Lecture 4: Applications II.

Diophantine approximation.

Given $x \in \mathbb{R}^d$, we would like to approximate it by $r \in \mathbb{Q}^d$ with small denominator.

example (Dirichlet): $\forall x \in \mathbb{R}^d \ \exists r \in \mathbb{Q}^d: \ \|x-r\| \leq \mathrm{den}(r)^{-\frac{d+1}{d}}$.

Question: $X = \{x : f_1(x) = \dots = f_s(x) = 0\}$, $x \in X(\mathbb{R})$, $\kappa > 0$.

Can we solve $\begin{cases} \|x-r\| \leq \varepsilon \\ \mathrm{den}(r) \leq \varepsilon^{-\kappa} \end{cases}$ with $r \in X(\mathbb{Q})$ $(r \in X(\mathbb{Z}[\frac{1}{p}]))$

for all sufficiently small ε?

M. Waldschmidt: elliptic curves / abelian varieties.

What about other varieties with group actions?

Elementary bound on κ:

Let $a_p(X) = \sup_{\text{cpt } \Omega \subset X(\mathbb{R})} \overline{\lim_{R \to \infty}} \dfrac{\log |\{r \in \Omega \cap X(\mathbb{Z}[\frac{1}{p}]) : \mathrm{den}(r) \leq R\}|}{\log R}$.

$\#(\mathbb{Z}[\frac{1}{p}]\text{-points in } \Omega \text{ with } \mathrm{den}(r) \leq R) \ll_\delta R^{a_p(X)+\delta}$, $\delta > 0$.

$\#(2\varepsilon)\text{-seperated set}) \gg \varepsilon^{-\dim(X)}$.

Hence, by the pegion-hole principle $\kappa \geq \dfrac{\dim(X)}{a_p(X)}$.

Let X be an affine variety$/\mathbb{Q}$ equipped with a transitive action of a simple algebraic group $G/\mathbb{Q}$.

$X(\mathbb{Z}[\frac{1}{p}]) \subset$ "classifying space".

$X(\mathbb{Z}[\frac{1}{p}]) \subset X(\mathbb{R}) \times X(\mathbb{Q}_p)$ $\quad (X(\mathbb{Q}) \subset \underbrace{X(\mathbb{A})}_{\text{adeles}})$.

$G(\mathbb{Z}[\frac{1}{p}]) \subset G(\mathbb{R}) \times G(\mathbb{Q}_p)$

discrete subgroup with finite covolume
(cf. $\mathbb{Z}[\frac{1}{p}] \subset \mathbb{R} \times \mathbb{Q}_p$).

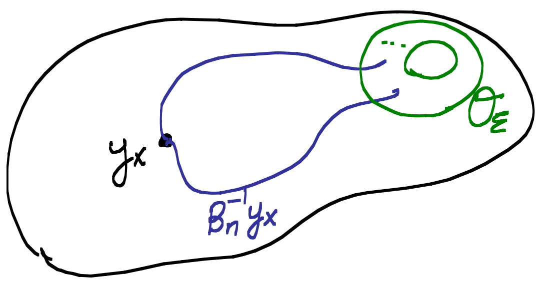
Consider the group action:

$$G(\mathbb{Q}_p) \circlearrowright Y := (G(\mathbb{R}) \times G(\mathbb{Q}_p)) / G(\mathbb{Z}[\tfrac{1}{p}]).$$

Generalised Dani correspondence

Dioph. approximation $X(\mathbb{Z}[\frac{1}{p}]) \subset X(\mathbb{R})$	Dynamics $G(\mathbb{Q}_p) \circlearrowright Y$
$x \in X(\mathbb{R})$	$y_x \in Y$
$\lVert x-r \rVert \le \varepsilon$	nbhd $\mathcal{O}_\varepsilon \subset Y$, $vol(\mathcal{O}_\varepsilon) \asymp \varepsilon^{\dim(X)}$
$\begin{cases} \lVert x-r \rVert \le \varepsilon \\ den(r) \le p^n \end{cases}$ has a solution $r \in X(\mathbb{Z}[\frac{1}{p}])$	$B_n^{-1} \cdot y_x \cap \mathcal{O}_\varepsilon \ne \emptyset$ where $B_n = \{ g \in G(\mathbb{Q}_p) : \lVert b^{-1} \rVert_p \le p^n \}$

Shrinking target property



$B_n^{-1} \cdot y_x \cap \mathcal{O}_\varepsilon \ne \emptyset$

Proof of Dani correspondence ($X=G$):

For $x \in G(\mathbb{R})$, we set $y_x = (x,e)\, G(\mathbb{Z}[\tfrac{1}{p}]) \in Y$

$\mathcal{O}_\varepsilon^\infty = \varepsilon$-nbhd of e in $G(\mathbb{R})$

$\mathcal{O}^p =$ bounded open subset in $G(\mathbb{Q}_p)$

$\mathcal{O}_\varepsilon = (\mathcal{O}_\varepsilon^\infty \times \mathcal{O}^p)\, G(\mathbb{Z}[\tfrac{1}{p}]) \subset Y$.

$B_n^{-1} y_x \cap \mathcal{O}_\varepsilon \neq \emptyset \Rightarrow \exists\, g \in B_n,\ w_\infty \in \mathcal{O}_\varepsilon^\infty,\ w_p \in \mathcal{O}^p,\ r \in G(\mathbb{Z}[\tfrac{1}{p}])$:

$$(x, b^{-1}) = (w_\infty \gamma,\ w_p \gamma)$$

$$\Downarrow$$

$$\gamma = w_\infty^{-1} x \underset{\varepsilon}{\approx} x$$

$$\gamma = w_p^{-1} b^{-1} \Rightarrow \|\gamma\|_p << p^n \Rightarrow \mathrm{den}(r) << p^n.$$

Mean ergodic Thm.

$L^2_{00}(Y) = \{ f \in L^2(Y) :\ f$ is orthogonal to automorphic characters of $G(\mathbb{R}) \times G(\mathbb{Q}_p) \}$.

Rmk: - $L^2_{00}(Y)$ has finite codimension in $L^2(Y)$.
- when G is simply connected, $L^2_{00}(Y) = L^2_0(Y)$.

(spherical) integrability exponent:
fix a (good) maximal compact subgroup $U_p \subset G(\mathbb{Q}_p)$.
$\pi_p : G(\mathbb{Q}_p) \curvearrowright L^2_{00}(Y)$ - unitary representation.

$$q_p(G) = \inf\{ q > 0 : \langle \pi_p(g) f_1, f_2 \rangle \in L^q(G(\mathbb{Q}_p)) \text{ for all } U_p\text{-inv. } f_1, f_2 \in L^2_{00}(Y) \}$$

Clozel: $q_p(G) < \infty$.

For $B \subset G(\mathbb{Q}_p)$, $0 < \mathrm{vol}(B) < \infty$, define an averaging operator:

$$\pi_p(B): L^2(Y) \to L^2(Y): f \longmapsto \frac{1}{\mathrm{vol}(B)} \int_B f(b^{-1}y)\,db.$$

<u>Mean Ergodic Thm:</u> Assuming that B is bi-invariant under U_p,

$$\forall f \in L^2_{00}(Y): \quad \|\pi_p(B)f\| \ll_\varepsilon \mathrm{vol}(B)^{-\frac{1}{q_p(G)}+\varepsilon} \|f\|_2, \quad \varepsilon > 0.$$

Mean Ergodic Thm $\Longrightarrow$ Shrinking Target Property.

<u>Thm</u> (Ghosh–G.–Nevo)
$X \subset \mathbb{C}^d$ affine alg. variety $/\mathbb{Q}$ equipped with transitive action of a simple alg. group $G \subset GL_d(\mathbb{C})$ defined over $\mathbb{Q}$.
Assume that $X(\mathbb{Z}[\frac{1}{p}])$ is not discrete in $X(\mathbb{R})$.
Then: 1) $\forall$ a.e. $x \in \overline{X(\mathbb{Z}[\frac{1}{p}])}$ $\forall \delta > 0$: $\forall \varepsilon \in (0, \varepsilon_0(x,\delta))$:

$$\begin{cases} \|x - r\| \le \varepsilon \\ \mathrm{den}(r) \le \left(\varepsilon^{-\frac{\dim(X)}{a_p(G)}-\delta}\right)^{\frac{q_p(G)}{2}} \end{cases}$$

has a solution $r \in X(\mathbb{Z}[\frac{1}{p}])$.

2) $\forall x \in \overline{X(\mathbb{Z}[\frac{1}{p}])}$ $\forall \delta > 0$: $\forall \varepsilon \in (0, \varepsilon_0(\delta))$:

$$\|x - r\| \le \varepsilon$$
$$\mathrm{den}(r) \le \left(\varepsilon^{-\frac{\dim(X)}{a_p(G)}-\delta}\right)^{q_p(G)}$$

has a solution $r \in X(\mathbb{Z}[\frac{1}{p}])$.

<u>Rmk:</u> If $X = G$ and $q_p(G) = 2$,
then (1) is the best possible (up to $\delta > 0$).

Corollaries

<u>Cor</u> $q_p(G) \geq 2$
(c.f. Burger–Sarnak: automorphic spectrum contains tempered spectrum)

<u>Proof:</u> If $q_p(G) < 2$, then $\kappa < \dfrac{\dim(G)}{a_p(G)}$.

<u>Cor.</u> $q_p(SL_d) \geq 2(d-1)$.

<u>Proof:</u> Consider $\mathbb{Z}[\frac{1}{p}]^d \subset \mathbb{R}^d$.
By elementary reasons, $\kappa \geq 1$.
Since $\mathbb{R}^d \setminus \{0\}$ is a homogeneous space of $SL_d(\mathbb{R})$,

$$\kappa \leq \left(\frac{\dim(\mathbb{R}^d)}{a_p(SL_d)} + \delta\right) \cdot \frac{q_p(SL_d)}{2} = \left(\frac{d}{d^2-d} + \delta\right) \cdot \frac{q_p(SL_d)}{2}.$$

<u>Cor</u> $q_p(SO_d) \geq \text{const} \cdot d$ when $p = 1 \pmod 4$.

<u>Proof:</u> Consider the Dioph. approximation on the sphere.