UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level 3 and Level M)

FUNCTIONAL ANALYSIS 34

MATH 36202 and MATH M6202

(Paper Code MATH 36202 and MATH M6202)

May-June 2015, 2 hours 30 minutes

This paper contains **five** *questions*
The best **FOUR** *answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. Let H be a Hilbert space over $\mathbb{R}$.
   (a) **(5 marks)**
   State and prove the Cauchy-Schwarz inequality for vectors x, y in H.
   (b) **(5 marks)**
   Suppose that $x_n \to x$ and $y_n \to y$ in H. Show that $\langle x_n, y_n\rangle \to \langle x, y\rangle$.
   (c) **(5 marks)**
   Let x, y be linearly independent vectors in H such that $\|x\| = \|y\| = 1$. Show that for every $0 < t < 1$ we have $\|tx + (1-t)y\| < 1$.
   (d) Let x_0 be a non-zero element in H.
   (i) **(2 marks)**
   Show that $H_0 = \{x \in H : \langle x, x_0\rangle = 0\}$ is a closed subspace of H.
   (ii) **(3 marks)**
   Given an element y in H, compute the distance from y to H_0.
   (e) (i) **(2 marks)**
   State the Bessel inequality.
   (ii) **(3 marks)**
   Let $(e_n)_{n\geq 1}$ be any collection of unit vectors in H such that the Bessel inequality holds for $(e_n)_{n\geq 1}$. Prove that the vectors e_n must be orthonormal.

2. (a) **(4 marks)**
   Show that the normed space ℓ^∞ is complete.
   (b) **(5 marks)**
   Show that a subspace of a Banach space is complete if and only if it is closed.
   (c) Let X be the subspace of ℓ^∞ consisting of infinite sequences $x = (x_n)$ of real numbers such that $x_n = 0$ for all but finitely many n's.
   (i) **(4 marks)**
   Show that X is not a complete subspace of ℓ^∞.
   (ii) **(5 marks)**
   What is the closure of X in ℓ^∞? (Justify your answer.)
   (iii) **(4 marks)**
   Let $T : X \to X$ be the linear operator defined by $(x_n) \mapsto (2^{-n}x_n)$. Show that T^{-1} exists, but $\|T^{-1}\| = \infty$.
   (d) **(3 marks)**
   State the Bounded Inverse Theorem. Why does (c)(iii) not contradict the Bounded Inverse Theorem?

3. Let X be a Banach space.
   (a) **(2 marks)**
   Give the definition of the dual space X^*.
   (b) **(4 marks)**
   Determine the dual space of ℓ^1, justifying your answer carefully.

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(c) **(3 marks)**
Let x_n be a sequence in X such that $\sum_{n\geq 1}\|x_n\| < \infty$. Show that the sum $\sum_{n=1}^{\infty} x_n$ converges in X.

(d) **(3 marks)**
Let x be an element in X such that $\|x\| = 1$, and $B = \{\|y\| \leq 1\}$ is the closed unit ball in X. Show that there exists f in X^* such that $f(x) = 1$ and $|f| \leq 1$ on B.

(e) **(5 marks)**
We say that a sequence x_n in X is a weak Cauchy sequence if for every f in X^* the sequence $f(x_n)$ is Cauchy. Show that a weak Cauchy sequence is bounded.

(f) (i) **(3 marks)**
Let $A_n : X \to X$ be a sequence of bounded linear operators. Define what it means that $A_n \to A$ in norm topology, in strong topology, in weak topology.

(ii) **(5 marks)**
Let $S : \ell^1 \to \ell^1$ be the operator defined by $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. Show that $S^n \to 0$ in strong topology, but not in norm topology.

4. Let H be a Hilbert space.

(a) **(2 marks)**
State the Riesz-Frechet theorem.

(b) **(2 marks)**
State what it means that a sequence x_n in H converges weakly in H.

(c) **(3 marks)**
Suppose that $x_n \to y_1$ and $x_n \to y_2$ weakly. Show that $y_1 = y_2$.

(d) (i) **(4 marks)**
Show that if $x_n \to x$ weakly and $A : H \to H$ is a bounded linear operator, then $Ax_n \to Ax$ weakly.

(ii) **(5 marks)**
Let $A : H \to H$ be a linear map with the following property: if $x_n \to x$ weakly, then $Ax_n \to Ax$ weakly. Show using the uniform boundedness principle that A is bounded.

(e) **(4 marks)**
Let $A : H \to H$ be a bounded linear operator. Show that if A has bounded inverse, then so does the adjoint operator A^* and $(A^*)^{-1} = (A^{-1})^*$.

(f) **(5 marks)**
Let $H = L^2([0,1])$. For a continuous function $\phi : [0,1] \to \mathbb{C}$, consider the linear operator $A_\phi : H \to H$ defined by $f \mapsto \phi f$. Compute $\|A_\phi\|$ and A_ϕ^*.

5. (a) **(3 marks)**
Let X be a complete metric space. Define what it means for a subset of X to be meager and state the Baire Category Theorem.

(b) **(3 marks)**
Let $\phi_n : [0,1] \to \mathbb{C}$ be a sequence of continuous functions such that $\|\phi_n\|_2 \to \infty$. Show that there exists a function f in $L^2([0,1])$ and a subsequence n_k such that $\int_0^1 \phi_{n_k}(x) f(x)dx \to \infty$.

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(c) (i) **(3 marks)**
Show that ℓ^1 is a dense subset of ℓ^2.

(ii) **(4 marks)**
Let $B_R = \{(x_k)_{k\geq 1} : \sum_{k\geq 1} |x_k| \leq R\}$. Show that B_R is closed in ℓ^2.

(iii) **(4 marks)**
Show that ℓ^1 has empty interior in ℓ^2 and that ℓ^1 is a meager subset of ℓ^2.

(d) Consider the space $H = C([0,1])$ with the norm $\|f\|_2 = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}$ and the sequence $f_n(x) = \cos(2\pi n x)$ in H.

(i) **(4 marks)**
Does it converge in H? (Justify your answer.)

(ii) **(4 marks)**
Show that for every $g \in H$, $\langle f_n, g\rangle \to 0$.

End of examination.