Lecture 1.

Ergodic Theory $\longleftrightarrow$ Number Theory

Dynamical System $\longleftarrow$ Symmetries

- Szemeredi Thm (Furstenberg)
- Oppenheim Conjecture (Margulis)
- Littlewood Conjecture (Einsiedler-Katok-Lindenstrauss)
- Sprindzuk Conjecture (Kleinbock-Margulis)

Ergodic Theory of semisimple groups $\longleftrightarrow$ Number Theory

Automorphic forms $\longleftrightarrow$ Ergodic Theory of semisimple groups $\longleftrightarrow$ Representation Theory

Number Theory:

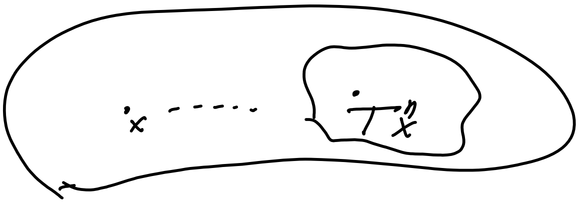
- counting integral/rational points on varieties
- sieving for almost prime,
- Diophantine approximation,
- expander graphs

Classical Ergodic Theory.

(X, μ) – probability space

$T: X \to X$ – measure-preserving transformation.

Basic Problem: distribution of orbits $\{x, Tx, \dots T^n x \dots\}$.



$$T \circlearrowright L^2(X): \quad f \longmapsto f(Tx)$$

$$\frac{1}{N} \sum_{n=0}^{N-1} T^n f \xrightarrow[N \to \infty]{} ?$$

Mean Ergodic Thm (von Neumann)

$\forall f \in L^2(X)$: $\quad \frac{1}{N} \sum_{n=0}^{N-1} T^n f \xrightarrow{L^2} Pf$,

where P is the orthogonal projection on $\{f : Tf = f\}$.

Pointwise Ergodic Thm (Birkhoff)

$\forall f \in L^2(X)$: $\forall$ a.e. $x \in X$: $\quad \frac{1}{N} \sum_{n=0}^{N-1} T^n f(x) \longrightarrow (Pf)(x)$.

ex. (decimal expansion)

$X=[0,1)$, $T: X \to X : x \mapsto 10x \pmod 1$, $f=\chi_{[\frac{i}{10},\frac{i+1}{10})}$.

$$\frac{1}{N}\sum_{i=0}^{N-1} f(T^i x) = \begin{pmatrix}\text{frequency of digit } i \\ \text{in decimal expansion of } x\end{pmatrix}$$

Rmk. In the classical ergodic theorems,

- convergence holds only a.e.,
- there is no rate of convergence.

Ergodic Theorems and counting.

Question: $\#\{\gamma \in SL_n(\mathbb{Z}) : \|\gamma\| < t\} \underset{t\to\infty}{\sim} ?$

Duke-Rudnick-Sarnak :
Eskin - Mcmullen

$$\underset{t\to\infty}{\sim} c_n \cdot t^{n^2-n}.$$

Embed $\underset{\Gamma}{\underset{\shortparallel}{SL_n(\mathbb{Z})}} \subset \underset{G}{\underset{\shortparallel}{SL_n(\mathbb{R})}}$

Then Γ is discrete and $\mathrm{vol}(G/\Gamma) < \infty$.

More generally, we consider:

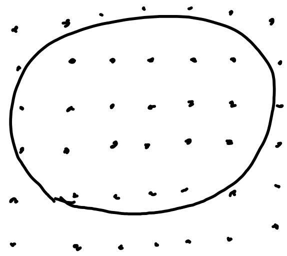
G – loc. compact group

Γ – discrete subgroup with $\mathrm{vol}(G/\Gamma) < \infty$.

$B_t \subset G$ – increasing family of compact domains

Compute $|\Gamma \cap B_t| \underset{t\to\infty}{\sim}$?

<u>example.</u> $\mathbb{Z}^d \subset \mathbb{R}^d$:



B_t

$$\#(\mathbb{Z}^d \cap B_t) = |B_t| + O(|\partial B_t|).$$

Difficulties: – $|B_t| \asymp |\partial B_t|$ (exponential volume growth)

– G/Γ is not compact.

Mean Ergodic Thm $\Rightarrow$ counting

$\{\mathcal{O}_\varepsilon\}_{0<\varepsilon<1}$ – basis of symmetric nbhds of e in G

<u>Def.</u> A family of increasing domains $\{B_t\}_{t\geq 1}$ is <u>admissible</u> if $\exists c>0$: $\forall \varepsilon \in (0,1)$ $\forall t \geq 1$:

$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon},$$

$$\mathrm{vol}(B_{t+\varepsilon}) \leq (1+c\varepsilon)\,\mathrm{vol}(B_t).$$

Mean Ergodic Thm. $\forall f \in L^2(G/\Gamma)$:

$$\left\| \frac{1}{|B_t|} \int_{B_t} f(g^{-1}x)\,dg - \int_X f \right\|_2 \xrightarrow[t\to\infty]{} 0.$$

Quantitative Mean Ergodic Thm.

$\exists \theta > 0$: $\forall f \in L^2(G/\Gamma)$:

$$\left\| \frac{1}{|B_t|} \int_{B_t} f(g^{-1}x)\,dg - \int_X f \right\|_2 \ll \mathrm{vol}(B_t)^{-\theta} \cdot \|f\|_2 .$$

Thm (G.–Nevo)

1) Assume admisibility & mean ergodic thm.

Then $|\Gamma \cap B_t| \sim \frac{\mathrm{vol}(B_t)}{\mathrm{vol}(G/\Gamma)}$ as $t \to \infty$.

2) assume admisibility, quant. mean erg. thm, $\mathrm{vol}(\mathcal{O}_\varepsilon) \gg \varepsilon^d$.

Then $|\Gamma \cap B_t| = \frac{\mathrm{vol}(B_t)}{\mathrm{vol}(G/\Gamma)} + O\left(\mathrm{vol}(B_t)^{1-\frac{\theta}{1+d}}\right).$

Proof: Let $\chi_\varepsilon = \frac{\chi_{\mathcal{O}_\varepsilon}}{\mathrm{vol}(\mathcal{O}_\varepsilon)}$ and $\overline{\chi}_\varepsilon(g\Gamma) = \sum_{\gamma\in\Gamma} \chi_\varepsilon(g\gamma)$.

We normalise the inv. measure so that $\mathrm{vol}(G/\Gamma) = 1$.

Then $\int_G \chi_\varepsilon = \int_{G/\Gamma} \overline{\chi}_\varepsilon = 1.$

Claim: $\forall x \in \mathcal{O}_\varepsilon$: $\displaystyle\int_{B_{t-c\varepsilon}} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg \le |\Gamma\cap B_t| \le \int_{B_{t+c\varepsilon}} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg$

$$\int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg = \sum_{\gamma\in\Gamma}\int_{B_t} \chi_\varepsilon(g^{-1}x\gamma)\,dg = \sum_{\gamma\in\Gamma}\frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(\mathcal{O}_\varepsilon)}.$$

$g^{-1}x\gamma\in\mathcal{O}_\varepsilon \Leftrightarrow g\in x\gamma\mathcal{O}_\varepsilon$

If $\gamma\in B_{t-c\varepsilon}$, then $x\gamma\mathcal{O}_\varepsilon \subset \mathcal{O}_\varepsilon B_{t-c\varepsilon}\mathcal{O}_\varepsilon = B_t$, and $\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t) = \mathrm{vol}(x\gamma\mathcal{O}_\varepsilon) = \mathrm{vol}(\mathcal{O}_\varepsilon)$. Hence,

$$|\Gamma\cap B_{t-c\varepsilon}| = \sum_{\gamma\in\Gamma\cap B_{t-c\varepsilon}}\frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(\mathcal{O}_\varepsilon)} \le \sum_{\gamma\in\Gamma}\frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(\mathcal{O}_\varepsilon)} = \int_{B_t}\overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg.$$

If $x\gamma\mathcal{O}_\varepsilon\cap B_t\ne\emptyset$, then $\gamma\in x^{-1}B_t\mathcal{O}_\varepsilon^{-1}\subset B_{t+c\varepsilon}$.

Hence,
$$\int_{B_t}\overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg = \sum_{\gamma\in\Gamma}\frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(\mathcal{O}_\varepsilon)}$$
$$= \sum_{\gamma\in\Gamma\cap B_{t+c\varepsilon}}\frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(\mathcal{O}_\varepsilon)} \le |\Gamma\cap B_{t+c\varepsilon}|.$$

By the Chebeshev inequality, for $\delta>0$,

$$\delta\cdot\mathrm{vol}\left(\left\{x\Gamma : \left|\frac{1}{\mathrm{vol}(B_t)}\int_{B_t}\overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg - 1\right|\ge\delta\right\}\right)^{1/2}$$
$$\le \left\|\frac{1}{\mathrm{vol}(B_t)}\cdot\int_{B_t}\overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg - 1\right\|_2 \ll \mathrm{vol}(B_t)^{-\theta}\cdot\|\overline{\chi}_\varepsilon\|_2.$$

$$\|\overline{\chi}_\varepsilon\|_2^2 = \int_{G/\Gamma} \Big|\sum_{\gamma\in\Gamma} \chi_\varepsilon(g\gamma)\Big|^2 dg.$$

If ε is sufficiently small, $\Theta_\varepsilon\gamma_1 \cap \Theta_\varepsilon\gamma_2 = \emptyset$ for $\gamma_1\neq\gamma_2\in\Gamma$,

and $\chi_\varepsilon(g\gamma_1)\chi_\varepsilon(g\gamma_2) = 0$ for $\gamma_1\neq\gamma_2\in\Gamma$.

Hence, $\|\overline{\chi}_\varepsilon\|_2^2 = \int_{G/\Gamma} \sum_{\gamma\in\Gamma} \chi_\varepsilon(g\gamma)^2 dg = \mathrm{vol}(\Theta_\varepsilon)^{-1}$, and

$$\mathrm{vol}\left(\left\{x\Gamma : \Big|\frac{1}{\mathrm{vol}(B_t)}\int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg - 1\Big| \geq \delta\right\}\right) \ll \delta^{-2}\,\mathrm{vol}(B_t)^{-2\theta}\,\mathrm{vol}(\Theta_\varepsilon)^{-1}.$$

We take δ such that

$$\mathrm{vol}\left(\left\{x\Gamma : \Big|\frac{1}{\mathrm{vol}(B_t)}\int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg - 1\Big| \geq \delta\right\}\right) < \mathrm{vol}(\Theta_\varepsilon\Gamma) = \mathrm{vol}(\Theta_\varepsilon).$$

Then $\exists x\in\Theta_\varepsilon$:

$$\Big|\frac{1}{\mathrm{vol}(B_t)}\int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg - 1\Big| < \delta,$$

$$(1-\delta)\,\mathrm{vol}(B_t) < \int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg < (1+\delta)\,\mathrm{vol}(B_t).$$

Hence, by the claim,

$$(1-\delta)(1-c^2\varepsilon)\mathrm{vol}(B_t) \leq (1-\delta)\mathrm{vol}(B_{t-c\varepsilon}) \leq |\Gamma\cap B_t| \leq (1+\delta)\mathrm{vol}(B_{t+c\varepsilon}) \leq (1+\delta)(1+c^2\varepsilon)\mathrm{vol}(B_t),$$

and $|\Gamma\cap B_t| = (1+O(\delta+\varepsilon))\cdot\mathrm{vol}(B_t)$.

Here, $\delta^{-2}\operatorname{vol}(B_t)^{-2\theta}\operatorname{vol}(\Theta_\varepsilon)^{-1} \asymp \operatorname{vol}(\Theta_\varepsilon)$,

$$\delta \asymp \operatorname{vol}(B_t)^{-\theta}\operatorname{vol}(\Theta_\varepsilon)^{-1} << \operatorname{vol}(B_t)^{-\theta}\cdot\varepsilon^{-d}.$$

To optimise the estimate, we set $\varepsilon = \operatorname{vol}(B_t)^{-\frac{\theta}{d+1}}$.

Then $|\Gamma\cap B_t| = \left(1+O\left(\operatorname{vol}(B_t)^{-\frac{\theta}{d+1}}\right)\right)\cdot \operatorname{vol}(B_t)$. $\square$

<u>Remark.</u> This estimate is uniform over families of subgroups such that

- the same rate in the mean ergodic theorem,
- $\Theta_\varepsilon\gamma_1\cap\Theta_\varepsilon\gamma_2=\emptyset$ for $\gamma_1\neq\gamma_2\in\Gamma$, with uniform ε.