Lecture 5.

Applications of spectral gap. I

1) Banach–Ruziewicz Problem

X = space with action of a group Γ.
$\mathcal{F}$ = Γ-inv. space of functions on X.

Def. An *invariant mean* on $\mathcal{F}$ is a linear map
$$M : \mathcal{F} \to \mathbb{C}$$
such that:
- $M(1) = 1$,
- $M(f) \geq 0$ for $f \geq 0$,
- $M(\pi_X(\gamma) f) = M(f)$ for $\gamma \in \Gamma$ and $f \in \mathcal{F}$.

Banach–Ruziewicz Problem.
Is the Lebsegue measure λ the only $SO(d+1)$-invariant mean on $L^\infty(S^d, \lambda)$?

$d = 1$: No (Banach),
$d \geq 4$: Yes (Margulis, Sullivan),
$d = 2, 3$: Yes (Drinfeld).

Thm. $\left(\begin{array}{c}\text{Rosenblatt}\\ \text{K. Schmidt}\end{array}\right)$ $\Gamma \curvearrowright (X, \mu)$ – measure-preserving action on prob. space with spectral gap.

Then μ is the unique invariant mean on $L^\infty(X, \mu)$.

Let M be an invariant mean.

Given $\varphi \in L^1(X)$ such that $\varphi \geq 0$, $\int_X \varphi \, d\mu = 1$, define $M_\varphi(f) = \langle \varphi, f \rangle$ for $f \in L^\infty(X)$.

Then M_φ is an invariant mean.

ex. $\exists$ a net of functions $\{\varphi_i\}$ as above such that $M_{\varphi_i}(f) \to M(f)$ for $f \in L^\infty(X)$ (i.e., $M_{\varphi_i} \to M$ in weak* topology in $L^\infty(X)^*$).

Hint: approximate $f \in L^\infty(X)$ by functions with finite image.

Take $\gamma \in \Gamma$.

Since $\langle \pi_X(\gamma)\varphi_i - \varphi_i, f \rangle = \langle \varphi_i, \pi_X(\gamma)^{-1} f - f \rangle \to 0$, we obtain $\pi_X(\gamma)\varphi_i - \varphi_i \to 0$ in weak* topology.

Let Ω_i be the convex hull of $\{\varphi_j\}_{j \succ i}$.

Lem. weak-closure(Ω_i) = norm-closure(Ω_i).

Hence, $\exists \tilde{\varphi}_i \in \Omega_i : \|\pi_X(\gamma)\tilde{\varphi}_i - \tilde{\varphi}_i\|_1 \to 0$,

$M_{\tilde{\varphi}_i} \to M$ in weak* topology.

Suppose that $\Gamma \curvearrowright L^2_0(X)$ has no (S,ε)-inv. vector ($S \subset \Gamma$ - finite, $\varepsilon > 0$).

Applying above argument to $L^1(X)^{|S|}$, we deduce that: $\exists \tilde{\varphi}_i \in L^1(X)$, $\tilde{\varphi}_i \geq 0$, $\int_X \tilde{\varphi}_i \, d\mu = 1$:

$\|\pi_X(\gamma)\tilde{\varphi}_i - \tilde{\varphi}_i\|_1 \to 0$ for all $\gamma \in S$,

$M_{\tilde{\varphi}_i} \to M$ in weak* topology.

Let $\psi_i = \tilde{\varphi}_i^{1/2}$. Then $\|\psi_i\|_2 = 1$, and

$$\|\pi_X(\gamma)\psi_i - \psi_i\|_2^2 = \int_X |\tilde{\varphi}_i(\gamma^{-1}x)^{1/2} - \tilde{\varphi}_i(x)^{1/2}|^2 d\mu(x)$$

$$\leq \int_X |\tilde{\varphi}_i(\gamma^{-1}x) - \tilde{\varphi}_i(x)| \, d\mu(x) = \|\pi_X(\gamma)\tilde{\varphi}_i - \tilde{\varphi}_i\|_1 \to 0.$$

Write $\psi_i = \overline{\psi_i} + c_i$ with $\overline{\psi_i} \in L^2_0(X)$, $c_i \geq 0$.

Then $\|\pi_X(\gamma)\overline{\psi_i} - \overline{\psi_i}\|_2 \to 0$ for $\gamma \in S$, and since $\nexists (S,\varepsilon)$-inv. vectors, $\|\overline{\psi_i}\|_2 \to 0$.

Then $\psi_i \to 1$ in $L^2(X)$ and $\tilde{\varphi}_i = \psi_i^2 \to 1$ in $L^1(X)$.

Hence, $M_{\tilde{\varphi}_i} \to$ (integration along μ). $\lrcorner$

Thm. (Margulis, Sullivan) $n \geq 4$.
The Lebesgue measure is the <u>only</u> invariant mean on $L^\infty(S^n)$.

Proof (sketch)
Let $Q(x) = x_1^2 + \dots + x_{n-1}^2 - \sqrt{2}\,(x_n^2 + x_{n+1}^2)$,

$$G = SO(Q) = \{ g \in SL_{n+1}(\mathbb{R}) : Q(gx) = Q(x) \}$$
$$\cup$$
$$\Gamma = G \cap SL_{n+1}(\mathbb{Z}[\sqrt{2}]).$$

G is a simple Lie group, $\operatorname{rank}(G) \geq 2$, so G has property T.
Γ is a lattice in $G \Rightarrow \Gamma$ has property T.

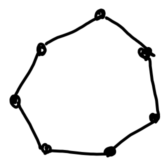
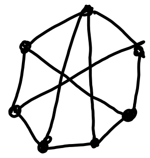
Let $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2}) : a + \sqrt{2}\,b \mapsto a - \sqrt{2}\,b$.
Then $\sigma(\Gamma) \subset SO(x_1^2 + \dots + x_{n-1}^2 + \sqrt{2}(x_n^2 + x_{n+1}^2))$
$\simeq SO(n+1)$.

$\sigma(\Gamma)$ — dense subgroup.

Then $\sigma(\Gamma)$ acts ergodically on S^n, and by property T, this action has spectral gap.
Hence, the inv. mean is <u>unique</u>. $\blacksquare$

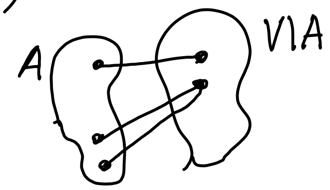
2) Expander graphs.

$\mathcal{G}=(V,E)$ - finite graph, V = set of vertices, E = set of edges.

$E\subset V\times V$, $(v,w)\in E \Rightarrow (w,v)\in E$.

For $A\subset V$, $\partial A=\{v\in V\setminus A : \exists \text{ edge from } v \text{ to } A\}$.



Def. The <u>expansion constant</u> is

$$c(\mathcal{G})=\min_{|A|\le \frac{|V|}{2}} \frac{|\partial A|}{|A|}.$$

$c(\mathcal{G})$ - big $\Rightarrow$ $\mathcal{G}$ is highly connected.

Def. A family of graphs $\mathcal{G}_n$ is <u>(k,ε)-expanders</u> if:

1) $|V_n|\to\infty$,
2) Every vertex has at most k neighbours,
3) $c(\mathcal{G}_n)\geqslant\varepsilon$.

Group actions $\rightsquigarrow$ Graphs:

Γ = discrete group

$\overset{\vee}{S}$ – finite symmetric generating set of Γ

V = finite set on which Γ acts transitively

$E = \{(v, sv) : v \in V, s \in S\}$.

Consider $\mathcal{G} = (V, E)$.

$$\mathcal{H} = L^2_0(V) = \{f : V \to \mathbb{C} : \sum_{v \in V} f(v) = 0\}$$

$$\langle f_1, f_2 \rangle = \frac{1}{|V|} \sum_{v \in V} f_1(v) \overline{f_2(v)}.$$

$$\pi(\gamma) : \mathcal{H} \to \mathcal{H} : f \mapsto f(\gamma^{-1} x).$$

$\llcorner$ unitary representation.

__Lem.__ Suppose that π has no (S, ε)-inv. vectors.
Then $c(\mathcal{G}) \geq \frac{\varepsilon^2}{4}$.

$\ulcorner$ Take a partition $V = A \sqcup B$, $|A| = a$, $|B| = b$,
and consider $F(v) = \begin{cases} b, & v \in A, \\ -a, & v \in B. \end{cases} \in \mathcal{H}$.

Then $\|F\|^2 = \frac{1}{|V|}(ab^2 + ba^2) = ab$, and

for $s \in S$, $F(s^{-1}v) - F(v) = \begin{cases} b+a, & s^{-1}v \in A \ \& \ v \in B, \\ -a-b, & s^{-1}v \in B \ \& \ v \in A \\ 0, & \text{otherwise} \end{cases}$,

$$\|\pi(s)F - F\|^2 = \frac{(a+b)^2}{|V|} \cdot |(sA \cap B) \cup (sB \cap A)| \leq (a+b) \cdot 2|\partial A|.$$

Hence, $\|\pi(s)F - F\|^2 \leq \frac{(a+b)\cdot 2|\partial A|}{ab} \cdot \|F\|^2$

$$\frac{(a+b)\cdot 2|\partial A|}{ab} = 2\left(\frac{1}{a}+\frac{1}{b}\right)\cdot|\partial A| \leq 4\cdot\frac{|\partial A|}{\min\{a,b\}}.$$

for all $s \in S$,

and $4\cdot\frac{|\partial A|}{\min\{a,b\}} \geq \varepsilon^2 \Rightarrow c(\mathcal{G}) \geq \frac{\varepsilon^2}{4}$. $\square$

$\Gamma = SL_n(\mathbb{Z})$

$\cup$

S – finite symmetric generating set such that (S,ε) has Kazhdan property.

$\Gamma_n < \Gamma$ – finite index subgroups, $|\Gamma:\Gamma_n| \to \infty$.

$V_n = \Gamma/\Gamma_n$, $\mathcal{G}_n$ = corresponding graphs.

<u>Thm</u> $\mathcal{G}_n$ is expander family.

$\ulcorner$ $\forall n$: $\Gamma \curvearrowright L^2_0(V_n)$ has no (S,ε)-inv. vector.

Hence, $c(\mathcal{G}_n) \geq \frac{\varepsilon^2}{2}$ by Lemma. $\square$