Lecture 3

Howe-Moore Property: $\forall$ unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ without G-fixed vectors
$\langle \pi(g)v, w \rangle \to 0$ as $g \to \infty$, $v, w \in \mathcal{H}$.

Thm (Cluckers, Cornulier, Louvet, Tessera, Valette)
$G \leq Gl_n(\mathbb{R})$ - closed Howe-Moore group.
Then G is a simple connected Lie group.

Spectral gap property.

$\pi : G \to \mathcal{U}(\mathcal{H})$ - unitary representation.
For a prob. measure ν on G, define
(Borel, regular)

averaging operators: $\pi_X(\nu) : L^2(X) \to L^2(X)$:

$$\langle \pi_X(\nu) f_1, f_2 \rangle = \int_G \langle \pi_X(g) f_1, f_2 \rangle \, d\nu(g).$$

We always have $\|\pi(\nu)\| \leq 1$.

Def. π has spectral gap if $\exists \nu \in Prob(G) : \|\pi(\nu)\| < 1$.
$G \curvearrowright (X, \mu)$ has spectral gap if $G \curvearrowright L^2_0(X)$ does.

(Here $L^2_0(X) = \{ f \in L^2(X) : \int_X f = 0 \}$.)

For $\nu_1, \nu_2 \in Prob(G)$, define $\nu_1 * \nu_2 \in Prob(G)$:

$$\int_G \varphi(x_1 \cdot x_2)\, d\nu_1(x_1)\, d\nu_2(x_2), \quad \varphi \in C_c(G),$$

and $\nu^* \in Prob(G)$ by $d\nu^*(x) = \overline{d\nu(x^{-1})}$.

<u>Properties:</u> $\pi(\nu_1) \cdot \pi(\nu_2) = \pi(\nu_1 * \nu_2)$,

$$\pi(\nu)^* = \pi(\nu^*).$$

<u>Rmk</u>: If $\|\pi(\nu)\| \leq \lambda < 1$, then $\|\pi(\nu^{*n})\| \leq \lambda^n \longrightarrow 0$.

<u>Def</u> $K \subset G$ - compact, $\varepsilon > 0$.
$v \in \mathcal{H}$ is called <u>(K, ε)-almost invariant</u> if $\|\pi(g)v - v\| < \varepsilon \|v\|$ for all $g \in K$ and $\varepsilon > 0$.

<u>Prop. 1</u> $\|\pi(\nu)\| < 1 \Rightarrow$ for some compact $K \subset G$, $\varepsilon > 0$, $\nexists$ no (nonzero) (K, ε)-inv. vectors.

Take compact $K_n \subset G$ such that $\nu(G \setminus K_n) < \frac{1}{n}$.
Suppose that $\exists v_n \in \mathcal{H} : \|v_n\| = 1$:
$$\|\pi(g)v_n - v_n\| \leq \frac{1}{n} \quad \text{for } g \in K_n.$$

Then
$$\|\pi(\nu)v_n - v_n\| \leq \int_G \|\pi(g)v_n - v_n\| \, d\nu(g)$$
$$\leq \int_{K_n} \|\pi(g)v_n - v_n\| d\nu(g) + \frac{2}{n} \leq \frac{3}{n} \to 0.$$

On the other hand,
$$\|\pi(\nu)v_n - v_n\| \geq \|v_n\| - \|\pi(\nu)v_n\| \geq 1 - \|\pi(\nu)\| > 0.$$

Hence, for large n, $\nexists$ $(K_n, \frac{1}{n})$-inv. vectors.

Prop. Suppose that π contains no (K,ε)-inv. vectors (for some compact $K \subset G$ and $\varepsilon > 0$) and $\nu \in \mathrm{Prob}(G)$, $\mathrm{supp}(\nu) \not\subseteq g_0 H$ for $g_0 \in G$ and closed proper subgroups H, and ν is absolutely continuous.

Then $\|\pi(\nu)\| < 1$.

- $\|\pi(\delta_g * \nu)\| = \|\pi(\nu)\|$, so without loss of generality, $e \in supp(\nu)$.
- $\|\pi(\nu^* * \nu)\| = \|\pi(\nu)\|^2$, so it is enough to show that $\|\pi(\nu^* * \nu)\| < 1$.

  Since $e \in supp(\nu)$, $supp(\nu^* * \nu) \supset supp(\nu)$ and $\overline{\langle supp(\nu^* * \nu) \rangle} = G$.

Now we assume that $\pi(\nu)$ is self-adjoint, positive, and $\overline{\langle supp(\nu) \rangle} = G$.

Suppose that $\|\pi(\nu)\| = \sup_{\|v\|=1} \langle \pi(\nu) v, v \rangle = 1$.

Then $\exists v_n \in \mathcal{H} : \|v_n\| = 1$ such that

$\langle \pi(\nu) v_n, v_n \rangle \to 1$ and $\|\pi(\nu) v_n - v_n\|^2 = 2(1 - Re\langle \pi(\nu) v_n, v_n \rangle) \to 0$.

Let $w_n = \frac{\pi(\nu) v_n}{\|\pi(\nu) v_n\|}$. Then

$$\|\pi(\nu) w_n - w_n\| \le \frac{\|\pi(\nu) v_n - v_n\|}{\|\pi(\nu) v_n\|} \to 0,$$

and $\langle \pi(\nu) w_n, w_n \rangle = \int_G \langle \pi(g) w_n, w_n \rangle \, d\nu(g) \to 1$.

Since $|\langle \pi(g) w_n, w_n \rangle| \le 1$, $\langle \pi(g) w_n, w_n \rangle \to 1$ for ν-a.e. $g \in G$,

and also $\|\pi(g) w_n - w_n\|^2 = 2(1 - Re\langle \pi(g) w_n, w_n \rangle) \to 0$.

Let $H = \{ g \in G : \lim_{n\to\infty} \|\pi(g) w_n - w_n\| = 0 \}$.

H is a measurable subgroup and $\nu(H) = 1$.

Since $\overline{\langle supp(\nu) \rangle} = G$, $\overline{H} = G$.

Since ν is absolutely continuous,

$d\nu(g) = f(g)\,dg$ for some $f \in L^1(G)$.

There exists a nbhd $\mathcal{U}$ of e in G:

$$\|u^{-1} \cdot f - f\|_1 < \varepsilon/2 \quad \text{for } u \in \mathcal{U}.$$

Since H is dense and K is compact,

$$K \subset \bigcup_{i=1}^{n} h_i \mathcal{U} \quad \text{for some } h_i \in H.$$

Then $\|\pi(h_i) w_n - w_n\| < \varepsilon/2$ for large n.

For every $g \in K$, $g = h_i u$ with $u \in \mathcal{U}$,

$$\|\pi(g) w_n - w_n\| \le \|\pi(u) w_n - w_n\| + \underbrace{\|\pi(h_i^{-1}) w_n - w_n\|}_{< \varepsilon/2},$$

$$w_n = \frac{1}{\|\pi(\nu) v_n\|} \cdot \int_G f(g) \pi(g) v_n \, dg,$$

$$\pi(u) w_n = \frac{1}{\|\pi(\nu) v_n\|} \int_G f(g) \pi(u) \pi(g) v_n \, dg = \frac{1}{\|\pi(\nu) v_n\|} \int_G (u^{-1} \cdot f)(g) \pi(g) v_n \, dg,$$

$$\|\pi(u)w_n - w_n\| = \frac{\|\pi(u)\pi(\nu)v_n - \pi(\nu)v_n\|}{\|\pi(\nu)v_n\|}$$

$$= \frac{\|\pi(u^{-1}f - f)v_n\|}{\|\pi(\nu)v_n\|} \leq \frac{\|u^{-1}f - f\|_1}{\|\pi(\nu)v_n\|} < \varepsilon/2$$

for sufficiently large n.

Then w_n is (K,ε)-inv. for $n \gg 0$. $\longrightarrow\!\!\!\!\!\times$

<u>Thm.</u> (Furman-Shalom) $X = \mathbb{R}^d/\mathbb{Z}^d$ - torus, $\Gamma \subset SL_d(\mathbb{Z})$.

Assume that Γ is totally irreducible (i.e., $\nexists$ finite union of proper subspaces of $\mathbb{C}^d$ which is Γ-invariant.)

Let $\nu \in Prob(\Gamma)$, $supp(\nu) \not\subset$ proper coset.

Then $\|\pi_X(\nu)|L^2_0(X)\| < 1$.

The Fourier transform:

$f \longmapsto \hat{f}$: $\hat{f}(k) = \langle f, e_k \rangle$, $e_k(x) = e^{2\pi i \langle k,x \rangle}$

defines isomorphisms $L^2(X) \simeq \ell^2(\mathbb{Z}^d)$ and $L^2_0(X) \simeq \ell^2(\mathbb{Z}^d \setminus \{0\}) = \mathcal{H}$.

The corresponding action $\mathcal{H}$ is given by

$$\{\varphi(k)\} \xrightarrow{\gamma} \{\varphi({}^t\gamma \cdot k)\}.$$

Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be the corresponding unitary representation. It is sufficient to show that $\|\pi(\nu)\| < 1$, and we may assume (as in the previous proof) that $\pi(\nu)$ is self-adjoint and positive.

Suppose that $\|\pi(\nu)\| = \sup_{\|f\|_2 = 1} \langle \pi(\nu) f, f \rangle = 1$.

Then $\exists f_n \in \mathcal{H}$: $\langle \pi(\nu) f_n, f_n \rangle = \sum_{\gamma \in \Gamma} \langle \pi(\gamma) f_n, f_n \rangle \nu(\gamma) \to 1$.
$\|f_n\|_2 = 1$

Since $|\langle \pi(\nu) f_n, f_n \rangle| \leq \langle \pi(\nu) |f_n|, |f_n| \rangle \leq 1$,

we may assume that $f_n \geq 0$.

Since $|\langle \pi(\gamma) f_n, f_n \rangle| \leq 1$, for every $\gamma \in \mathrm{supp}(\nu)$,

$$\langle \pi(\gamma) f_n, f_n \rangle \to 1,$$

$$\|\pi(\gamma) f_n - f_n\|_2^2 = 2 - 2 \langle \pi(\gamma) f_n, f_n \rangle \to 0.$$

Since $\Gamma = \langle \mathrm{supp}(\nu) \rangle$, $\|\pi(\gamma) f_n - f_n\|_2 \to 0$ for all $\gamma \in \Gamma$.

By Cauchy-Schwarz inequality,

$$\|\pi(\gamma)f_n^2 - f_n^2\|_1 \leq \|\pi(\gamma)f_n - f_n\|_2 \cdot \|\pi(\gamma)f_n + f_n\|_2 \quad (*).$$
$$\leq 2\|\pi(\gamma)f_n - f_n\|_2 \longrightarrow 0.$$

Consider the prob. measures $\mu_n = \sum_{x \in \mathbb{Z}^d \setminus \{0\}} f_n^2(x) \cdot \delta_x$ on $\mathbb{C}^d$ and $\overline{\mu_n}$ on $\mathbb{P}(\mathbb{C}^d)$ which are projections of μ_n.

Since $\mathbb{P}(\mathbb{C}^d)$ is compact, $\{\overline{\mu_n}\}$ has a convergent subsequence : $\overline{\mu_{n_i}} \longrightarrow \overline{\mu}$.

By $(*)$, $\overline{\mu}$ is Γ-invariant.

The following lemma gives a contradiction.

Lem. (Furstenberg)

Let $G < SL_d(\mathbb{C})$ be a closed noncompact, totally irreducible subgroup.

Then $\nexists$ G-inv. prob. measure on $\mathbb{P}(\mathbb{C}^d)$.

Let μ be a G-inv. measure on $\mathbb{P}(\mathbb{C}^d)$.

Since G is non-compact, $\exists g_n \in G$: $\|g_n\| \to \infty$.

Let $u_n = \frac{g_n}{\|g_n\|}$. Then $\det(u_n) = \|g_n\|^{-d} \longrightarrow 0$.

Passing to a subsequence, $u_n \to u \in M_d(\mathbb{C})$: $\|u\|=1$, $\det(u)=0$.
Let $V=[\mathrm{Ker}(u)] \subset \mathbb{P}(\mathbb{C}^d)$ and $W=[\mathrm{Im}(u)] \subset \mathbb{P}(\mathbb{C}^d)$.
We write $\mu=\mu_1+\mu_2$ with $\mu_1=\mu|_V$, $\mu_2=\mu|_{V^c}$.
For $x \in V^c$, $g_n \cdot x = u_n \cdot x \longrightarrow u \cdot x$, so that
$$\mu = g_n \cdot \mu = \left(\lim_{n\to\infty} g_n \cdot \mu_1\right) + u \cdot \mu_2 .$$
Passing to a subsequence,
$g_n \cdot \mu_1 \longrightarrow \mu_1^\infty$ = a measure on $\mathbb{P}(\mathbb{C}^d)$,
$g_n \cdot V \longrightarrow V^\infty$ = a projective subspace of $\mathbb{P}(\mathbb{C}^d)$.
Then $\mathrm{supp}(\mu_1^\infty) \subset V^\infty$ and $\mathrm{supp}(u\cdot\mu_2) \subset W$.
Hence, $\mathrm{supp}(\mu) \subset V^\infty \cup W$.
Let $F \subset V^\infty \cup W$ be the minimal set which is a union of proj. subspaces such that $\mu(F)=1$.
Then $\forall g \in G$: $\mu(gF \cap F)=1$ and by minimality, $gF=F$. This contradicts total irreducibility. $\square$

<u>Prop.</u> G - locally compact σ-compact group
(X,μ) - <u>compact</u> homogeneous space with invariant prob. measure.
Then $G \curvearrowright X$ has spectral gap.

If not, $\forall$ cpt $K \subset G$ and $\varepsilon > 0$ $\exists$ (K,ε)-inv. vector in $L^2_0(X)$.
Since G is σ-compact, $\exists \varphi_n \in L^2_0(X) : \|\varphi_n\|_2 = 1$.
$\max\limits_{g \in K} \|\pi_X(g)\varphi_n - \varphi_n\|_2 \longrightarrow 0$ for all compact K. $(*)$

Let $f \in C_c(G)$ with $\int\limits_X f = 1$ and $\psi_n = \pi_X(f)\varphi_n$.

$$\psi_n(x) = \int\limits_G f(g)\varphi_n(g^{-1}x)\,dg,$$

$$\psi_n(h\cdot x) = \int\limits_G f(g)\varphi_n(g^{-1}hx)\,dg = \int\limits_G f(hg)\varphi_n(g^{-1}x)\,dg.$$

$\{\psi_n\}$ is uniformly bounded and equicontinuous.

Hence, $\underset{\text{(uniformly)}}{\psi_n \to \psi} \in C(X)$. By $(*)$, $\|\psi_n\|_2 \to 1$,
and ψ is G-invariant. $\Rightarrow \psi = \text{const}$.
On the other hand, $\int\limits_X \psi = 0$. $\longrightarrow\!\!\!\!\!\!\times$

<u>Thm (Bekka - Cornulier)</u> G - a Lie group, and $X = G/H$ homogeneous space with invariant prob. measure.
Then $G \curvearrowright X$ has spectral gap.

Thm (Bekka–Lubotzky) $G = \mathrm{Aut}(k\text{-regular tree})$.
Then $\exists$ discrete subgroup $\Gamma \subset G$, $\mathrm{vol}(G/\Gamma) < \infty$,
and $G \curvearrowright G/\Gamma$ has <u>no</u> spectral gap.