Lecture 13

Automorphic spectrum & Burger–Sarnak argument.

$G < GL_n(\mathbb{C})$ – simple alg. group defined over $\mathbb{Q}$, $G(\mathbb{R})$ is noncompact.

For simplicity assume that G is simply connected.

$\Gamma_\ell = \{\gamma \in G(\mathbb{Z}) : \gamma = I \pmod{\ell}\}$,

$X_\ell = G(\mathbb{R})/\Gamma_\ell$,

$\tau_\ell : G(\mathbb{R}) \curvearrowright L^2(X_\ell)$.

$\widehat{G(\mathbb{R})} = \{$irreducible unitary representations of $G(\mathbb{R})\}$

$\bigcup$

$\widehat{G(\mathbb{R})}^{Aut} = \{\pi : \pi$ is weakly contained in $\bigoplus_{\ell \geq 1} \tau_\ell\}$

Question: Describe $\widehat{G(\mathbb{R})}^{Aut}$?

<u>Thm.</u> (Burger–Sarnak) Let $H \subset G$ – simple algebraic group, defined over $\mathbb{Q}$.

Then $\forall \pi \in \widehat{G(\mathbb{R})}^{Aut}$: $\pi|_{H(\mathbb{R})} \in \widehat{H(\mathbb{R})}^{Aut}$.

Hecke operators.

Take a prime p, coprime to ℓ, such that $G(\mathbb{Q}_p)$ is noncompact.

Let $\Lambda_{\ell,p} = \{ \lambda \in G(\mathbb{Z}[\frac{1}{p}]) : \lambda \equiv I \pmod{\ell} \}$.

Then $\Lambda_{\ell,p} \underset{diag}{\subset} G(\mathbb{R}) \times G(\mathbb{Q}_p)$ is a lattice.

$$Y_{\ell,p} = (G(\mathbb{R}) \times G(\mathbb{Q}_p)) / \Lambda_{\ell,p}.$$

By Strong Approximation Property, $\Lambda_{\ell,p} \to G(\mathbb{Q}_p)$ is dense.

$U_p = G(\mathbb{Z}_p)$ – compact open subgroup in $G(\mathbb{Q}_p)$.

Lem. $G(\mathbb{R})/\Gamma_e \simeq U_p \backslash (G(\mathbb{R}) \times G(\mathbb{Q}_p))/\Lambda_{e,p}$.

Consider the map $g \longmapsto U_p(g,e)\Lambda_{e,p}$, $g \in G(\mathbb{R})$.
It is $G(\mathbb{R})$-equivariant.
$\forall h \in G(\mathbb{Q}_p)$: $h = u\lambda$ for $u \in U_p$ and $\lambda \in \Lambda_{e,p}$.
Then $(g, u\lambda) = (e,u)\cdot(g\lambda^{-1}, e)\cdot(\lambda,\lambda)$, so that
this map is surjective.
If $U_p(g,e)\Lambda_{e,p} = U_p \cdot \Lambda_{e,p}$, then $(g\lambda, u\lambda) \in U_p \times \{e\}$.
and $g \in U_p \cap \Lambda_{e,p} = \Gamma_e$. $\square$

In particular, $L^2(Y_{e,p})^{U_p} \simeq L^2(X_e)$.

Let $B_n = \{b \in G(\mathbb{Q}_p) : \|b\|_p = p^n\}$, and

$$\pi_Y(B_n) : L^2(Y_{e,p}) \longrightarrow L^2(Y_{e,p})$$
$$\varphi \longmapsto \frac{1}{|B_n|}\int_{B_n} \varphi(b^{-1}y)\,db.$$

$$
\begin{tikzcd}
L^2(Y_{e,p})^{U_p} \arrow[r, "\pi_Y(\beta_n)"] & L^2(Y_{e,p})^{U_p} \\
L^2(X_e) \arrow[u, phantom, "\cong" description] \arrow[r, "T_n"] & L^2(X_e) \arrow[u, phantom, "\cong" description]
\end{tikzcd}
$$

<u>Hecke operators</u>

<u>Lem.</u> $T_n f(g\Gamma_e) = \frac{1}{|\Omega_n|} \cdot \sum_{\lambda \in \Omega_n} f(g\lambda\Gamma_e)$,

where $\Omega_n = (B_n \cap \Lambda_{e,p})/\Gamma_e$.

We have U_p-coset decomposition:

$$B_n = \bigsqcup_{\lambda \in \Omega_n} \lambda U_p .$$

For U_p-inv. φ,

$$\pi_Y(\beta_n)\varphi\big((g,e)\Lambda_{e,p}\big) = \frac{1}{|\Omega_n|} \cdot \sum_{\lambda \in \Omega_n} \varphi\big((g,\lambda^{-1})\Lambda_{e,p}\big)$$

$$= \frac{1}{|\Omega_n|} \cdot \sum_{\lambda \in \Omega_n} \varphi\big((g\lambda,e)\Lambda_{e,p}\big).$$

We knows that $\pi_Y(\beta_n)\varphi \xrightarrow[n\to\infty]{L^2} \int_{Y_{e,\rho}} \varphi$, so that

$$T_n f \xrightarrow[n\to\infty]{L^2} \int_{X_e} f. \qquad (*)$$

<u>Prop.</u> For $f \in C_c(X_e)$,

$$T_n f \longrightarrow \int_{X_e} f$$

uniformly on compact sets.

$\ulcorner$ $\forall \varepsilon > 0$: $\exists$ nbhd $\mathcal{O}$ of e:

$|f(ux) - f(x)| < \varepsilon$ for all $u \in \mathcal{O}$ and $x \in X_e$.

Then also $|T_n f(ux) - T_n(x)| < \varepsilon$.

Take $\rho_x \in C_c(X_e)$, $\operatorname{supp}(\rho_x) \subset \mathcal{O}x$, $\rho_x \geq 0$, $\int_{X_e} \rho_x = 1$.

Then $|\langle T_n f, \rho_x \rangle - T_n f(x)| < \varepsilon$.

By $(*)$, $\langle T_n f, \rho_x \rangle \longrightarrow \int_{X_e} f$ as $n \to \infty$.

Hence, $T_n f(x) \longrightarrow \int_{X_e} f$. $\lrcorner$

Proof of Thm.

Take $f \in C_c(X_e)$ and consider the matrix coefficient:

$$c(h) = \int_{X_e} \underbrace{f(h^{-1}x)\overline{f(x)}}_{F(x)}\, dx$$

We need to show that

$$c(h) \approx \sum_i d_i \langle \pi_i(h) v_i, v_i \rangle,$$

where π_i's are automorphic representations of $H(\mathbb{R})$.

We use that $T_n F(x) \longrightarrow c(h)$ uniformly on compact sets, so that

$$\int_{H(\mathbb{R})/\Delta} T_n F(x)\, dx \longrightarrow c(h), \quad \text{where } \Delta = H(\mathbb{R}) \cap \Gamma_e.$$

dx ← prob. measure on $H(\mathbb{R})/\Delta$

$$\int_{H(\mathbb{R})/\Delta} T_n F(x)\, dx = \frac{1}{|\Omega_n|} \cdot \int_{H(\mathbb{R})/\Delta} \sum_{\lambda \in \Omega_n} F(x\lambda\Gamma_e)\, dx$$

We have coset decomposition: $\Omega_n = \bigsqcup_i \Delta \delta_i$.

Set $x_i = \delta_i \Gamma_e$ and $\Delta_i = \mathrm{Stab}_\Delta(x_i)$.

$$\int_{H(\mathbb{R})/\Delta} \sum_{\lambda \in \Omega_n} F(x\lambda\Gamma_e)\,dx = \sum_i \int_{H(\mathbb{R})/\Delta_i} F(x \cdot x_i)\,dx$$

$$= \sum_i \underbrace{\int_{H(\mathbb{R})/\Delta_i} f(h^{-1}x x_i)\,\overline{f(x x_i)}\,dx}_{\text{matrix coefficients for } L^2(H(\mathbb{R})/\Delta_i)}.$$

Since $\Delta_i = \Delta \cap \delta_i \Gamma_e \delta_i^{-1}$ with $\delta_i \in G(\mathbb{Z}[\frac{1}{p}])$, it contains a congruence subgroup, so that $L^2(H(\mathbb{R})/\Delta_i)$ are automorphic. $\square$