and only if there exists $\delta > 0$ such that the inequality $\Delta(g_t \Lambda_\alpha) \le e^{-\delta t}$ has solutions $t_i \to \infty$.

A vector $\alpha \in \mathbb{R}^d$ is called *singular* if for every $\varepsilon > 0$ and sufficiently large $N \in \mathbb{N}$, the system of inequalities

$$\|q\alpha - p\| < \frac{\varepsilon}{N^{1/d}}, \quad 0 < q < N$$

has a solution $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$.

(12) Prove that a vector $\alpha \in \mathbb{R}^d$ is singular if and only if the orbit $\{g_t \Lambda_\alpha : t \ge 0\}$ is divergent (that is, $\Delta(g_t \Lambda_\alpha) \to 0$ as $t \to \infty$).

(13) Deduce that the set of singular vectors has Lebesgue measure zero.