Lecture 12

Selberg property.

$G < GL_N(\mathbb{C})$ – simple algebraic group defined over $\mathbb{Q}$

$\Gamma(\ell) = \{\gamma \in G(\mathbb{Z}) : \gamma = I \pmod{\ell}\}$

↖ congruence subgroups

Consider the family of unitary representations:

$$\pi_\ell : G(\mathbb{R}) \curvearrowright L_0^2(G(\mathbb{R})/\Gamma(\ell))$$

<u>Thm</u> (Selberg property) $\boxed{\sup_{\ell \geq 1} q(\pi_\ell) < \infty.}$

History: 1) $G = SL_2$: – Selberg (Kloosterman sums)

– Gelbart-Jacquet, Luo-Rudnick-Sarnak Kim-Sarnak (Langland's functoriality)

<u>Conj.</u> (Selberg) $q(\pi_\ell) = 2$

Known: $q(\pi_\ell) \leq \frac{64}{25}$ (Kim-Sarnak)

2) $\underset{\sim}{SL(1,D)}$, D - quaternion algebra:

Jasquet-Langlands correspondence

If D is ramified $/\mathbb{R}$, $q(\pi_\infty) = 2$ (by Deligne).

3) Rank $G(\mathbb{R}) \geq 2$:

Kazhdan property $\Rightarrow$ Selberg property

4) Subgroups $\rightsquigarrow$ Groups

Burger - Sarnak

5) Unitary groups

Clozel.

In this lecture, we prove the following partial case of the Selberg property (following Sarnak-Xue):

Fix $a, b \in \mathbb{N}$ such that $ax^2 + by^2 = z^2$ has no solutions in $\mathbb{Z}$.

$$\Gamma = \left\{ \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix} : \begin{array}{c} x_0, x_1, x_2, x_3 \in \mathbb{Z} \\ \det = 1 \end{array} \right\}$$

$$\cup$$

$$\Gamma_\ell = \{ x_0 = 1 \pmod{\ell},\ x_1, x_2, x_3 = 0 \pmod{\ell} \}.$$

Then $X_\ell = SL_2(\mathbb{R})/\Gamma_\ell$ is compact.

Thm. For $\pi_\ell : SL_2(\mathbb{R}) \curvearrowright L^2_0(X_\ell)$,

$$\sup_{\ell \geq 1} q(\pi_\ell) < \infty.$$

Direct decomposition

G – locally compact second countable group

Γ – discrete cocompact subgroup

Consider $\pi : G \curvearrowright L^2(G/\Gamma)$.

Thm $L^2(G/\Gamma) = \bigoplus_i \mathcal{H}_i$ where $\mathcal{H}_i$'s irreducible G-inv. subspace, and each representation appears only finitely often.

For $f \in C_c(G)$, consider the operator:

$$\pi(f) : \varphi \longmapsto \int_G f(g)\,\pi(g)\varphi \, dg.$$

$$\pi(f)\varphi(h) = \int_G f(g)\varphi(\underbrace{g^{-1}h}_{y})\,dg = \int_G f(hy^{-1})\varphi(y)\,dy$$

$$= \int_{G/\Gamma} \underbrace{\Big(\sum_{\gamma\in\Gamma} f(h\gamma^{-1}y^{-1})\Big)}_{K(h,y)\in C(G/\Gamma\times G/\Gamma)}\varphi(y)\,dy.$$

Hence, $\pi(f)$ is integral operator with kernel $K(\cdot,\cdot)$.

In particular, $\pi(f)$ is compact.

Now we always assume that $\overline{f(g^{-1})} = f(g)$

$\Downarrow$

($\pi(f)$ is self-adjoint)

Then for every inv. subspace $\mathcal{H}$,

$$\mathcal{H} = \bigoplus_{i\geq 1} \mathcal{H}_{f,\lambda_i}$$

where $\mathcal{H}_{f,\lambda_i}$ are λ_i-eigenspaces of $\pi(f)$,

and $\dim(\mathcal{H}_{f,\lambda_i}) < \infty$ for $\lambda_i \neq 0$.

<u>Claim.</u> $\forall$ G-inv. $\mathcal{H}\neq 0$ $\exists f$: $\mathcal{H}_{f,\lambda}\neq 0$ for some $\lambda\neq 0$.

$\ulcorner$If not, then $\pi(f)|_{\mathcal{H}} = 0$ for all f.

Take a sequence f_n with $\int_G f_n = 1$, $f_n \to$ "Dirac measure at e."

Then $\pi(f) \to id$, which is a contradiction.$\lrcorner$

Now fix G-inv. subspace $\mathcal{H} \neq 0$, and choose G-inv $S \subset \mathcal{H}$ such that $S_{f,\lambda} \neq 0$ and has minimal dimension. Let T be the minimal G-inv. subspace containg $S_{f,\lambda}$.
Then T is irreducible.

This proves that every inv. subspace contains an irreducible subspace, and hence

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i$$

where $\mathcal{H}_i$'s are irreducible.

Suppose that $\mathcal{H}_i \simeq \mathcal{H}_j$ and take f such that $\pi(f)|_{\mathcal{H}_i} \neq 0$. Then $(\mathcal{H}_i)_{f,\lambda} \neq 0$ for some $\lambda \neq 0$.

$$(\mathcal{H}_j)_{f,\lambda} \neq 0$$

This implies that multiplicities are finite.

$\square$

Irreducible representations of $SL_2(\mathbb{R})$

1) Principle series

$$\mathcal{H}_s^{\pm} = \{ u : \mathbb{R}^2 \to \mathbb{C} : \ u(tx) = |t|^{s-1} \operatorname{sign}(t) u(x), \ t \in \mathbb{R} \}, \quad s \in i\mathbb{R}.$$

$$\|u\| = \left(\int_{|x|=1} |u(x)|^2 dx \right)^{1/2}$$

$$\pi_s^{\pm}(g) u(x) = u(g^{-1}x)$$

Exercise: 1) $\pi_s^{\pm}$ is unitary representation

2) $u_0 = (x_1^2 + x_2^2)^{\frac{s-1}{2}}$ is unique unit $SO(2)$-inv. vector

3) $|\langle \pi_s^{\pm}(g) u_0, u_0 \rangle| \leq \Xi(g)$ ← Harish-Chandra function

In particular, $\pi_s^{\pm} \prec$ "regular representation".

2) Complementary series.

$$\mathcal{H}_s = \mathcal{H}_s^{+} \text{ (as above)}, \quad s \in (0,1)$$

$$\langle u, v \rangle = \iint_{|x|=|y|=1} \frac{u(x)\overline{v(y)}}{|x_1 y_2 - x_2 y_1|^{s+1}}$$

$$\sigma_s(g) u(x) = u(g^{-1}x).$$

Exercise: 1) σ_s is a unitary representation

2) For $a_t = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}$,

$$\langle \sigma_s(a_t)u_0, u_0\rangle \asymp \int_0^{2\pi} \left(e^{2t}\cos^2\theta + e^{-2t}\sin^2\theta\right)^{\frac{s-1}{2}} d\theta$$

$$\sim \text{const}\cdot e^{(s-1)t} \quad \text{as } t\to\infty.$$

Multiplicities.

$G = SL_2(\mathbb{R})$

$X_\ell = G/\Gamma_\ell$

$\tau_\ell : G \circlearrowright L^2(X_\ell)$

$L^2(X_\ell) = \bigoplus_\pi m_\ell(\pi)\cdot \mathcal{H}_\pi$

Aim: Estimate $m_\ell(\pi)$ for complementary series?

Idea: Take $f\in C_c(G)$ such that $\pi(f)\geq 0$.

$\mathrm{Tr}(\pi(f)) = \sum_i \langle \pi(f)e_i, e_i\rangle$,

where $\{e_i\}$ is an orthonormal basis of $L^2(X_\ell)$.

Then:

$$\mathrm{Tr}(\tau_\ell(f)) = \sum_\pi m_\ell(\pi)\,\mathrm{Tr}(\pi(f)) \geq m_\ell(\sigma_s)\,\mathrm{Tr}(\sigma_s(f)).$$

Hence, $m_\ell(\sigma_s) \leq \dfrac{\mathrm{Tr}(\tau_\ell(f))}{\mathrm{Tr}(\sigma_s(f))}$.

Take $SO(2)$-biinvariant $F : G \to \mathbb{C}$:

$$F(a_t) = \chi_{[0,R]}(t) \underbrace{\langle \sigma_s(a_t) u_0, u_0 \rangle}_{\varphi_s(a_t)}.$$

$$f = F^* * F.$$

<u>Lem. 1.</u> $\operatorname{Tr}(\sigma_s(f)) = \|F\|_2^4 \asymp e^{4sR}$.

Let $\{e_i\}_{i \geq 1}$ be an orthonormal basis, $e_1 = u_0$.

Then $\sigma_s(f) e_i = \lambda_i e_1$ and $\langle \sigma_s(f) e_i, e_j \rangle = 0$ for $(i,j) \neq (1,1)$.

$$\langle \sigma_s(f) e_1, e_1 \rangle = \| \sigma_s(F) e_1 \|^2 = \langle \sigma_s(F) e_1, e_1 \rangle^2.$$

$$\langle \sigma_s(F) e_1, e_1 \rangle = \int_G F(g) \underbrace{\langle \sigma_s(g) u_0, u_0 \rangle}_{\varphi_s(g)} dg = \|F\|_2^2$$

Hence, $\operatorname{Tr}(\sigma_s(f)) = \sum_{i \geq 1} \langle \sigma_s(f) e_i, e_i \rangle = \|F\|_2^2$

The Haar measure is $dk_1 \sinh(2t)\, dt\, dk_2$.

Since $\varphi_s(a_t) \asymp e^{(s-1)t}$,

$$\|F\|_2^2 = \int_0^R e^{2(s-1)t} \sin(2t)\, dt \asymp e^{2sR}.$$

Lem. 2 $\quad f(a_t) \ll \begin{cases} e^{2sR-t}, & t\le 2R, \\ 0, & t>2R. \end{cases}$

$$f(a_t) = \int_G \overline{F(g^{-1})}\, F(g^{-1}a_t)\,dg = \int_G \overline{F(g)}\, F(ga_t)\,dg$$

$$= \int_{K\times A^+\times K} \overline{F(k_1 a_r k_2)}\, F(k_1 a_r k_2 a_t)\, \sinh(2r)\, dk_1\, dr\, dk_2$$

$$\ll \int_0^R e^{(s-1)r} \left(\int_K F(a_r k a_t)\, dk \right) \cdot e^{2r}\, dr$$

$$\ll \int_0^R e^{(s+1)r} \left(\int_{k:\ t(a_r k a_t)\le R} e^{(s-1)\,t(a_r k a_t)}\, dk \right) dr$$

We use that

$$\|k_1 a_u k_2\|^2 = e^{2u} + e^{-2u} = 2\cosh(2u),$$

$$\|a_r k_\theta a_t\|^2 = \left\| \begin{pmatrix} e^{r+t}\cos\theta & e^{r-t}\sin\theta \\ -e^{-r+t}\sin\theta & e^{-r-t}\cos\theta \end{pmatrix} \right\|^2 = 2\cosh 2(t+r)\cos^2\theta + 2\cosh 2(t-r)\sin^2\theta$$

Hence, $t(a_r k_\theta a_t) \le R \iff \cosh 2(t+r)\cos^2\theta + \cosh 2(t-r)\sin^2\theta \le \cosh(2R)$

$$\Downarrow$$

$$|\cos\theta| \ll e^{R-t-r}.$$

If $t>2R$, $\cosh 2(t+r)\cos^2\theta + \cosh 2(t-r)\sin^2\theta > \cosh(2R)\cdot(\cos^2\theta+\sin^2\theta)$

$$= \cosh(2R),$$

and $f(a_t)=0$.

Since $e^{2t(a_r k_\theta a_t)} \gg \cosh 2t(a_r k_\theta a_t) \gg e^{2(t+r)}(\cos\theta)^2$,

$$\int_0^R e^{(s+1)r}\left(\int_{k:\, t(a_r k a_t)\le R} e^{(s-1)t(a_r k a_t)}\,dk\right)dr$$

$$\ll \int_0^R e^{(s+1)r}\int_{|\cos\theta|<e^{R-t-s}} e^{(s-1)(t+r)}\cdot|\cos\theta|^{s-1}\,d\theta\,dr$$

$$\ll \int_0^R e^{(s+1)r}\cdot e^{(s-1)(t+r)}\cdot e^{s(R-t-r)}\,dr \asymp e^{2sR-t}.$$

Thm. $\boxed{m_\ell(\sigma_s) \le C_\varepsilon\cdot|\Gamma_1:\Gamma_\ell|^{1-s}.}$

$$\mathrm{Tr}(\tau_\ell(f)) = \sum_{\gamma\in\Gamma_\ell}\int_{G/\Gamma_\ell} f(x\gamma x^{-1})\,dx$$

$$= \sum_{\gamma\in\Gamma_\ell}\sum_{\delta\in\Gamma_1/\Gamma_\ell}\int_{G/\Gamma_1} f(x\delta\gamma\delta^{-1}x^{-1})\,dx$$

$$= |\Gamma_1:\Gamma_\ell|\cdot\sum_{\gamma\in\Gamma_\ell}\int_{G/\Gamma_1} f(x\gamma x^{-1})\,dx$$

$$\underset{\text{Lem.2}}{\ll} |\Gamma_1:\Gamma_\ell|\cdot\sum_{\gamma\in\Gamma_\ell}\int_{G/\Gamma_1} e^{2sR-t(x\gamma x^{-1})}\,dx.$$

Since G/Γ_1 is compact, $t(x\gamma x^{-1}) \geq t(\gamma) - c$ for uniform $c > 0$.

Hence, $\mathrm{Tr}(\tau_\ell(f)) \ll |\Gamma_1 : \Gamma_\ell| \cdot e^{2sR} \cdot \sum_{\gamma \in \Gamma_\ell : t(\gamma) \leq 2R} e^{-t(\gamma)}$.

Note that $t(g) \sim \log \|g\|$.

By Lem. 3 below, $\#\{\gamma \in \Gamma_\ell : \|\gamma\| \leq T\} = O_\varepsilon\left(\frac{T^{2+\varepsilon}}{\ell^3} + T^{1+\varepsilon}\right)$,

so that $\sum_{\gamma \in \Gamma_\ell : t(\gamma) \leq 2R} e^{-t(\gamma)} \ll_\varepsilon \frac{e^{(1+\varepsilon)2R}}{\ell^3} + e^{\varepsilon \cdot 2R}$.

Note that $|\Gamma_1 : \Gamma_\ell| \asymp \ell^3$.

Take R so that $e^{2R} \asymp |\Gamma_1 : \Gamma_\ell|$.

Then $m_\ell(\sigma_s) \leq \frac{\mathrm{Tr}(\tau_\ell(f))}{\mathrm{Tr}(\sigma_s(f))} \ll_\varepsilon \frac{|\Gamma_1 : \Gamma_\ell| \cdot e^{(2s+\varepsilon)R}}{e^{4sR}}$

$\ll_\varepsilon |\Gamma_1 : \Gamma_\ell|^{1-s+\varepsilon}$ for all $\varepsilon > 0$. $\square$

<u>Lem. 3.</u> $\#\{\gamma \in \Gamma_\ell : \|\gamma\| < T\} \ll_\varepsilon \frac{T^{2+\varepsilon}}{\ell^3} + T^{1+\varepsilon}$ for all $\varepsilon > 0$.

We need to estimate the number of solutions:

$$\begin{cases} x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2 = 1 \\ x_0 = 1\ (\ell),\quad x_1, x_2, x_3 = 0\ (\ell) \\ |x_i| \leq T \end{cases}$$

We have $x_0^2 = 1 \pmod{\ell} \overset{\ell+2}{\Longrightarrow} x_0 = 1 \pmod{\ell^2}$.

We have $O\left(\frac{T}{\ell^2}+1\right)$ possibilities for x_0,

$O\left(\frac{T}{\ell}+1\right)$ possibilities for x_3.

Then $a x_1^2 + b x_2^2 = \xi \neq 0$, and there are $O_\varepsilon(\xi^\varepsilon)$ solutions.

We obtain $\left(\frac{T}{\ell^2}+1\right)\left(\frac{T}{\ell}+1\right)T^\varepsilon = \frac{T^{2+\varepsilon}}{\ell^3} + O(T^{1+\varepsilon})$.

<u>Lem. 4.</u> For $\ell \geq 5$, dimension of nontrivial representation of $SL_2(\mathbb{Z}_\ell)$ is $\geq \frac{\ell-1}{2}$.

<u>Cor</u> $\exists s_0 \in (0,1)$: σ_s with $s \in (s_0, 1)$ doesn't occur in τ_ℓ.

There is an action of $\Gamma_1/\Gamma_\ell \simeq SL_2(\mathbb{Z}_\ell)$ on $L^2(X_\ell)$ which commutes with the action of G, and the set of fixed points is $L^2(X_1)$.

We know that $G \curvearrowright L^2(X_1)$ has spectral gap, so it remains to check the claim for $\mathcal{H} = L^2(X_1)^\perp$.

Suppose that $m \cdot \mathcal{H}_{\sigma_s} \subset \mathcal{H}$ with $m > 0$. Then $SL_2(\mathbb{Z}_\ell)$ acts on $m\mathcal{H}_{\sigma_s}$ and $(m\mathcal{H}_{\sigma_s})^{SO(2)}$ without fixed vectors. Hence,

$$m = \dim (m \mathcal{H}_{\sigma_s})^{SO(2)} \geq \frac{\ell - 1}{2}.$$

On the other hand,

$$m \ll_\varepsilon |\Gamma_1 : \Gamma_\ell|^{1-s+\varepsilon} \ll \ell^{3(1-s+\varepsilon)} \quad \text{for all } \varepsilon > 0.$$

If $s > \frac{2}{3}$, we get a contradiction. $\square$