Lecture 1: Kunze-Stein inequality

Let $\varphi, \psi : G \to \mathbb{C}$ be functions on a locally compact group.

$$(\varphi * \psi)(g) = \int_G \varphi(h)\, \psi(h^{-1} g)\, dh.$$

Lem. $\|\varphi * \psi\|_2 \le \|\varphi\|_1 \cdot \|\psi\|_2$ for all $\varphi \in L^1(G)$, $\psi \in L^2(G)$.

Can this inequality be improved: $1 \rightsquigarrow p>1$?

example: If φ & ψ are characteristic functions of compact sets $A, B \subset G$,

$$\|\varphi * \psi\|_2^2 = \int_{A\times A} \langle \chi_B(a_1^{-1} g), \chi_B(a_2^{-1} g) \rangle \, da_1 da_2$$

$$= \int_{A\times A} \mathrm{vol}(a_1 B \cap a_2 B)\, da_1 da_2.$$

Clearly, $\int_{A\times A} \mathrm{vol}(a_1 B \cap a_2 B)\, da_1 da_2 \le \underbrace{\mathrm{vol}(A)^2}_{\|\varphi\|_1^2} \cdot \underbrace{\mathrm{vol}(B)}_{\|\psi\|_2^2}$,

If $G = \mathbb{R}$ and $B = [-N, N]$, $N \to \infty$, then

$$\mathrm{vol}(a_1 B \cap a_2 B) \simeq \mathrm{vol}(B).$$

Hence, for $\mathbb{R}$ (and, more generally, amenable groups), this is the best that we can do.

Kunze-Stein Inequality (Cowling)

G = a connected semisimple Lie group with finite center.

$\forall \varphi \in L^1(G) \cap L^p(G)$ $\forall \psi \in L^2(G)$, $1 \le p < 2$

$$\boxed{\|\varphi * \psi\|_2 \le C_p \cdot \|\varphi\|_p \cdot \|\psi\|_2.}$$

1) Herz majoration principle.

$$G = K \cdot P \quad \left(\begin{array}{l} K = \text{maximal compact subgroup} \\ P = \text{minimal parabolic subgroup} \end{array} \right)$$

Invariant measure on G is given by

$$\int_G \varphi(g)dg = \int_{K\times P} \varphi(kp)dk\,dp, \quad \varphi : G \to \mathbb{C},$$

where dk is invariant measure on K, and dp is right invariant measure on P.

Then $\int_P \psi(p_0 p)\,dp = \Delta(p_0) \int_P \psi(p)\,dp, \quad \psi : P \to \mathbb{C},$

where Δ is the modular function of P.

We introduce principle series representations:

$$\mathcal{H}_0 = \left\{ \varphi : G \to \mathbb{C} : \begin{array}{l} \varphi(gp) = \varphi(g)\,\Delta(p)^{1/2} \text{ for all } g \in G \text{ and } p \in P \\ \|\varphi\| := \left(\int_K |\varphi(k)|^2 dk \right)^{1/2} < \infty \end{array} \right\}$$

$$\pi_0(g)\varphi(x) = \varphi(g^{-1}x)$$

Lemma. π_0 is a unitary representation of G.

Proof. We write $g = K(g)P(g)$ with $K(g) \in K$ and $P(g) \in P$.
Let $\psi \in C_c(P)$ with $\int_P \psi(p)\,dp = 1$. Then

$$\begin{aligned}
\int_K |\varphi(g^{-1}k)|^2 dk &= \int_K |\varphi(K(g^{-1}k))|^2 \cdot \Delta(P(g^{-1}k))\,dk \\
&= \int_{K\times P} |\varphi(K(g^{-1}k))|^2 \cdot \Delta(P(g^{-1}k)) \cdot \psi(p)\,dk\,dp \\
&= \int_{K\times P} |\varphi(K(g^{-1}kp))|^2 \cdot \psi(P(g^{-1}kp))\,dk\,dp \\
&= \int_G |\varphi(K(g^{-1}h))|^2 \cdot \psi(P(g^{-1}h))\,dh = \int_G |\varphi(K(h))|^2 \cdot \psi(P(h))\,dh = \int_K |\varphi(k)|^2 dk.
\end{aligned}$$

$\lambda: G \curvearrowright L^2(G)$ – the regular representation.

$\lambda(g)\varphi(x) = \varphi(g^{-1}x)$, $\varphi \in L^2(G)$.

Lem. Given $\psi, \eta \in L^2(G)$, there exist $\overline{\psi}, \overline{\eta} \in \mathcal{H}_0$ such that $\|\psi\|_2 = \|\overline{\psi}\|_2$, $\|\eta\|_2 = \|\overline{\eta}\|$,

$$|\langle \lambda(g)\psi, \eta\rangle| \leq |\langle \pi_0(g)\overline{\psi}, \overline{\eta}\rangle|, \quad g \in G.$$

Proof: Let $\overline{\psi}(g) = \left(\int_P |\psi(gp)|^2 dp\right)^{1/2}$ and $\overline{\eta}(g) = \dots$

Then $\overline{\psi}, \overline{\eta} \in \mathcal{H}_0$ and $\|\overline{\psi}\| = \left(\int_{K\times P} |\psi(kp)|^2 dk dp\right)^{1/2} = \|\psi\|_2$

and similarly $\|\overline{\eta}\| = \|\eta\|_2$.

By the Cauchy–Schwarz inequality,

$$|\langle \lambda(g)\varphi, \eta\rangle| = \left|\int_{K\times P} \psi(g^{-1}kp)\eta(kp)\, dp\, dk\right|$$

$$\leq \int_K \left(\int_P |\psi(g^{-1}kp)|^2 dp\right)^{1/2} \cdot \left(\int_P |\eta(kp)|^2 dp\right)^{1/2} dk$$

$$= |\langle \pi_0(g)\overline{\psi}, \overline{\eta}\rangle|, \text{ as required.}$$

For a unitary representation $\pi: G \curvearrowright \mathcal{H}$ and $\varphi \in L^1(G)$, we set $\pi(\varphi): \mathcal{H} \to \mathcal{H}: v \longmapsto \int_G \pi(g) v\, dg$.

Kunze–Stein Ineq. $\Leftrightarrow \underset{1 \leq p < 2}{\|\lambda(\varphi)\| \leq C_p \|\varphi\|_p} \Longleftrightarrow \underset{1 \leq p < 2}{\|\pi_0(\varphi)\| \leq C_p \cdot \|\varphi\|_p}$

2) Complex interpolation.

Consider a family of representations $\pi_z : G \circlearrowright \mathcal{H}_z$

$$\mathcal{H}_z = \left\{ \varphi : G \to \mathbb{C} : \begin{array}{l} \varphi(gp) = \varphi(g)\Delta(p)^{\frac{1}{2}(1+z)} \text{ for } g \in G, p \in P \\ \|\varphi\|_q = \left(\int_K |\varphi(k)|^q dk \right)^{1/q} < \infty \end{array} \right\}$$

$$\pi_z(g)\varphi(x) = \varphi(g^{-1}x)$$

This representation is isometric if $\frac{q}{2} \cdot (1 + \mathrm{Re}(z)) = 1$, which we always assume.

<u>Aim:</u> $\forall \varphi \in L^1(G) \cap L^p(G)$ $\forall \psi, \eta \in \mathcal{H}_0$ $\boxed{|\langle \pi_0(\varphi)\psi, \eta\rangle| \leq c_p \|\varphi\|_p.}$ (?)
$1 \leq p < 2$ $\quad \|\psi\|_2 = \|\eta\|_2 = 1$

Let $\psi_z = |\psi|^z \cdot \psi$ and $\overline{\eta_z} = |\eta|^{-z} \cdot \overline{\eta}$. Then $\|\psi_z\|_q = 1$ and $\|\eta_z\|_{q'} = 1$ where $\frac{1}{q} + \frac{1}{q'} = 1$.

Let $F_z(\varphi) = \langle \pi_z(\varphi)\psi_z, \eta_z\rangle$ with $\varphi \in L^1(G)$.

By the Hölder inequality,

$$(*) \quad |\langle \pi_z(\varphi)\psi_z, \eta_z\rangle| \leq \|\pi_z(\varphi)\psi_z\|_q \cdot \|\eta_z\|_{q'} \leq \|\varphi\|_1.$$

Now suppose that $G = SL(2, \mathbb{R})$.

Then by the Plancherel formula,

$$(**) \quad \left(\int_{\mathbb{R}} \underbrace{|\langle \pi_{iy}(\varphi)\psi_{iy}, \eta_{iy}\rangle|}_{\leq \|\pi_{iy}(\varphi)\|_{HS}}{}^2 \cdot y \cdot \tanh\left(\frac{\pi y}{2}\right) dy \right)^{1/2} \leq C \cdot \|\varphi\|_2$$

for $\varphi \in L^1(G) \cap L^2(G)$.

Hilbert-Schmidt norm

Let $\alpha(z) = \frac{z}{z-2}$. Then

$$|\alpha(iy)|^2 \leq y \cdot \tanh\left(\frac{\pi y}{2}\right), \quad y \in \mathbb{R},$$

$$\frac{Re(z)}{3} \leq |\alpha(z)| \leq 1, \quad -1 \leq Re(z) \leq 1.$$

We prove (?) by interpolating:

$$(*) \quad |F_z(\varphi)| \leq \|\varphi\|_1, \quad \varphi \in L^1(G),$$

$$(**) \quad \left(\int_{\mathbb{R}} |\alpha(iy)|^2 \cdot |F_{iy}(\varphi)|^2 dy\right)^{1/2} \leq C \cdot \|\varphi\|_2, \quad \varphi \in L^1(G) \cap L^2(G).$$

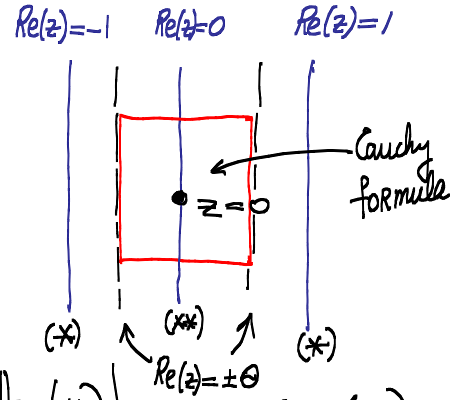
It is sufficient to prove (?) for simple φ with $\|\varphi\|_p = 1$.

Let $h \in L^p(\mathbb{R})$ be a simple function with $\|h\|_p = 1$, and $h_z = h \cdot |h|^{\frac{p}{2}(1+z)-1}$. Also, $\varphi_z = \varphi \cdot |\varphi|^{\frac{p}{2}\cdot(1+z)-1}$.

We have:

$Re(z) = 0$: $\quad \|h_z\|_2 = \|\varphi_z\|_2 = 1$

$Re(z) = 1$: $\quad \|h_z\|_1 = \|\varphi_z\|_1 = 1$.



For $0 \leq Re(z) \leq 1$,

$$\begin{aligned} |\alpha(z+iy)\, F_{z+iy}(\varphi_z) h_z(y)| &\leq |F_{z+iy}(\varphi_z)| \cdot |h_z(y)| \quad \text{by } (*) \\ &\leq \|\varphi_z\|_1 \cdot (|h_0(y)| + |h_1(y)|) \\ &\leq C(\varphi) \cdot (|h_0(y)| + |h_1(y)|). \end{aligned}$$

Since the last term is a simple function,

$$S(z) = \int_{\mathbb{R}} \alpha(z+iy)\, F_{z+iy}(\varphi_z) h_z(y)\, dy$$

is well defined, analytic on $0 < Re(z) < 1$, and bounded on $0 \leq Re(z) \leq 1$.

For $Re(z)=0$, by the Cauchy-Schwartz inequality,

$$|S(z)| \le \left(\int_{\mathbb{R}} |\alpha(z+iy)|^2 \cdot |F_{z+iy}(\varphi_z)|^2 dy\right)^{1/2} \cdot \|h_z\|_2$$

$$\le C \cdot \|\varphi_z\|_2 = C, \quad \text{by } (**).$$

For $Re(z)=1$,

$$|S(z)| \le \left(\sup_{y\in\mathbb{R}} |F_{z+iy}(\varphi_z)|\right) \cdot \|h_z\|_1 \underset{(*)}{\le} \|\varphi_z\|_1 \cdot \|h_z\|_1 = 1.$$

Hence, by the Phragmen-Lindelöf principle,

$$S(\theta) \le C^{1-\theta} \quad \text{for } \theta \in [0,1].$$

We pick θ such that $\frac{p}{2}(1+\theta)=1$. Then

$$\left|\int_{\mathbb{R}} \alpha(\theta+iy) \cdot F_{\theta+iy}(\varphi)\, h(y)\, dy\right| \le C^{1-\theta}.$$

Taking supremum over h, we obtain

$$\left(\int_{\mathbb{R}} |\alpha(\theta+iy)|^{p'} |F_{\theta+iy}(\varphi)|^{p'} dy\right)^{1/p'} \le C^{1-\theta}.$$

Since $|\alpha(z)| \ge \frac{Re(z)}{3}$, $\left(\int_{\mathbb{R}} |F_{\theta+iy}(\varphi)|^{p'} dy\right)^{1/p'} \le C(\theta)$.

Similarly, $\left(\int_{\mathbb{R}} |F_{-\theta+iy}(\varphi)|^{p'} dy\right)^{1/p'} \le C(\theta)$.

Hence, by the Cauchy formula,

$$F_0(\varphi) = \frac{1}{2\pi i}\int_{Re(z)=\theta} \frac{F_z(\varphi)}{z} dz + \frac{1}{2\pi i}\int_{Re(z)=-\theta} \frac{F_z(\varphi)}{z} dz$$

Finally,

$$\left|\int_{Re(z)=\pm\theta} \frac{F_z(\varphi)}{z} dz\right| \le \left(\int_{Re(z)=\pm\theta} |F_z(\varphi)|^{p'} dz\right)^{1/p'} \cdot \left(\int_{Re(z)=\pm\theta} \frac{dz}{|z|^p}\right)^{1/p} < \infty$$

for $p \in (1,2)$. $\blacksquare$