Lecture 4

Kazhdan property T.

Def A group G has <u>property T</u> if $\exists$ compact $K \subset G$ $\exists \varepsilon > 0$: for every unitary representation $\pi: G \to \mathcal{U}(\mathcal{H})$ without inv. vectors, $\nexists$ (K, ε)-invariant vectors.

ex. 1) $\mathbb{R}^d$ and $\mathbb{Z}^d$ don't have property T.
Consider the regular representation $\rho: \mathbb{R}^d \to \mathcal{U}(L^2(\mathbb{R}^d))$:
$\rho(g)f(x) = f(x-g)$. Let $B_n = [-n, n]^d$ and
$\varphi_n = \frac{\chi_{B_n}}{|B_n|^{1/2}}$. Then $\|\varphi_n\|_2 = 1$, $\|\rho(g)\varphi_n - \varphi_n\|_2 \to 0$ (uniformly on compact sets),
but $\nexists$ inv. vectors.

2) nonabelian free group doesn't have property T.

<u>Prop.</u> (uniform spectral gap)
$\pi: G \to \mathcal{U}(\mathcal{H})$ – unitary representation without (K, ε)-inv. vectors.

$\nu \in \mathrm{Prob}(G)$: $d\nu(g) = f(g)dg$, $f \geq 0$, $f \in C_c(G)$, $\int_G f = 1$.
$f > 0$ on $K \cup K^{-1} \cup \{1\}$

Then $\exists \delta = \delta(\nu, \varepsilon) \in (0,1)$: $\|\pi(\nu)\| < \delta$.

Let $\sigma = \nu^* * \nu$. Since $\|\pi(\sigma)\| = \|\pi(\nu)^*\pi(\nu)\| = \|\pi(\nu)\|^2$, it is sufficient to estimate $\|\pi(\sigma)\|$.

We write $d\sigma(g) = \tilde{f}(g)\,dg$.

Let $Q = KUK^{-1} \cup \{1\}$ and $\alpha = \min(\tilde{f}|_{Q^2}) > 0$.

Since $Q \subset Q^2$, for $x \in Q$, $\tilde{f} \geq \frac{\alpha}{2}(\chi_{xQ} + \chi_Q)$.

We have
$$\|\pi(\tilde{f} - \tfrac{\alpha}{2}(\chi_{xQ} + \chi_Q))\| \leq \|\tilde{f} - \tfrac{\alpha}{2}(\chi_{xQ} + \chi_Q)\|_1 = 1 - \alpha|Q|.$$

Take $v \in \mathcal{H} : \|v\| = 1$ and $x \in Q : \|\pi(x)\pi(\chi_Q)v - \pi(\chi_Q)v\| \geq \varepsilon\|\pi(\chi_Q)v\|$.

By the parallelogram identity,
$$\|\pi(\chi_{xQ} + \chi_Q)v\|^2 = 2\|\underbrace{\pi(\chi_{xQ})v}_{\pi(x)\pi(\chi_Q)v}\|^2 + 2\|\pi(\chi_Q)v\|^2 - \|\pi(\chi_{xQ} - \chi_Q)v\|^2$$
$$= 4\|\pi(\chi_Q)v\|^2 - \|\pi(x)\pi(\chi_Q)v - \pi(\chi_Q)v\|^2$$
$$\leq 4\|\pi(\chi_Q)v\|^2 - \varepsilon^2\|\pi(\chi_Q)v\|^2 \leq (4 - \varepsilon^2)|Q|^2.$$

Hence, by the triangle inequality,
$$\|\pi(\tilde{f})v\| \leq \|\pi(\tilde{f} - \tfrac{\alpha}{2}(\chi_{xQ} + \chi_Q))v\| + \|\pi(\tfrac{\alpha}{2}(\chi_{xQ} + \chi_Q))v\|$$
$$\leq \underbrace{1 - \alpha|Q| + \tfrac{\alpha}{2}\sqrt{4 - \varepsilon^2}\cdot|Q|}_{\delta} < 1 \text{ for all } v, \|v\| = 1,$$

So $\|\pi(\tilde{f})\| < \delta$.

Thm (Kazhdan) $SL_d(\mathbb{R})$, $d \geq 3$, has property T.

Rmk. In general, if G is a connected simple Lie group with finite center, then

G has property $T \iff G$ is not locally isomorphic to $SO(n,1)$, $SU(n,1)$.

Unitary representations of $\mathbb{R}^d$:

$\pi : \mathbb{R}^d \to \mathcal{U}(\mathcal{H})$ – unitary representation.

Then there exists unique projection-valued measure

$$P : \{\text{Borel subsets of } \mathbb{R}^d\} \to \{\text{orthogonal projections on } \mathcal{H}\}$$

$$\pi(a) = \int_{\mathbb{R}^d} e^{i\langle a,u\rangle}\, dP(u)$$

Proof of Thm. Consider the subgroup

$$\underbrace{SL_{d-1}(\mathbb{R})}_{G} \ltimes \underbrace{\mathbb{R}^{d-1}}_{A} = \left(\begin{array}{c|c} * & * \\ \hline 0 \dots 0 & 1 \end{array}\right) \subset SL_d(\mathbb{R}).$$

Given a unitary representation $\pi : SL_d(\mathbb{R}) \to \mathcal{U}(\mathcal{H}_\pi)$,

$$\pi(a) = \int_{\mathbb{R}^{d-1}} e^{i\langle a,u\rangle}\, dP_\pi(u), \quad a \in A.$$

For $g \in G$ and $a \in A$,

$$\pi(g)^{-1}\pi(a)\pi(g) = \pi(g^{-1}ag) = \pi(g^{-1}(a))$$

$$\pi(g)^{-1}\pi(a)\pi(g) = \int_{\mathbb{R}^{d-1}} e^{i\langle a,u\rangle} d\left(\pi(g)^{-1}P_\pi(u)\pi(g)\right)$$

$$\pi(g^{-1}(a)) = \int_{\mathbb{R}^{d-1}} e^{i\langle g^{-1}(a),u\rangle} dP_\pi(u) = \int_{\mathbb{R}^{d-1}} e^{i\langle a,({}^tg)^{-1}(u)\rangle} dP_\pi(u).$$

Hence, by uniqueness of the spectral measure,

$$\pi(g)^{-1}\circ P_\pi \circ \pi(g) = ({}^tg)^{-1}_{*} P_\pi,$$

$$\boxed{\pi(g)^{-1}P_\pi(B)\pi(g) = P_\pi({}^tgB)}$$ for Borel $B \subset \mathbb{R}^{d-1}$.

Suppose that $SL_d(\mathbb{R})$ does <u>not</u> have property T. Let K be a compact generating set of G. Then $\exists$ representations $\pi_n : G \to \mathcal{U}(\mathcal{H}_{\pi_n})$, without fixed vectors, and unit vectors $v_n \in \mathcal{H}_{\pi_n}$ such that

$$\sup_{g\in K} \|\pi_n(g)v_n - v_n\| \longrightarrow 0.$$

Consider the corresponding sequence of probability measures on $\mathbb{R}^{n-1}$:

$$\mu_n(B) = \langle P_{\pi_n}(B) v_n, v_n \rangle, \quad \text{Borel } B \subset \mathbb{R}^{d-1}.$$

Then uniformly on $g \in K$,

$$|\mu_n(g^t B) - \mu_n(B)| = |\langle P_{\pi_n}(B)\pi_n(g)v_n, \pi_n(g)v_n\rangle - \langle P_{\pi_n}(B)v_n, v_n\rangle|$$

$$\leq |\langle P_{\pi_n}(B)(\pi_n(g)v_n - v_n), \pi_n(g)v_n\rangle| + |\langle P_{\pi_n}(B)v_n, \pi_n(g)v_n - v_n\rangle|$$

$$\leq 2\,\|\pi_n(g)v_n - v_n\| \longrightarrow 0.$$

If $\mu_n(\{0\}) \neq 0$, then $P_{\pi_n}(\{0\}) \neq 0$, and $\mathcal{H}_{\pi_n}$ contains an A-inv. vector.

Then by Moore ergodicity Thm (Lecture 2), $\mathcal{H}_{\pi_n}$ contains a $SL_d(\mathbb{R})$-inv. vector.

Hence, $\mu_n(\{0\}) = 0$.

Let $\overline{\mu_n}$ be the projections of μ_n on $\mathbb{P}(\mathbb{R}^{d-1})$.

By weak compactness, $\overline{\mu_{n_i}} \longrightarrow \bar{\mu} \in \mathrm{Prob}(\mathbb{P}(\mathbb{R}^{d-1}))$.

Then $\bar{\mu}$ is $SL_{d-1}(\mathbb{R})$-invariant, which is impossible (see, for instance, Furstenberg Lemma, Lecture 3) $\blacksquare$

Thm. G = locally compact σ-compact group
H = closed subgroup, $\mathrm{vol}(G/H) < \infty$.
Then

G has property T $\Longleftrightarrow$ H has property T.

Induced representation: $\pi : H \to \mathcal{U}(\mathcal{H})$ - unitary representation.

Define $\hat{\pi} = \mathrm{Ind}_H^G(\pi) : G \to \mathcal{U}(\hat{\mathcal{H}})$:

$$\hat{\mathcal{H}} = \left\{ f : G \to \mathcal{H} : \begin{array}{l} f(gh) = \pi(h)^{-1} f(g),\ h \in H, \\ \|f\|^2 = \int_{G/H} \|f(g)\|^2 d\mu(gH) < \infty \end{array} \right\}$$

prob. measure on G/H.

$$\hat{\pi}(g) f(x) = f(g^{-1}x).$$

If $f \in C_c(G)$ and $v \in \mathcal{H}$, then

$$F(x) = \int_H f(xh)\pi(h) v \, dh \in \hat{\mathcal{H}}.$$

In fact, the span of such functions is dense in $\hat{\mathcal{H}}$.

Lem. 1. $\hat{\mathcal{H}} \ni G$-inv. vector $\Rightarrow$ $\mathcal{H} \ni G$-inv. vector.

Lem. 2 For $f \in C_c(G)$, define $\bar{f}(gH) = \int_H f(gh)\,dh \in C_c(G/H)$.
The map $f \to \bar{f} : C_c(G) \to C_c(G/H)$ is onto.

Proof of Thm.

$\Longrightarrow$ Suppose that $\forall$ compact $K \subset H$, $\forall \varepsilon > 0$:
$\exists$ a unitary representation $\pi : H \to \mathcal{U}(\mathcal{H})$
with (K,ε)-inv. unit vectors, but no inv. vectors.
We show that the same is true for G.
Let $Q \subset G$ – compact and $\varepsilon > 0$.
Take real-valued $f \in C_c(G)$, $f \geq 0$.
Let $S = \mathrm{supp}(f)$, $K = (S^{-1}QS \cup S^{-1}S) \cap H$, and
$\pi : H \to \mathcal{U}(\mathcal{H})$ is a representation as above
with a (K,ε)-inv. vector $v \in \mathcal{H}$, $\|v\| = 1$.

Consider the induced representation: $\hat{\pi} : G \to \mathcal{U}(\hat{\mathcal{H}})$,
and $F(x) = \int_H f(xh)\, \pi(h) v \, dh \in \hat{\mathcal{H}}$.

$$\|\hat{\pi}(g)F - F\|^2 = 2\left(\|F\|^2 - \mathrm{Re}\langle \hat{\pi}(g)F, F\rangle\right),$$

$$\|F\|^2 = \int_{G/H} \int_{H\times H} f(xh_1) f(xh_2) \langle \pi(h_1)v, \pi(h_2)v\rangle \, dh_1 dh_2 d\mu(x).$$

$$\langle \hat{\pi}(g)F, F\rangle = \int_{G/H} \int_{H\times H} f(g^{-1}xh_1) f(xh_2) \langle \pi(h_1)v, \pi(h_2)v\rangle \, dh_1 dh_2 d\mu(x).$$

Let $\bar{f}(gH) = \int_H f(gh)\,dh \in C_c(G/H)$.

If $f(xh_1)f(xh_2) \neq 0$, then $h_2^{-1}h_1 \in S^{-1}S$, and

$$2(1 - \mathrm{Re}\langle \pi(h_1)v, \pi(h_2)v\rangle) = \|\pi(h_2^{-1}h_1)v - v\|^2 < \varepsilon^2.$$

Hence,

$$\left| \|F\|^2 - \|\bar{f}\|^2_{L^2(G/H)} \right| \leq \int_{G/H} \int_{H\times H} f(xh_1)f(xh_2) \left| \mathrm{Re}\langle \pi(h_1)v, \pi(h_2)v\rangle - 1\right| dh_1\,dh_2\,d\mu(x)$$

$$\leq \frac{\varepsilon^2}{2} \cdot \|\bar{f}\|^2_{L^2(G/H)}.$$

Similarly, if $g \in Q$ and $f(g^{-1}xh_1)f(xh_2) \neq 0$, then $h_2^{-1}h_2 \in S^{-1}QS$, and

$$\left| \langle \hat{\pi}(g)F, F\rangle - \langle g^{-1}\cdot \bar{f}, \bar{f}\rangle_{L^2(G/H)} \right| \leq \frac{\varepsilon^2}{2} \cdot \|\bar{f}\|^2_{L^2(G/H)}.$$

We choose $f \in C_c(G)$, so that $\|\bar{f} - 1\|_{L^2(G/H)} < \varepsilon$ (see Lemma 2).

Then $\|F\| \approx \|\bar{f}\|_{L^2(G/H)} \approx 1$, $\langle \pi(g)F, F\rangle \approx \langle g^{-1}\bar{f}, \bar{f}\rangle_{L^2(G/H)} \approx 1$, and $\|F\|^2 \approx \langle \pi(g)F, F\rangle$, which gives almost inv. vectors.

Since G has property T for suitable (Q, ε), $\hat{\mathcal{H}} \ni G$-inv. vector.

Then $\mathcal{H} \ni H$-inv. vector, which is contradiction.

$\Leftarrow$ Suppose that H has property T, and (K, ε) are the corresponding parameters.

Let $Q \subset G$ be compact such that

$$K \subset Q, \quad \mu(QH) \approx 1.$$

We claim that G has property T with parameters $(Q, \varepsilon/4)$.

Let $\pi : G \to U(\mathcal{H})$ is a unitary representation with $(Q, \frac{\varepsilon}{4})$-invariant unit vector v.

We need to show that $\exists$ nonzero G-inv. vector.

Write $v = v' + v''$ for $v' \in \mathcal{H}^H$ and $v'' \in (\mathcal{H}^H)^\perp$.

For $h \in K$, $\|\pi(h)v'' - v''\| = \|\pi(h)v - v\| \leq \varepsilon/4$.

Since $\nexists$ H-inv. vectors in $(\mathcal{H}^H)^\perp$, $\exists h \in K$:

$\|\pi(h)v'' - v''\| \geq \varepsilon \|v''\|$.

Hence, $\|v - v'\| = \|v''\| \leq 1/4$. In particular, $\|v'\| \geq \frac{3}{4}$.

Let $w = \int_{G/H} \pi(x) v' \, d\mu(x)$.

Since μ is invariant, w is G-invariant.

For $g \in Q$, $\|\pi(g)v' - v'\| \leq \|\pi(g)v'' - v''\| + \|\pi(g)v - v\|$

$$\leq 2\|v''\| + \frac{\varepsilon}{4} \leq \frac{1}{2} + \frac{\varepsilon}{4}.$$

$$\|w - v'\| = \left\| \int_{G/H} (\pi(x)v' - v') \, d\mu(x) \right\|$$
$$\leq \int_{QH} \|\pi(x)v' - v'\| \, d\mu(x) + 2\mu((QH)^c)$$
$$\leq \frac{1}{2} + \frac{\varepsilon}{4} + 2\underbrace{\mu((QH)^c)}_{\approx 0}.$$

Since $\|v'\| \geq \frac{3}{4}$, $w \neq 0$. $\square$

<u>Cor.</u> $SL_d(\mathbb{Z})$, $d \geq 3$, has property T.

<u>Cor.</u> $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{R})$ don't have property T.

<u>ex.</u> $SL_d(\mathbb{R}) \ltimes \mathbb{R}^d$, $d \geq 3$, has property T.