Lecture 7.

$G = SL_2(\mathbb{R})$

$G = KA^+K$, $K = SO(2)$, $A = \{a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \geq 1\}$.

Thm π = unitary representation of $\underbrace{SL_2(\mathbb{R})}_{G} \ltimes \underbrace{\mathbb{R}^2}_{A}$ without $\mathbb{R}^2$-inv. vectors

Then there exists a set of $SO(2)$-eigenvectors v which span is dense, such that

$$|\langle \pi(a_t) v_1, v_2 \rangle| \leq c(v_1, v_2) \cdot t^{-1}, \quad t \geq 1.$$

Recall that

$$\pi(a) = \int_{\mathbb{R}^2} e^{i\langle a, u\rangle} dP(u), \quad a \in A,$$

where P is a proj. valued measure,

$$\pi(g)^{-1} P(B) \pi(g) = P({}^t g B), \quad g \in G, \ B \subset \mathbb{R}^2.$$

Let $\Omega_s = \{u \in \mathbb{R}^2 : s^{-1} \leq \|u\| \leq s\}$, $s > 1$,

$$\mathcal{H}_s = \mathrm{Im}\,(P(\Omega_s)).$$

For $k \in K$, $P(\Omega_s)\pi(k) = \pi(k) P({}^t k \Omega_s) = \pi(k) P(\Omega_s)$.

Hence, $\mathcal{H}_s$ is K-invariant.

<u>Lem.</u> $\mathcal{H}_s$ = closure of span of K-eigenvectors

Hint: Fourier analysis on $\mathbb{R}/\mathbb{Z}$.

For $v \in \mathcal{H}$, $P(\Omega_s)v \xrightarrow[s\to\infty]{} P(\mathbb{R}^2\setminus\{0\})v = v$.

(since no $\mathbb{R}^2$-fixed vectors)

Hence, $\bigcup_{s>1} \mathcal{H}_s$ is dense in $\mathcal{H}$.

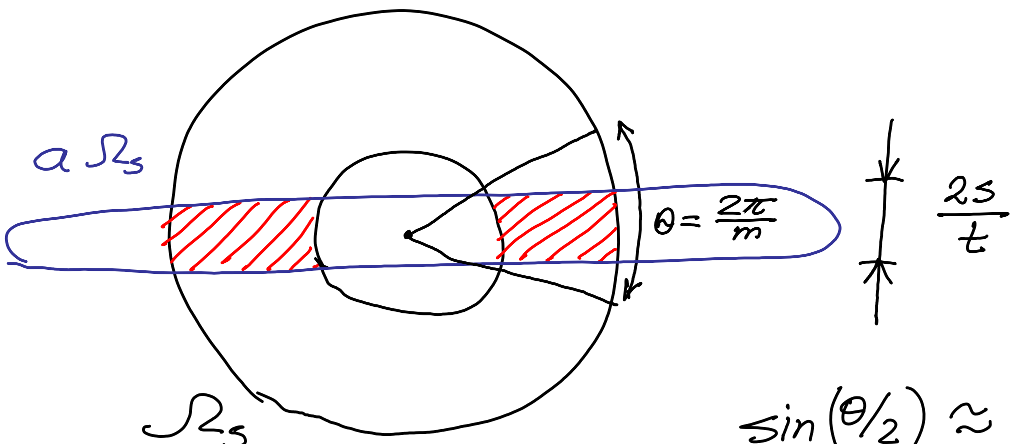
It remains to prove the claim for $\mathcal{H}_s$.

For $v_1, v_2 \in \mathcal{H}_s$,

$$
\begin{aligned}
\langle \pi(a)v_1, v_2\rangle &= \langle \pi(a)P(\Omega_s)v_1, P(\Omega_s)v_2\rangle \\
&= \langle P(a^{-1}\Omega_s)\pi(a)v_1, P(\Omega_s)v_2\rangle \\
&= \langle P(a^{-1}\Omega_s)\pi(a)v_1, P(a^{-1}\Omega_s)P(\Omega_s)v_2\rangle \\
&= \langle \pi(a)v_1, P(a^{-1}\Omega_s \cap \Omega_s)v_2\rangle \\
&= \ldots \text{ similarly } \ldots \\
&= \langle \pi(a)P(a\Omega_s \cap \Omega_s)v_1, P(a^{-1}\Omega_s \cap \Omega_s)v_2\rangle.
\end{aligned}
$$

By Cauchy–Schwarz inequality,

$$|\langle \pi(a)v_1, v_2\rangle| \leq \|P(a\Omega_s \cap \Omega_s)v_1\| \cdot \|P(a^{-1}\Omega_s \cap \Omega_s)v_2\|.$$



$$\sin(\theta/2) \approx \frac{s/t}{s^{-1}} = \frac{s^2}{t}$$

$$\theta \approx 2 \cdot \sin^{-1}\left(\frac{s^2}{t}\right)$$

$$\Omega_s = S_1 \sqcup \ldots \sqcup S_m$$

disjoint sectors with angle θ.

$$P(\Omega_s) = P(S_1) + \ldots + P(S_m).$$

Suppose that $\pi(k)v = \chi(k)v$, $k \in K$.

χ – character of K.

Then $\pi(k_\theta)P(S_i)v = P(k_\theta S_i)\pi(k_\theta)v = \chi(k_\theta)P(S_{i+1})v.$

$$\Downarrow$$

$$\|P(S_i)v\| = \|P(S_{i+1})v\|.$$

Hence, $\|P(S_i)v\| \leq \frac{\|v\|}{\sqrt{m}} = \sqrt{\frac{\theta}{2\pi}}\,\|v\| \approx \sqrt{\frac{2\sin^{-1}(s^2/t)}{2\pi}}\,\|v\|$,

Pythagorean Thm

and $\|P(a^{-1}\Omega_s \cap \Omega_s)v\| \leq 2\cdot(\text{—"—})$,

so that $\langle \pi(a_t)v_1, v_2\rangle \leq c(s)\cdot t^{-1}\cdot\|v_1\|\cdot\|v_2\|$
for K-eigenfunctions in $\mathcal{H}_s$.

Def A unitary representation $\pi: G \to \mathcal{U}(\mathcal{H})$
is L^p-integrable if
$$\langle \pi(g)v_1, v_2\rangle \in L^p(G)$$
for a dense family of vectors $v_1, v_2 \in \mathcal{H}$.

Cor. In above theorem,
$\pi|_{SL_2(\mathbb{R})}$ is $L^{2+\varepsilon}$, $\varepsilon > 0$.

Def. π_1, π_2 = unitary representations of G.
We say that $\pi_1 \prec \pi_2$ (*weakly contained*) if
$$\|\pi_1(f)\| \leq \|\pi_2(f)\| \text{ for all } f \in L^1(G).$$

Thm. G = loc. compact group
π = $L^{2+\varepsilon}$ unitary rep., $\varepsilon > 0$.
Then $\pi \prec \rho$, ρ = the regular representation on $L^2(G)$.

$\ulcorner$ It is enough to check that
$$\|\pi(f)\| \leq \|\rho(f)\| \text{ for all } f \in C_c(G).$$
Moreover, it is enough to check this for $f^* * f$.
So we may assume that $\pi(f)$ is self-adjoint.
We use that
$$\|\pi(f)\| = \sup_{v \in \mathcal{H}_0} \overline{\lim_{n\to\infty}} |\langle \pi(f)^n v, v\rangle|^{1/n}$$
where $\mathcal{H}_0$ is a dense subset of $\mathcal{H}$.
$$\langle \pi(f)^n v, v\rangle = \langle \pi(f^{*n})v, v\rangle = \int_G f^{*n}(g) \underbrace{\langle \pi(g)v, v\rangle}_{\varphi(g)} dg.$$
Let $\Omega = \operatorname{supp}(f)$. Then $\operatorname{supp}(f^{*n}) \subset \Omega^n$.
$$|\langle \pi(f)^n v, v\rangle| = \int_{\Omega^n} |f^{*n}(g)\varphi(g)| dg \leq \left(\int_{\Omega^n} |f^{*n}(g)|^2 dg\right)^{1/2} \cdot \left(\int_{\Omega^n} |\varphi(g)|^2 dg\right)^{1/2}$$
Cauchy-Schwarz ineq.

$$\leq \|f^{*n}\|_2 \cdot \left(\int_{\Omega^n} |\varphi(g)|^{2+\varepsilon} dg \right)^{1/2+\varepsilon} \cdot |\Omega^n|^{\frac{\varepsilon}{2+\varepsilon}}$$

Hölder ineq. with $p = \frac{2+\varepsilon}{2}$ and $p' = \frac{2+\varepsilon}{\varepsilon}$

<u>Lem.</u> $\exists C>0 : |\Omega^n| \leq C^n$.

Without loss of generality, $\mathrm{Int}(\Omega) \neq \emptyset$.
Then by compactness, $\Omega^2 \subset \bigcup_{i=1}^{\ell} \omega_i \Omega$,
and $\Omega^n \subset \bigcup_{i_1 \ldots i_{n-1}} \omega_{i_1} \ldots \omega_{i_{n-1}} \Omega$.

If $\varphi \in L^{2+\varepsilon}(G)$,
$$\overline{\lim_{n\to\infty}} |\langle \pi(f)^n v, v \rangle|^{1/n} \leq \overline{\lim_{n\to\infty}} \|f^{*n}\|^{1/n} \cdot C^{\frac{\varepsilon}{2+\varepsilon}}.$$

Hence, $\|\pi(f)\| \leq \overline{\lim_{n\to\infty}} \|f^{*n}\|^{1/n}$.

We have $(f_1 * f_2)(y) = \int_G f_1(x) f_2(x^{-1}y)\, dx = \rho(f_1) f_2(y)$,
so $f^{*n} = \rho(f)^{n-1} f$, and
$$\|f^{*n}\|^{1/n} \leq \|\rho(f)\|^{1-\frac{1}{n}} \cdot \|f\|^{1/n} \xrightarrow[n\to\infty]{} \|\rho(f)\|.$$

$\pi : G \to \mathcal{U}(\mathcal{H})$ – unitary representation

$$C(\pi) = \text{convex}\langle \underbrace{\langle \pi(g)u, u\rangle}_{\varphi_u(g)} : u \in \mathcal{H},\ \|u\| = 1 \rangle \subset L^\infty(G) \cap C(G).$$

<u>Thm</u> $\pi_1 \prec \pi_2 \Rightarrow C(\pi_1) \subset \overline{C(\pi_2)}$
in topology of uniform convergence on compact sets.

Γ We have: $\forall f \in L^1(G)$: $\|\pi_1(f)\| \le \|\pi_2(f)\|$.

$\forall u \in \mathcal{H}_1 : \|u\| = 1$:

$$\begin{array}{ccc}
\operatorname{Re}\langle \pi_1(f)u, u\rangle & \le & \sup\limits_{\|v_1\| = \|v_2\| = 1} \operatorname{Re}\langle \pi_2(f)v_1, v_2\rangle \\
\| & & \| \\
\operatorname{Re}\langle f, \varphi_u\rangle & & \sup\limits_{\|v_1\| = \|v_2\| = 1} \operatorname{Re}\langle f, \varphi_{v_1, v_2}\rangle \\
 & & \wedge\!\| \\
 & & \sup\limits_{\varphi \in C(\pi_2)} \operatorname{Re}\langle f, \varphi\rangle
\end{array}$$

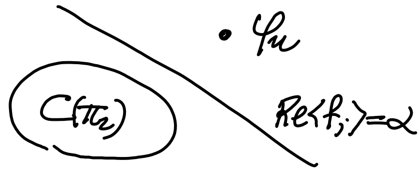
Hence, $\operatorname{Re}\langle f, \varphi_u\rangle \le \sup\limits_{\varphi \in C(\pi_2)} \operatorname{Re}\langle f, \varphi\rangle$.

We equip $L^\infty(G)$ with weak* topology.

Then $L^\infty(G)^* = L^1(G)$, and

by Hahn–Banach Thm.,

$$\varphi_u \in \overline{C(\pi_2)}^{\,weak*}.$$



Now Thm. follows from the following. $\blacksquare$

Prop. $\varphi_n, \varphi \in C(\pi)$, $\varphi_n \xrightarrow{weak*} \varphi$.
Then $\varphi_n \to \varphi$ uniformly on compact sets.

Basic properties: For $\varphi \in \mathcal{C}(\pi)$,

1) $\varphi(g^{-1}) = \overline{\varphi(g)}$,

2) $|\varphi(x)-\varphi(y)| \leq \sqrt{2-2\,Re\,\varphi(yx^{-1})}$.

Suppose that $\varphi_n \xrightarrow{weak*} \varphi$ and $f \in L^1(G)$.
Then $f*\varphi_n \to f*\varphi$ uniformly on compact sets.

Indeed, $(f*\varphi_n)(x) = \int_G f(y)\varphi_n(y^{-1}x)\,dy$

$$= \int_G f(xy)\underbrace{\varphi_n(y^{-1})}_{\overline{\varphi_n(y)}}\,dy$$

$$= \langle x\cdot f, \varphi_n\rangle.$$

The set $\{x\cdot f : x \in K\}$ is compact in $L^1(G)$, so $\langle x\cdot f, \varphi_n\rangle \to \langle x\cdot f, \varphi\rangle$ uniformly on $x \in K$.

We approximate φ_n & φ by $f*\varphi_n$ and $f*\varphi$.

Take $\varepsilon>0$ and a nbhd V of e in G:

$$|\varphi - 1| \leq \varepsilon \text{ on } V.$$

Then for sufficiently large n,

$$\left|\int_V (\varphi_n - \varphi)\right| \leq \varepsilon |V|,$$

and

$$\left|\int_V (1-\varphi_n)\right| \leq \left|\int_V (1-\varphi)\right| + \left|\int_V (\varphi - \varphi_n)\right| \leq 2\varepsilon |V|.$$

We have:

$$\left|(\chi_V * \varphi)(x) - |V|\varphi(x)\right| \leq \int_V |\varphi(y^{-1}x) - \varphi(x)|\, dy$$

$$\leq \int_V \sqrt{2 - 2\,\mathrm{Re}\,\varphi(y)}\, dy \leq \left(\int_V (2 - 2\,\mathrm{Re}\,\varphi(y))\,dy\right)^{1/2} \cdot |V|^{1/2}$$

$$\ll \sqrt{\varepsilon}\cdot |V|,$$

and similarly for φ_n.

Hence, for $f = \frac{\chi_V}{|V|}$,

$$|\varphi_n - \varphi| \leqslant |\varphi_n - f * \varphi_n| + |f * \varphi_n - f * \varphi| + |f * \varphi - \varphi|$$
$$\ll \sqrt{\varepsilon} + |f * \varphi_n - f * \varphi|.$$

This implies the claim. $\square$