Lecture 2.

G – locally compact, σ-compact group

(X,μ) – probability space

We assume that $L^2(X)$ is separable.

$G \times X \to X$ – measurable measure-preserving action
($\mu(g^{-1}A) = \mu(A)$ for $g \in G$, $A \subset X$).

For $g \in G$, define
$$\pi_X(g) : L^2(X) \to L^2(X)$$
$$f \longmapsto f(g^{-1}x).$$

Note that: – $\pi_X(g_1 g_2) = \pi_X(g_1)\pi_X(g_2)$

– $\|\pi_X(g)f\|_2 = \left(\int_X |f(g^{-1}x)|^2 d\mu(x)\right)^{1/2} = \|f\|_2$

(i.e. $\pi_X(g)$ is unitary operator).

$$\pi_X : G \to \underset{\text{group of unitary operators}}{\mathcal{U}(L^2(X))} \text{ - homomorphism}$$

Lem. $\forall f \in L^2(X)$: the map $G \to L^2(X) : g \mapsto \pi_X(g)f$ is continuous.

It sufficient to show continuity at e.
By Fubini Thm, the function
$$\langle \pi_X(g)f, f\rangle = \int_X f(g^{-1}x)\overline{f(x)}\, d\mu(x)$$
is measurable.

The the set $A=\{g\in G: \|\pi_X(g)f-f\|_2<\varepsilon/2\}$ is measurable.

1) $\|\pi_X(g)f-f\|_2=\|f-\pi_X(g^{-1})f\|_2 \Rightarrow A=A^{-1}$.

2) $\forall g_1,g_2\in G$: $\|\pi_X(g_1g_2)f-f\|_2=\|\pi_X(g_2)f-\pi_X(g_1^{-1})f\|_2$

$$= \|\pi_X(g_2)f-f\|_2+\|f-\pi_X(g_1^{-1})f\|_2<\varepsilon.$$

Hence, $A^2\subset\{g\in G: \|\pi_X(g)f-f\|_2<\varepsilon\}$. $(*)$

Since $\pi_X(G)f\subset L^2(X)$ is separable, $\{\pi_X(g_n)f\}_{n\geq 1}$ is dense in $\pi_X(G)f$ for some $\{g_n\}_{n\geq 1}\subset G$.

Then $\forall g\in G$: $\exists g_n$: $\|\pi_X(g)f-\pi_X(g_n)f\|_2<\varepsilon/2$

$$\|\pi_X(g)f-\pi_X(g_n)f\|_2 = \|\pi_X(g_n^{-1}g)f-f\|_2,$$

so $g_n^{-1}g\in A$. Hence, $G=\bigcup_{n\geq 1} g_nA$.

In particular, $|A|>0$.

<u>Lem.</u> $|A|>0 \Rightarrow A\cdot A^{-1}\supset$ a nbhd of e in G.

By $(*)$, $\{g: \|\pi_X(g)f-f\|_2<\varepsilon\}\supset\{\text{nbhd of } e\}$.

Rmk. The map $G \to \mathcal{U}(L^2(X))$ is <u>not</u> always continuous w.r.t. the norm topology.

A <u>unitary representation</u> of G is a homomorphism
$$\pi : G \to \mathcal{U}(\mathcal{H}), \quad \mathcal{H} - \text{Hilbert space},$$
such that the maps $g \mapsto \pi(g)v$, $v \in \mathcal{H}$, are continuous.

Measure-preserving actions $\longrightarrow$ Unitary representations

Howe–Moore Thm.

$T : (X,\mu) \to (X,\mu)$ – measure-preserving transformation of a prob. space.

1) T is <u>ergodic</u>: $\forall f \in L^2(X)$: $f \circ T = f \Rightarrow f = \text{const}$.

$\Uparrow$

2) T is <u>weak mixing</u>: $\forall f \in L^2(X)$: $f \circ T = \lambda f \Rightarrow f = \text{const}$.

$\Uparrow$

3) T is <u>mixing</u>: $\forall f_1, f_2 \in L^2(X)$: $\langle f_1 \circ T^n, f_2 \rangle \xrightarrow[n\to\infty]{} \int_X f_1 \cdot \int_X f_2$.

For one transformation, all these notitions are distinct.

$G \overset{\alpha}{\curvearrowright} (X,\mu)$ — measure-preserving action

1) α is <u>ergodic</u>: $\forall f \in L^2(X)$: $\pi_X(G)f = f \Rightarrow f = \text{const}$.

2) α is <u>weak-mixing</u>: $\forall f \in L^2(X)$: $\langle \pi_X(G) f \rangle$ is fin. dimensional
$\Downarrow$
$f = \text{const}$.

3) α is <u>mixing</u>: $\forall f_1, f_2 \in L^2(X)$: $\langle \pi_X(g) f_1, f_2 \rangle \to \int_X f_1 \cdot \int_X f_2$
as $g \to \infty$.

(i.e., $\forall \varepsilon > 0$ $\exists$ compact $K \subset G$: $\forall g \in G \setminus K$:
$\left| \langle \pi_X(g) f_1, f_2 \rangle - \int_X f_1 \int_X f_2 \right| < \varepsilon$.)

<u>Thm.</u> (Howe–Moore) $G = SL_d(\mathbb{R})$.
Then ergodicity $\Rightarrow$ mixing.
(More generally, G = simple connected Lie group with finite center)

<u>Thm.</u> (Howe–Moore) $\pi: G \to \mathcal{U}(\mathcal{H})$ - unitary representation.
Assume that $\mathcal{H} \not\ni$ G-fixed vectors.
Then $\forall v, w \in \mathcal{H}$: $\langle \pi(g) v, w \rangle \to 0$ as $g \to \infty$.

<u>Lem.</u> (Cartan decomposition)
$G = KA^+K$ where $K = SO_d(\mathbb{R})$,
$A^+ = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} : a_1 \geq a_2 \geq \dots \geq a_d > 0 \right\}$.

$g \in G$, $Q(x) = \|gx\|^2$ is a pos. definite quad. form.
It can be diagonalised using $k \in SO_d(\mathbb{R})$.
Then $\exists a \in A^+$: $Q(k^{-1}a^{-1}x) = \|x\|^2 = \|gk^{-1}a^{-1}x\|^2$.
Hence, $gk^{-1}a^{-1} \in SO_d(\mathbb{R})$.

<u>Proof of Thm.</u> Suppose by contradiction that for $v, w \in \mathcal{H}$ and $g_n \in G$, $g_n \to \infty$,

$$\langle \pi(g_n) v, w \rangle \not\to 0.$$

Write $g_n = k_n a_n l_n$ with $k_n, l_n \in K$ and $a_n \in A^+$.
Note that K is compact, so $a_n \to \infty$.
Passing to a subsequence, we may assume that $\{k_n\}$ and $\{l_n\}$ converge.
Then $\pi(l_n)v \to \tilde{v} \in \mathcal{H}$, $\pi(k_n)^{-1} w \to \tilde{w} \in \mathcal{H}$, and

$$\langle \pi(\underbrace{k_n a_n l_n}_{g_n}) v, w \rangle - \langle \pi(a_n)\tilde{v}, \tilde{w} \rangle$$

$$= \langle \pi(a_n)\pi(l_n)v, \pi(k_n)^{-1} w \rangle - \langle \pi(a_n)\tilde{v}, \tilde{w} \rangle$$

$$= \langle \pi(a_n)(\pi(l_n)v - \tilde{v}), \pi(k_n)^{-1} w \rangle + \langle \pi(a_n)\tilde{v}, \pi(k_n)^{-1} w - \tilde{w} \rangle.$$

$$\|\pi(a_n)(\pi(l_n)v - \tilde{v})\| = \|\pi(l_n)v - \tilde{v}\| \to 0,$$

$$\|\pi(k_n)^{-1} w\| = \|w\|,$$

$$\|\pi(a_n)\tilde{v}\| = \|\tilde{v}\|,$$

$$\|\pi(k_n)^{-1} w - \tilde{w}\| \to 0.$$

Hence, by Cauchy-Schwarz inequality,

$$\langle \pi(\underbrace{k_n a_n l_n}_{g_n})v, w\rangle - \langle \pi(a_n)\tilde{v}, \tilde{w}\rangle \longrightarrow 0,$$

and $\langle \pi(a_n)\tilde{v}, \tilde{w}\rangle \not\longrightarrow 0$.

Weak convergence: $v_n \in \mathcal{H}$: $v_n \xrightarrow{\text{weak}} v$ if $\langle v_n, w\rangle \to \langle v, w\rangle$ for all $w \in \mathcal{H}$.

Banach-Alaoglu Thm: Closed bounded subsets in $\mathcal{H}$ are compact in weak topology.

Proof for $G = SL_2(\mathbb{R})$:

Suppose that for some $a_n \in A^+$, $a_n \to \infty$, and $v, w \in \mathcal{H}$, we have $\langle \pi(a_n)v, w\rangle \not\to 0$.

Write $a_n = \begin{pmatrix} t_n & 0 \\ 0 & t_n^{-1}\end{pmatrix}$, with $t_n \to \infty$.

Since $\|\pi(a_n)v\| = \|v\|$, the sequence $\pi(a_n)v$ has a subsequence which converges in weak topology.

Hence, we may assume that $\pi(a_n)v \xrightarrow{\text{weak}} \tilde{v} \in \mathcal{H}$.

Then $\langle \pi(a_n)v, w\rangle \longrightarrow \langle \tilde{v}, w\rangle \neq 0$.

Let $u_s = \begin{pmatrix} 1 & s \\ 0 & 1\end{pmatrix}$. Note that $a_n^{-1} u_s a_n = \begin{pmatrix} 1 & s/t_n^2 \\ 0 & 1\end{pmatrix} \to e$.

Hence, $\pi(u_s)\tilde{v} = \underset{n\to\infty}{w\text{-}\lim}\, \pi(u_s)\pi(a_n)v = \underset{n\to\infty}{w\text{-}\lim}\, \pi(a_n)\pi(u_{s/t_n^2})v$.

Since $\|\pi(a_n)\pi(u_{s/t_n^2})v - \pi(a_n)v\| = \|\pi(u_{s/t_n^2})v - v\| \to 0$,

$$\pi(u_s)\tilde{v} = \underset{n\to\infty}{w\text{-}\lim}\, \pi(a_n)v = \tilde{v}.$$

Hence, $\tilde{v}$ is fixed by the group $U = \begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}$.

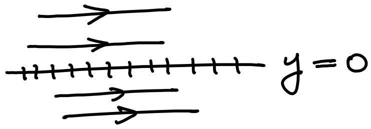
Consider the function:

$$F(g) = \langle \pi(g)\tilde{v}, \tilde{v} \rangle, \quad g \in G.$$

Then $F(U \cdot g \cdot U) = F(g)$.

We consider F as a function on $G/U \simeq \mathbb{R}^2 \setminus \{0\}$.

$U \circlearrowright \mathbb{R}^2$



Orbits of U $\begin{cases} \text{lines } y = c,\ c \neq 0, \\ \text{points } \{(x,0)\}. \end{cases}$

Since F is U-inv., $F = \text{const}$ on $y = c$, $c \neq 0$.

By continuity, $F = \text{const}$ on $y = 0$.

Then $\langle \pi(a_t)\tilde{v}, \tilde{v} \rangle = F(a_t e_1) = F(t e_1) = F(e_1) = \|\tilde{v}\|^2$.

This gives equality in the Cauchy-Schwartz inequality, so that $\pi(a_t)\tilde{v} = \lambda \tilde{v}$ and $\lambda = 1$.

Hence, $\tilde{v}$ is (AU)-invariant, and

F is (AU)-biinvariant.

Since $AU \cdot e_2 = \{y > 0\}$ and $AU(-e_2) = \{y < 0\}$,

$F = \text{const}$ on $\{y > 0\}$ and on $\{y < 0\}$.

Hence, $F = \text{const}$ (by continuity):

$$\langle \pi(g)\tilde{v}, \tilde{v} \rangle = \|\tilde{v}\|^2 \text{ for all } g \in G.$$

As above, by Cauchy-Schwarz inequality,

$\pi(g)\tilde{v} = \tilde{v}$ for all $g \in G$, and $\tilde{v} = 0$ $\times$.

Proof for $G = SL_n(\mathbb{R})$:

Let $\alpha_i(a) = \frac{a_i}{a_{i+1}}$ for $a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} \in A^+$.

The set $\left\{ (a_1, \dots a_d) : \begin{array}{l} a_1 \geq \dots \geq a_d > 0 \\ a_1 \dots a_d = 1 \\ a_i \leq t a_{i+1} \end{array} \right\}$ is bounded.

Hence, if $a_n \to \infty$, then $\max_i (\alpha_i(a_n)) \to \infty$

Suppose that $\langle \pi(a_n) v, w \rangle \not\to 0$ for $a_n \in A^+$, $a_n \to \infty$, $v, w \in \mathcal{H}$.

Passing to a subsequence, we may assume that

$$\pi(a_n) v \xrightarrow{\text{weak}} \tilde{v} \in \mathcal{H},$$

$$\alpha_i(a_n) \longrightarrow \infty \quad \text{for some } i.$$

Let $U_i = \left(\begin{array}{c|c} I & * \\ \hline 0 & I \end{array} \right)\}i$. We have:

$$a^{-1} \left(\begin{array}{c|c} I & u_{\ell k} \\ \hline 0 & I \end{array} \right) a = \left(\begin{array}{c|c} I & \frac{a_k}{a_\ell} u_{\ell k} \\ \hline 0 & I \end{array} \right), \quad \frac{a_k}{a_\ell} = \frac{a_k}{a_{k-1}} \cdots \frac{a_{\ell+1}}{a_\ell} \leq \frac{a_{i+1}}{a_i},$$

hence, $\frac{a_k^{(n)}}{a_\ell^{(n)}} \to 0$.

As in SL_2-case, $U_i \cdot \tilde{v} = \tilde{v}$.

Let $G_{\ell k} = \begin{array}{c} \\ \ell \\ k \end{array} \begin{array}{c} \begin{array}{cc} \ell & k \end{array} \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{array} \simeq SL_2(\mathbb{R})$.

Then by SL_2-case, $G_{\ell k} \cdot \tilde{v} = \tilde{v}$.
for $1 \leq \ell \leq i$ and $i+1 \leq k \leq d$.

Since these groups generate G, $G \cdot \tilde{v} = \tilde{v}$ and $\tilde{v} = 0$ $\blacksquare$