Lecture 11

Diophantine approximation.

$y \in \mathbb{R}^d$, $y \approx r \in \mathbb{Q}^d$

How good is the quality of approximation?

Fix (non-increasing) $\psi : \mathbb{R}^+ \to (0,1)$.

Def. x is ψ-approximable if

$$\left\| y - \frac{p}{q} \right\| \leq \frac{\psi(q)}{q}$$

has infinitely many solutions

$(p, q) \in \mathbb{Z}^d \times \mathbb{N}$.

$$W_\psi(\mathbb{Q}^d) = \{ \psi\text{-approx. vectors} \}$$

Note that $W_\psi(\mathbb{Q}^d) = \varlimsup_{(p,q)} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right)$.

By Borel-Cantelli Lemma, if

$$\sum_{(p,q):\, \frac{p}{q} \in [0,1)^d} \left| B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \right| \asymp \sum_{q \geq 1} q^d \cdot \left(\frac{\psi(q)}{q}\right)^d = \sum_{q \geq 1} \psi(q)^d < \infty,$$

then $W_\psi(\mathbb{Q}^d)$ has measure 0.

$\underline{\text{Thm}}$ (Khinchin) If $\sum_{q\geq 1} \psi(q)^d = \infty$, then $W_\psi(\mathbb{Q}^d)$ has full measure.

$\underline{\text{Question}}$ (Lang) Khinchin's Thm for more general varieties?

$\underline{\text{Setting:}}$ $G < GL_N(\mathbb{C})$ - simply connected simple algebraic group$/\mathbb{Q}$

(e.g. $G = SL_N(\mathbb{C})$)

p - prime

$G(\mathbb{Z}[\frac{1}{p}]) \subset G(\mathbb{R})$

Assume that $G[\mathbb{Z}[\frac{1}{p}])$ is not discrete

($\Rightarrow$ $\overline{G(\mathbb{Z}[\frac{1}{p}])} = G(\mathbb{R})$ (strong approximation property))

$$\boxed{\text{Khinchin's Thm ?}}$$

$\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$ $\longrightarrow$ $\mathbb{Z}[\frac{1}{p}]^d \hookrightarrow$?

(discrete, cocompact)

p-adic numbers:

For $r \in \mathbb{Q}$, write $r = p^n \cdot \frac{\ell}{s}$, ℓ, s are coprime to p.

Define p-adic norm: $|r|_p = p^{-n}$.

Basic properties: $|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$,

$|r_1 + r_2|_p \leq \max\{|r_1|_p, |r_2|_p\}$.

$\mathbb{Q}_p$ = completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

$\cup$

$\mathbb{Z}_p = \{x : |x|_p \leq 1\}$ – compact open ring

$\forall x \in \mathbb{Q}_p$: $x = \sum_{i=s}^{\infty} x_i \cdot p^i$ with $x_i = 0, 1, \dots p-1$.

<u>Lem.</u> $\mathbb{Z}[\frac{1}{p}] \underset{diag}{\hookrightarrow} \mathbb{R} \times \mathbb{Q}_p$ is a discrete cocompact subgroup.

If $\mathbb{Z}[\frac{1}{p}]$ is not discrete, then $\exists r_n \in \mathbb{Z}[\frac{1}{p}] \setminus \{0\}$: $(r_n, r_n) \to (0,0)$. Then $|r_n|_\infty \to 0 \Rightarrow den(r_n) \to 0$,

$|r|_p \to 0 \Rightarrow den(r_n) \to \infty$,

which is a contradiction.

We have $\mathbb{R} = [0,1) + \mathbb{Z}$ and $\mathbb{Q}_p = \mathbb{Z}_p + \mathbb{Z}[\frac{1}{p}]$.

Then $\mathbb{R} \times \mathbb{Q}_p = (\mathbb{R} \times \mathbb{Z}_p) + \mathbb{Z}[\frac{1}{p}] = ([0,1) \times \mathbb{Z}_p) + \mathbb{Z}[\frac{1}{p}]$.

bounded

FOR $G = SL_N$ (more generally, simply connected simple group, isotropic over $\mathbb{Q}_p$),

$$\Gamma \overset{\text{def}}{=} G(\mathbb{Z}[\tfrac{1}{p}]) \underset{\text{diag}}{\hookrightarrow} G(\mathbb{R}) \times G(\mathbb{Q}_p)$$

$$X = (G(\mathbb{R}) \times G(\mathbb{Q}_p)) / G(\mathbb{Z}[\tfrac{1}{p}]).$$

Then:
- $G(\mathbb{Z}[\tfrac{1}{p}])$ is discrete in $G(\mathbb{R}) \times G(\mathbb{Q}_p)$,
- $G(\mathbb{Z}[\tfrac{1}{p}])$ is dense in $G(\mathbb{Q}_p)$,
  (for $G = SL_N$, this is because G is generated by unipotent one-parameter subgroup
  in general, this is (deep) strong approximation property),
- $\mathrm{vol}(X) < \infty$.

$$\boxed{\text{Diophantine approximation for } G(\mathbb{Z}[\tfrac{1}{p}]) \subset G(\mathbb{R})} \longleftrightarrow \boxed{\underline{\text{Dynamics}} \quad G(\mathbb{Q}_p) \circlearrowright X}$$

Thm (property τ / Clozel)

The representation $\pi_p : G(\mathbb{Q}_p) \curvearrowright L^2_0(X)$ has spectral gap.

examples:

1) $G = SL_2$:

Conj (Ramanujan) $q(\pi_p) = 2$.

Known: $q(\pi_p) \leq \frac{64}{25}$ (Kim–Sarnak)

2) G = form of SL_2, anisotropic over $\mathbb{Q}$

$q(\pi_p) = 2$ (Deligne)

3) $G(\mathbb{Q}_p)$ has higher rank

This follows from property T.

Thm (Ghosh – G. – Nevo)

Assume that for bounded $\Omega \subset G(\mathbb{R})$ and $\alpha > n(T_{\mathbb{Q}}) \cdot \dim(G)$,

$$\sum_{r \in G(\mathbb{Z}[\frac{1}{p}]) \cap \Omega} \psi(\mathrm{den}(r))^{\alpha} = \infty.$$

Then $W_\psi(G(\mathbb{Z}[\frac{1}{p}]))$ has full measure.

Fix bounded $\Omega \subset G(\mathbb{R})$,

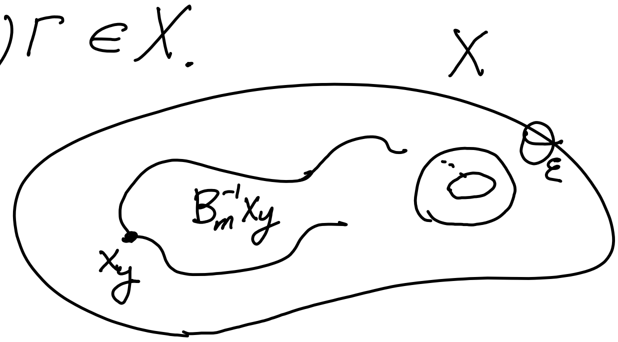
$c = \sup_{y \in \Omega} \|y\|$,

$\tilde{\Theta}_\varepsilon = \{ g \in G(\mathbb{R}) : \|g - e\|_\infty < \varepsilon / c \} \times G(\mathbb{Z}_p)$

$\Theta_\varepsilon = \tilde{\Theta}_\varepsilon \Gamma \subset X$

$B_m = \{ b \in G(\mathbb{Q}_p) : \|b\|_p = p^m \}$ $\left(\begin{array}{l} \text{for } \gamma \in G(\mathbb{Z}[\frac{1}{p}]) \\ \|\gamma\|_p = p^m \Leftrightarrow \mathrm{den}(\gamma) = p^m \end{array} \right)$

For $y \in \Omega$, define $x_y = (y^{-1}, e)\Gamma \in X$.



Lem. (shrinking targets)

$$B_m^{-1} \cdot x_y \cap \Theta_\varepsilon \neq \emptyset \Rightarrow \begin{cases} \|y - \gamma\|_\infty < \varepsilon \\ \mathrm{den}(\gamma) = p^m \end{cases} \text{ has solution } \gamma \in G(\mathbb{Z}[\tfrac{1}{p}]).$$

For some $b \in B_m$ and $\gamma \in \Gamma$,

$$(e, b^{-1})\cdot(y^{-1}, e)\cdot(\gamma,\gamma) \in \widetilde{\Theta}_\varepsilon \Rightarrow \begin{array}{l} \|y^{-1}\gamma - e\|_\infty < \varepsilon/c \\ b^{-1}\gamma \in G(\mathbb{Z}_p) \Rightarrow \|\gamma\|_p = p^m \Rightarrow \mathrm{den}(\gamma) = p^m \end{array}$$

Then $\|\gamma - y\|_\infty \leq \|y\|_\infty \cdot \|y^{-1}\gamma - e\|_\infty < \varepsilon.$

Let β_m = prob. measure supported on B_m.

Thm (mean ergodic thm) $\forall f \in L^2(X)$:

$$\left\| \pi_X(\beta_m) f - \int_X f \, d\mu \right\|_2 \leq C_\varepsilon \cdot |B_m|^{-\frac{1}{2n(\pi_p)}+\varepsilon} \cdot \|f\|_2$$

for every $\varepsilon > 0$.

Let $a_m = |\Gamma \cap B_m \cap \Omega|$.

The series $\sum_{m\geq 1} a_m \cdot \psi(p^m)^\alpha \begin{cases} \text{converges} & \alpha > \alpha_0, \\ \text{diverges} & \alpha < \alpha_0. \end{cases}$

Note that $\alpha_0 > n(\pi_p)\dim(G)$.

For simplicity assume that $\alpha_0 < \infty$.

Take $\alpha < \alpha_0$, $\alpha \approx \alpha_0$.

Let $\Phi_m = \Theta_{\psi(p^m)/c}$, $\varphi_m = c_m \cdot \chi_{\Phi_m}$ with $c_m = a_m \cdot \psi(p^m)^{\alpha - d}$,

where $d = \dim(G)$.

Then: 1) $\sum_{m\geq 1} \int_X \varphi_m = \infty$,

2) $F_k = \sum_{m\geq k} \left| \pi_X(\beta_m)\varphi_m - \int_X \varphi_m \right| \in L^2(X)$.

1): $\sum_{m\geq 1} \int_X \varphi_m = \sum_{m\geq 1} c_m \,|\Phi_m| \asymp \sum_{m\geq 1} a_m \cdot \psi(p^m)^{\alpha} = \infty$

because $\alpha < \alpha_0$.

2): We note that $a_m = |\Gamma \cap \Omega \cap B_m| << |B_m|$.

By the Ergodic Theorem,

$$\|F_k\|_2 << \sum_{m\geq k} |B_m|^{-\frac{1}{2n}+\varepsilon} \cdot \|\varphi_m\|_2 = \sum_{m\geq k} |B_m|^{-\frac{1}{2n}+\varepsilon} c_m \,|\Phi_m|^{1/2}$$

$$<< \sum_{m\geq k} |B_m|^{1-\frac{1}{2n}+\varepsilon} \cdot \psi(p^m)^{\alpha-\frac{d}{2}}$$

$$= \sum_{m\geq k} |B_m|^{1-\frac{1}{2n}+2\varepsilon} \psi(p^m)^{\alpha-\frac{d}{2}} \cdot |B_m|^{-\varepsilon}.$$

By the Hölder inequality with $r = \left(1-\frac{1}{2n}+2\varepsilon\right)$, $\bar{r} = \left(\frac{1}{2n}-2\varepsilon\right)^{-1}$,

$$\|F_k\|_2 << \left(\sum_{m\geq k} |B_m| \cdot \psi(p^m)^{r(\alpha-\frac{d}{2})}\right)^{1/r} \cdot \underbrace{\left(\sum_{m\geq k} |B_m|^{-\varepsilon \bar{r}}\right)^{1/\bar{r}}}_{<\infty,\ \text{because } |B_m| \text{ grows exponentially.}}$$

Since $\alpha_0 > n \cdot d$, $\frac{\alpha_0 - d/2}{1 - 1/2n} > \alpha_0$ and $\frac{\alpha - d/2}{1 - 1/2n + 2\varepsilon} > \alpha_0$,

where $\alpha \approx \alpha_0$ and $\varepsilon \approx 0$.

Then $F_k \in L^2(X)$.

Let $X_m = \{x : B_m^{-1} x \cap \varphi_m = \emptyset\}$.

On $\bigcap_{m \geq k} X_m$, $F_k = \sum_{m \geq k} \int_X \varphi_m = \infty$.

Since $F_k \in L^2(X)$, $\left| \bigcap_{m \geq k} X_m \right| = 0$.

Hence, $X_\infty = \underline{\lim} X_m$ has measure 0.

Let $\Omega' = \{y \in \Omega : \underset{(y^{-1}, e)\Gamma}{\overset{\parallel}{x_y}} \notin X_\infty\}$.

Then $((\Omega \setminus \Omega')^{-1} \times G(\mathbb{Z}_p))\Gamma \subset X_\infty$, and

Ω' has full measure in Ω.

For $x \in \Omega'$; $x_y \in X_m$ infinitely often

$$\Updownarrow$$

$$B_m^{-1} \cdot x_y \cap \mathcal{O}_{\psi(p^m)} \neq \emptyset.$$

Hence, $\Omega' \subset W_\psi(G(\mathbb{Z}[\frac{1}{p}]))$ by the lemma. $\square$