Functional Analysis Exercise sheet 4 — solutions

1. If $x, y \in A^\perp$, then for every $v \in A$,

$$\langle \alpha x + \beta y, v \rangle = \alpha \langle x, v \rangle + \beta \langle y, v \rangle = 0.$$

Hence, $A^\perp$ is a subspace.

Let $x_n \in x$ and $x_n \in A^\perp$. Then for every $v \in A$,

$$\langle x, v \rangle = \lim_{n\to\infty} \langle x_n, v \rangle = 0.$$

Hence, $x \in A^\perp$. This shows that $A^\perp$ is closed.

2. For every $x \in A$ and $y \in A^\perp$, we have $\langle x, y \rangle = 0$, so that $A \subset (A^\perp)^\perp$. By the previous problem, $(A^\perp)^\perp$ is a closed subspace. Hence, it follows that $\mathrm{span}(A) \subset (A^\perp)^\perp$, and $\overline{\mathrm{span}(A)} \subset (A^\perp)^\perp$.

Let $V = \overline{\mathrm{span}(A)}$. It is a closed subspace. We claim that $V^\perp = A^\perp$. Since $A \subset V$, it is clear from the definition of orthogonal complement that $A^\perp \supset V^\perp$. Conversely, let $x \in A^\perp$. Then $\langle x, y \rangle = 0$ for every $y \in A$, and it follows that also $\langle x, y \rangle = 0$ for every $y \in \mathrm{span}(A)$ and for every $y \in \overline{\mathrm{span}(A)}$, that is, $x \in V^\perp$. This proves that $V^\perp = A^\perp$.

We have the orthogonal decomposition $H = V \oplus V^\perp$. Let $x \in (A^\perp)^\perp$ and $x = v + w$ with $v \in V$ and $w \in V^\perp = A^\perp$. Then

$$0 = \langle w, x \rangle = \langle w, v \rangle + \langle w, w \rangle = \langle w, w \rangle .$$

Hence, $w = 0$. This shows that $(A^\perp)^\perp \subset V = \overline{\mathrm{span}(A)}$.

3. We can take M to be any proper dense subspace of ℓ^2. (For example, one can take to consist of sequences $(x_n)_{n\geq 1}$ such that $x_n = 0$ for all but finitely many n.) Then given $x \in M^\perp$, we can a sequence $x_n \in M$ such that $x_n \to x$. We obtain $\langle x, x \rangle = \lim_{n\to\infty} \langle x, x_n \rangle = 0$, so that $x = 0$. This shows that $M^\perp = 0$. Since M is proper, $M \oplus M^\perp \neq \ell^2$.

4. For every y, $\|Ay\| \leq \|A\|\|y\|$. Hence,

$$\|A(Bx)\| \leq \|A\|\|Bx\| \leq \|A\|\|B\|\|x\|.$$

This implies that $\|AB\| \leq \|A\|\|B\|$.

In general, it is not true that $\|AB\| = \|A\|\|B\|$. For instance, consider

$$A : \ell^2 \to \ell_2 : (x_1, x_2, \ldots) \mapsto (x_2, 0, \ldots)$$

Then $\|A\| = 1$, but $A^2 = 0$.

5. First, we show that $\|T_x\| = \|x\|_\infty$. We have

$$|T_x(y)| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \le \sum_{n=1}^{\infty} |x_n||y_n| \le \|x\|_\infty \sum_{n=1}^{\infty} |y_n| = \|x\|_\infty \|y\|_1.$$

Hence, $\|T_x\| \le \|x\|_\infty$. To prove the opposite inequality, we observe that for every $\epsilon > 0$, there exists k such that $|x_k| > \|x\|_\infty - \epsilon$. Then $|T_x(e_k)| = |x_k| > \|x\|_\infty - \epsilon$ and $\|e_k\|_1 = 1$. Thus, $\|T_x\| \ge \|x\|_\infty - \epsilon$ for every $\epsilon > 0$, which implies the claim.

Suppose that there exists $y \in \ell^1$ such that $\|y\|_1 = 1$ and $|T_x(y)| = \|x\|_\infty$. Then in the above inequality, we must have $|x_n||y_n| = \|x\|_\infty |y_n|$ for all n. Since $\|y\|_1 = 1$, $y_n \ne 0$ for some n. Hence, it follows that for some n, $|x_n| = \|x\|_\infty$. Conversely, suppose that $|x_n| = \|x\|_\infty$ for some n. Then $|T_x(e_n)| = \|x\|_\infty$.

Thus, such y exists if and only if $\sup_{n\ge 1} |x_n|$ is achieved.

6. Since

$$\|A_\phi f\|_2^2 = \int_0^1 |\phi(x) f(x)|^2\, dx \le \|\phi\|_\infty^2 \int_0^1 |f(x)|^2\, dx = \|\phi\|_\infty^2 \|f\|_2^2,$$

it follows $\|A_\phi\| \le \|\phi\|_\infty$.

We claim that $\|A_\phi\| \le \|\phi\|_\infty$. Let $x_0 \in [0,1]$ be such that $|\phi(x_0)| = \|\phi\|_\infty$. For every $\epsilon > 0$, there exists $\delta > 0$ such that $|\phi(x) - \phi(x_0)| < \epsilon$ provided that $|x - x_0| < \delta$. We take a non-zero continuous function f supported on δ-neighbourhood of x_0. Then

$$\begin{aligned} \|A_\phi f\|_2^2 &= \int_0^1 |\phi(x) f(x)|^2\, dx \ge \int_0^1 (|\phi(x_0)| - \epsilon)^2 |f(x)|^2\, dx \\ &= (\|\phi\|_\infty - \epsilon)^2 \|f\|_2^2 \end{aligned}$$

for every $\epsilon > 0$. This implies the claim.

7. For every $y \in \ell^1$, we define $f_y(x) = \sum_{n=1}^{\infty} y_n x_n$. Then

$$|f_y(x)| = \sum_{n=1}^{\infty} |y_n||x_n| \le \|y\|_1 \|x\|_\infty$$

Hence, the series converges absolutely, and f_y is well-defined. Moreover, it is clear that f_y is linear, and $\|f_y\| \le \|y\|_1$.

We claim that $\|f_y\| = \|y\|_1$. For every $\epsilon > 0$, there exists N such that $\sum_{n=1}^{N} |y_n| \ge \|y\|_1 - \epsilon$. Let $x \in c_0$ be such that $x_n = \frac{\bar{y}_n}{|y_n|}$ for $n \le N$ (if $y_n = 0$, we set $x_n = 0$), and $x_n = 0$ for $n > N$. Then $f_y(x) = \sum_{n=1}^{N} |y_n| \ge \|y\|_1 - \epsilon$. Also, $\|x\|_\infty = 1$ for sufficiently large N. This implies that $\|f_y\| \ge \|y\|_1 - \epsilon$ for every $\epsilon > 0$.

We have a norm-preserving linear map $\ell^1 \to c_0$: $y \mapsto f_y$. If $f_y = 0$, then $\|y\|_1 = \|f_y\| = 0$. Hence, this map is injective. It remains to show that this map is surjective.

Given $x \in c_0$, we define $x^{(N)} \in c_0$ as follows: $x_n^{(N)} = x_n$ for $n \leq N$ and $x_n^{(N)} = 0$ for $n > N$. Since $x_n \to 0$,

$$\|x - x^{(N)}\|_\infty = \sup_{n>N} |x_n| \to 0$$

as $N \to \infty$. In particular, it follows that the space c_{fin}, the space generated by e_n's, is dense in c_0.

Take any $f \in (c_0)^*$. Let $y_n = f(e_n)$. Given $N \geq 1$, we define $x_n = \frac{\bar{y}_n}{|y_n|}$ for $n \leq N$ (if $y_n = 0$, we set $x_n = 0$), and $x_n = 0$ for $n > N$. Then $x = (x_n)_{n\geq 1}$ is in c_0 and $\|x\|_\infty \leq 1$. We have

$$f(x) = \sum_{n=1}^{N} x_n f(e_n) = \sum_{n=1}^{N} |y_n|,$$

and $|f(x)| \leq \|f\|\|x\|_\infty \leq \|f\|$. Hence, $y = (y_n)_{n\geq 1} \in \ell^1$.

Finally, we claim that $f = f_y$. Indeed, $f(e_n) = f_y(e_n)$, so that $f = f_y$ on c_{fin}. Hence, since c_{fin} is dense in c_0, it follows that $f = f_y$ on c_0.