# Lecture 2: Spectral gap & Ergodic Theorems

**Aim:** $G$ – a locally compact group

$G \curvearrowright (X,\mu)$ – a measure-preserving action on a probability space

For measurable $B \subset G$, $0 < \mathrm{vol}(B) < \infty$, we define an averaging operator:

$$\left(f \in L^2(X)\right) \longmapsto \frac{1}{\mathrm{vol}(B)} \int_B f(g^{-1}x)\,dg \in L^2(X).$$

What is the limit:

$$\frac{1}{\mathrm{vol}(B)} \int_B f(g^{-1}x)\,dg \quad \text{as } \mathrm{vol}(B) \to \infty\,?$$

(ergodic theorem)

To the action $G \curvearrowright (X,\mu)$, we associate a unitary representation:

$$G \curvearrowright L^2_0(X) = \{f \in L^2(X) : \int_X f\,d\mu = 0\}.$$

## Spectral gap.

Let $\pi : G \curvearrowright \mathcal{H}$ be a unitary representation of $G$.

$\beta$ – an absolutely continuous symmetric probability measure on $G$ such that $\overline{\langle \mathrm{supp}(\beta) \rangle} = G$.

As above, we have consider the averaging operator:

$$\pi(\beta) : \mathcal{H} \longrightarrow \mathcal{H} : v \longmapsto \int_G \pi(g) v\, dg.$$

**Def.** $\pi$ has *spectral gap* if $\|\pi(\beta)\| < 1$.

**Rmk:** This is independent of a choice of $\beta$.

An action $G \curvearrowright (X,\mu)$ has spectral gap if $G \curvearrowright L^2_0(X)$ has spectral gap.

## Examples.

1) $\lambda : G \curvearrowright L^2(G)$ – the regular representation.

$\lambda$ has spectral gap $\Leftrightarrow$ $G$ is not amenable (*Kersten*)

When $G$ is a simple connected Lie group with finite centre, take $\beta = \frac{\chi_B}{vol(B)} d\,Vol$. Then by *Kunze–Stein* inequality,

$$\|\lambda(\beta) f\|_2 = \left\| \frac{\chi_B}{vol(B)} * f \right\|_2 \leq c_p \left\| \frac{\chi_B}{vol(B)} \right\|_p \cdot \|f\|_2 = c_p \cdot vol(B)^{\frac{1}{p}-1} \|f\|_2,$$

for $1 \leq p < 2$.

Hence, $\|\lambda(B)\| \ll_\varepsilon vol(B)^{-\frac{1}{2}+\varepsilon}$, $\varepsilon > 0$.

2) $G$ – a connected simple Lie group with finite centre, $Rank(G) \geq 2$.

Then $\sup_\pi \|\pi(\beta)\| < 1$, where $\pi$ runs over all unitary representations without $G$-inv. vectors.

(*Kazhdan*)

3) $G$ – a Lie group

$\Lambda$ – a closed subgroup with $vol(G/\Lambda) < \infty$.

Then $G \curvearrowright L^2_0(G/\Lambda)$ has spectral gap

(*Bekka–Cornulier*)

4) $\pi$ - a unitary representation of simple connected Lie group with finite centre.

$\pi$ has spectral gap $\Rightarrow$ $\pi|_H$ has spectral gap for any closed nonamenable subgroup $H$ of $G$.
(Nevo, Shalom)

## Integrability exponents.

$\pi : G \curvearrowright \mathcal{H}$ - a unitary representation.

We introduce the integrability exponent of $\pi$:

$$q(\pi) = \inf\{ q > 0 : \langle \pi(g)u_1, u_2 \rangle \in L^q(G) \text{ for } u_1, u_2 \in \text{dense subspace of } \mathcal{H} \}.$$

Borel-Wallach, Cowling, Howe-Moore:

If $G$ is a simple connected Lie group with finite centre,

$\pi$ has spectral gap $\Leftrightarrow$ $q(\pi) < \infty$.

Prop. 1 (Godement) $\pi : G \curvearrowright \mathcal{H}$ - a unitary representation.
Assume that $\langle \pi(g)u_1, u_2 \rangle \in L^2(G)$ for $u_1, u_2 \in$ dense subspace of $\mathcal{H}$.
Then $\pi \subset \bigoplus_{i=1}^{\infty} \lambda$. In particular, $\|\pi(\beta)\| \le \|\lambda(\beta)\|$.

Sketch: Let $\mathcal{H}_0$ be countable dense subset with the above property. We introduce a map

$$\mathcal{H}_0 \longrightarrow \bigoplus_{v \in \mathcal{H}_0} L^2(G) : w \longmapsto \bigoplus_{v \in \mathcal{H}_0} \langle \pi(g)v, w \rangle.$$

This map is densely defined, equivariant, and closed.
To show that it extends to $\mathcal{H}$, one needs to use a version of Schur's Lemma.

Prop 2 (Cowling-Haagerup-Howe)

$\pi : G \curvearrowright \mathcal{H}$ - unitary representation.

$$q(\pi) = 2 \implies \|\pi(\beta)\| \leqslant \|\lambda(\beta)\|.$$

Prop. 3 (Nevo's transfer principle)

$G \curvearrowright (X, \mu)$ - a measure-preserving action.

$\pi : G \curvearrowright L^2_0(X)$ - the corresponding unitary representation.

Assume that $q(\pi) < \infty$.

$$n(\pi) = \begin{cases} \text{least even integer} \geqslant \frac{q(\pi)}{2}, & q(\pi) > 2 \\ 1, & q(\pi) \leqslant 2. \end{cases}$$

Then $\|\pi(\beta)\| \leqslant 2 \|\lambda(\beta)\|^{\frac{1}{n(\pi)}}$.

Proof. By the Hölder inequality,

$$q(\pi^{\otimes n}) \leqslant \frac{q(\pi)}{n} \leqslant 2.$$

Hence, by Prop. 2, $\|\pi^{\otimes n}(\beta)\| \leqslant \|\lambda(\beta)\|$.

For every $f \in L^2_0(X)$, real-valued,

$$\|\pi(\beta) f\|_2^{2n} = \left( \int_{G \times G} \langle \pi(g_1) f, \pi(g_2) f \rangle \, d\beta(g_1)\, d\beta(g_2) \right)^n$$

Jensen inequality

$$\leqslant \int_{G \times G} \langle \pi(g_1) f, \pi(g_2) f \rangle^n \, d\beta(g_1)\, d\beta(g_2)$$

$$= \int_{G \times G} \langle \pi^{\otimes n}(g_1) f^{\otimes n}, \pi^{\otimes n}(g_2) f^{\otimes n} \rangle \, d\beta(g_1)\, d\beta(g_2)$$

$$= \|\pi^{\otimes n}(\beta) f^{\otimes n}\|_2^2 \leqslant \|\lambda(\beta)\|^2 \cdot \|f\|_2^{2}.$$

For general $f = f_1 + i f_2$ with $f_i$'s real-valued, we have $\|\pi(\beta) f\|_2 \leq 2 \|\lambda(\beta)\| \cdot \|f\|_2$. $\square$

<u>Cor.</u> $G$ – a connected simple Lie group with finite centre.
$G \curvearrowright (X,\mu)$ – a measure-preserving action on a prob. space, with spectral gap.

Then $\forall B \subset G$, $0 < \mathrm{vol}(B) < \infty$ $\forall f \in L^2(X)$:

$$\left\| \frac{1}{\mathrm{vol}(B)} \int_B f(g^{-1}x)\, dg - \int_X f\, d\mu \right\|_2 \ll_\varepsilon \mathrm{vol}(B)^{-\frac{1}{2n(\pi)}+\varepsilon} \|f\|_2, \quad \varepsilon > 0,$$

where $\pi : G \curvearrowright L^2_0(X)$.

<u>Proof:</u> Let $\beta = \frac{\chi_B}{\mathrm{vol}(B)}\, d\mathrm{Vol}$.
By Prop. 3, $\|\pi(\beta)\| \leq \|\lambda(\beta)\|^{\frac{1}{n(\pi)}}$.
By the Kunze–Stein inequality,
$$\|\lambda(\beta)\| \ll_\varepsilon \mathrm{Vol}(B)^{-\frac{1}{2}+\varepsilon}, \quad \varepsilon > 0.$$
$\square$