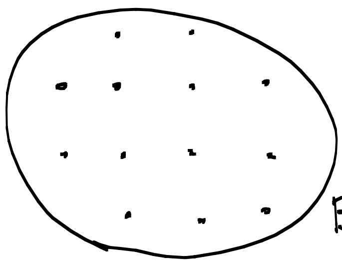
Lecture 3: Applications I.

1) Counting lattice points.

G = locally compact group
Γ = discrete subgroup with $\mathrm{vol}(G/\Gamma) < \infty$.

[Given an increasing family of compact domains $B_t \subset G$, compute the asymptotics of $|\Gamma \cap B_t|$ as $t \to \infty$.



B_t

We expect that $|\Gamma \cap B_t| \sim \dfrac{\mathrm{vol}(B_t)}{\mathrm{vol}(G/\Gamma)}$.

examples: $\mathbb{Z}^d \subset \mathbb{R}^d$, $SL_d(\mathbb{Z}) \subset SL_d(\mathbb{R})$, $SL_d(\mathbb{Q}) \subset SL_d(\mathbb{A})$.
adèles

Approaches:

1) Spectral expansion (Delsarte, Huber, Patterson, ... Duke-Rudnick-Sarnak)
(gives more information, uses more information)

2) Decay of matrix coefficients (Margulis, Eskin-Mcmullen)
(decay of matrix coefficients is rare, gives error term (Sobolev norms))

3) Unipotent flows (Eskin-Mozes-Shah)
(applies to more general varieties, no/poor? error term)

4) Mean ergodic theorem (G.-Nevo)
(might be used for general G, gives error term (L^2-norms))
$\Downarrow$
better rates

Mean ergodic thm $\Rightarrow$ counting.

Fix a basis $\{\mathcal{O}_\varepsilon\}_{0<\varepsilon<1}$ of symmetric nbhds of e in G.

Def. A family of increasing domains $\{B_t\}_{t\geq 1}$ is called admissible if $\exists c>0$: $\forall \varepsilon\in(0,1)$ $\forall t\geq 1$:

$$\mathcal{O}_\varepsilon\cdot B_t\cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon},$$
$$\mathrm{vol}(B_{t+\varepsilon})\leq (1+c\varepsilon)\,\mathrm{vol}(B_t). \qquad \text{(A)}$$

Rmk: This condition can be further relaxed.

Mean Ergodic Thm: $\forall f\in L^2(G/\Gamma)$:

$$\left\| \frac{1}{\mathrm{vol}(B_t)}\int_{B_t} f(g^{-1}x)\,dg - \int_X f \right\|_2 \xrightarrow[t\to\infty]{} 0. \qquad \text{(ET)}$$

Quantitative mean ergodic thm: $\exists \theta>0$: $\forall f\in L^2(G/\Gamma)$:

$$\left\| \frac{1}{\mathrm{vol}(B_t)}\int_{B_t} f(g^{-1}x)\,dg - \int_X f \right\|_2 \ll \mathrm{vol}(B_t)^{-\theta}\|f\|_2. \qquad \text{(QET)}$$

Thm (G.–Nevo)

1) Assume (A) & (ET). Then
$$|\Gamma\cap B_t| \sim \frac{\mathrm{vol}(B_t)}{\mathrm{vol}(G/\Gamma)} \quad \text{as } t\to\infty.$$

2) Assume (A) & (QET) & $\mathrm{vol}(\mathcal{O}_\varepsilon)\gg \varepsilon^d$. Then
$$|\Gamma\cap B_t| = \frac{\mathrm{vol}(B_t)}{\mathrm{vol}(G/\Gamma)} + O\left(\mathrm{vol}(B_t)^{1-\frac{\theta}{1+d}}\right).$$

Proof of (1): Set $\chi_\varepsilon = \frac{\chi_{\mathcal{O}_\varepsilon}}{\mathrm{vol}(\mathcal{O}_\varepsilon)}$ and $\overline{\chi_\varepsilon}(g\Gamma) = \sum_{\gamma\in\Gamma} \chi_\varepsilon(g\gamma)$.

We normalise the volume: $\mathrm{vol}(G/\Gamma)=1$. Then

$$\int_G \chi_\varepsilon = \int_{G/\Gamma} \overline{\chi_\varepsilon} = 1.$$

Claim: $\forall x \in \mathcal{O}_\varepsilon$:

$$\int_{B_{t-c\varepsilon}} \overline{\chi_\varepsilon}(g^{-1}x\Gamma)\,dg \leq |\Gamma\cap B_t| \leq \int_{B_{t+c\varepsilon}} \overline{\chi_\varepsilon}(g^{-1}x\Gamma)\,dg.$$

Note that:

$$\int_{B_t} \overline{\chi_\varepsilon}(g^{-1}x\Gamma)\,dg = \sum_{\gamma\in\Gamma} \int_{B_t} \chi_\varepsilon(g^{-1}x\gamma)\,dg \underset{\uparrow}{=} \sum_{\gamma\in\Gamma} \frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(\mathcal{O}_\varepsilon)}$$

$g^{-1}x\gamma \in \mathcal{O}_\varepsilon \Leftrightarrow g\in x\gamma\mathcal{O}_\varepsilon$

If $\gamma\in B_{t-c\varepsilon}$, then $x\gamma\mathcal{O}_\varepsilon \subset B_t$ and $\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t) = \mathrm{vol}(\mathcal{O}_\varepsilon)$.

Hence, $|\Gamma\cap B_{t-c\varepsilon}| = \sum_{\gamma\in\Gamma\cap B_{t-c\varepsilon}} \frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(B_t)} \leq \int_{B_t} \overline{\chi_\varepsilon}(g^{-1}x\Gamma)\,dg$.

If $x\gamma\mathcal{O}_\varepsilon\cap B_t \neq \emptyset$, then $\gamma\in x^{-1}B_t\mathcal{O}_\varepsilon^{-1} \subset B_{t+c\varepsilon}$.

Hence, $\int_{B_t} \overline{\chi_\varepsilon}(g^{-1}x\Gamma)\,dg = \sum_{\gamma\in\Gamma\cap B_{t+c\varepsilon}} \frac{\mathrm{vol}(x\gamma\mathcal{O}_\varepsilon\cap B_t)}{\mathrm{vol}(\mathcal{O}_\varepsilon)} \leq |\Gamma\cap B_{t+c\varepsilon}|$.

This proves the claim.

By (ET) and Chebeshev inequality,

$$\delta\cdot \mathrm{vol}\left(\left\{x\Gamma : \left|\frac{1}{\mathrm{vol}(B_t)}\cdot\int_{B_t} \overline{\chi_\varepsilon}(g^{-1}x\Gamma)\,dg - 1\right| \geq \delta\right\}\right)^{1/2}$$

$$\leq \left\|\frac{1}{\mathrm{vol}(B_t)}\cdot\int_{B_t} \overline{\chi_\varepsilon}(g^{-1}x\Gamma)\,dg - 1\right\|_2 \xrightarrow[t\to\infty]{} 0.$$

Hence, $\forall t \geq t(\varepsilon,\delta)$:

$$\mathrm{vol}\left(\left\{x\Gamma : \left|\frac{1}{\mathrm{vol}(B_t)}\int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg - 1\right| \geq \delta\right\}\right) < \mathrm{vol}(\mathcal{O}_\varepsilon\Gamma),$$

and $\exists x \in \mathcal{O}_\varepsilon$:

$$\left|\frac{1}{\mathrm{vol}(B_t)}\cdot\int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg - 1\right| < \delta,$$

$$(1-\delta)\mathrm{vol}(B_t) < \int_{B_t} \overline{\chi}_\varepsilon(g^{-1}x\Gamma)\,dg < (1+\delta)\,\mathrm{vol}(B_t)$$

Finally, by the claim and (A),

$$(1-\delta)(1-c^2\varepsilon)\mathrm{vol}(B_t) \leq (1-\delta)\mathrm{vol}(B_{t-c\varepsilon}) \leq |\Gamma \cap B_t| \leq (1+\delta)\,\mathrm{vol}(B_{t+c\varepsilon}) \leq (1+\delta)(1+c\varepsilon^2)\mathrm{vol}(B_t).$$

This holds for all ε,δ and $t \geq t_0(\varepsilon,\delta)$.

2) Primes / almost primes in orbits.

<u>Question:</u> Given $f \in \mathbb{Z}[x_1,\dots x_d]$, does $f(\mathbb{Z}^d)$ contain infinitely many primes?
(assume: f is irreducible, $\gcd(f(\mathbb{Z}^d))=1$).

<u>examples:</u> 1) $d=1$, $\deg(f)=1$: the Dirichlet thm,

2) (Halberstam–Richert) $d=1$:

$$\#\left\{x\in\mathbb{Z} : \begin{array}{l}|x|\leq T\\ f(x) \text{ is } r\text{-prime}\end{array}\right\} \gg \frac{T}{\log T}$$

$p_1\dots p_r$

for $r = \deg(f)+1$.

Sarnak Programme:

$\Gamma =$ "large" subgroup of $GL_d(\mathbb{Z})$

$v \in \mathbb{Z}^d$

$\mathcal{O} = \Gamma \cdot v$

$f \in \mathbb{Z}[x_1, \dots x_d]$.

Does $f(\mathcal{O})$ contain infinitely many primes/r-primes?

example $\left(\begin{array}{l} \text{Liu–Sarnak}, d=3 \\ \text{G.–Nevo}, d>3 \end{array} \right)$ $X = \{Q(x) = b\}$, $b \in \mathbb{Z}, \neq 0$

nondegenerate, indefinite quadratic form$/\mathbb{Z}$.

Take $v \in X(\mathbb{Z})$, $\mathcal{O} = SO_Q(\mathbb{Z}) \cdot v$, $f \in \mathbb{Z}[x_1, \dots x_d]$ such that $f|_X$ is irreducible$/\mathbb{C}$.

Then $\exists r \geq 1$: $\#\left\{x \in \mathcal{O} : \begin{array}{l} \|x\| \leq T \\ f(x) \text{ is } r\text{-prime} \end{array}\right\} \gg \frac{\#\{x \in \mathcal{O} : \|x\| \leq T\}}{\log T}$.

explicit

Sieve methods have been developed in great generality.

To apply them, we need to know

1) "Independence" of reductions of $\mathcal{O}$ mod ℓ_1 & ℓ_2 for coprime ℓ_1 and ℓ_2.

(cf. Chinese reminder Thm, Strong approximation).

2) Counting estimate on reduction of $\mathcal{O} \pmod{\ell}$:

$$\boxed{|\mathcal{O}(T,\ell)| = \rho(\ell) \cdot |\mathcal{O}(T)| + O\left(\ell^{\alpha} \cdot |\mathcal{O}(T)|^{1-\beta}\right) \quad \text{with uniform } \alpha, \beta > 0,}$$

where $\mathcal{O}(T) = \{x \in \mathcal{O} : \|x\| < T\}$,

$\mathcal{O}(T,\ell) = \{x \in \mathcal{O} : \|x\| < T, f(x) = 0 \pmod{\ell}\}$.

Rmk: an estimate on average of $|\Theta(T;\ell)|$ is also sufficient.

Let $\Gamma(\ell)=\{\gamma\in\Gamma:\ \gamma=id \pmod{\ell}\}$.

Then $|\Theta(T;\ell)| = \sum_{i=1}^{N_\ell} |\Gamma(\ell)x_i \cap \{\|\cdot\|\le T\}|$.

Thm (G.–Nevo)

G = simply connected simple algebraic group $/\mathbb{Q}$ (assume that $G(\mathbb{R})$ is noncompact)

$\Gamma(\ell)=\{\gamma\in G(\mathbb{Z}):\ \gamma=id \pmod{\ell}\}$

$B_t\subset G(\mathbb{R})$ - admissible family of sets.

Then $\exists\,\theta>0:\ \forall t,\ \gamma\in\Gamma(1),\ \ell\ge 1.$

$$|\Gamma(\ell)\gamma\cap B_t| = \frac{vol(B_t)}{vol(G/\Gamma)} + O\left(vol(B_t)^{1-\frac{\theta}{1+\dim(G)}}\right).$$

This result is based on:

Thm (property (τ) / Clozel)

The representations $\pi_\ell: G(\mathbb{R}) \curvearrowright L^2_0(G(\mathbb{R})/\Gamma(\ell))$ have uniform spectral gap. Namely,

$$q := \sup_\ell q(\pi_\ell) < \infty$$

the integrability exponent.

Rmk. $\Theta = \frac{1}{2n_e(q)}$.