Lecture 9

Unipotent flows and application.

Let G be a closed subgroup of $SL_d(\mathbb{R})$, and Γ is a discrete subgroup such that $\Gamma\backslash G$ has finite measure.

A one-parameter subgroup $\mathcal{U}=\{u(t)\}\subset G$ is called <u>unipotent</u> if $u(t)=\exp(t\cdot N)$ where N is a nilpotent matrix.
(here, $\exp(x)=I+x+\frac{x^2}{2!}+\dots$).

Our interest is the dynamical system:

$$X=\Gamma\backslash G \circlearrowleft u(t): \quad x\longmapsto x\cdot u(t),$$

which is a generalisation of the horocycle flow.

<u>Thm</u> (Ratner)

1) (<u>topological version</u>) For every $x\in X$, $\overline{x\mathcal{U}}=xF$, where F is a closed connected subgroup of G, containing $\mathcal{U}$, and xF supports finite F-invariant measure.

More generally, if H is a closed subgroup of G generated by unipotent subgroups, then

$$\forall x\in X: \quad \overline{x\cdot H}=xF$$

where F is a closed connected subgroup of G, containing H, and xF supports finite F-inv. measure.

2) (measurable version) Every ergodic $\mathcal{U}$-invariant probability measure on X is an F-invariant probability measure supported on closed orbit xF, $x \in X$.

3) (equidistribution) For every $x \in X$,

$$\frac{1}{T}\int_0^T f(xu_t)\,dt \xrightarrow[T\to\infty]{} \int_X f\,d\mu$$

where μ is the F-invariant probability measure supported on $\overline{x\mathcal{U}} = xF$.

Oppenheim Conjecture.

$$Q(x_1, \dots x_d) = \sum_{i,j=1}^{d} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{R}, \quad a_{ij} = a_{ji}.$$

Assume that: 1) Q is nondegenerate ($\Leftrightarrow \det(a_{ij}) \neq 0$),

2) Q is indefinite ($\Leftrightarrow Q(\mathbb{R}^d) = \mathbb{R}$)

3) Q is not a scalar multiple of a form with rational coefficients.

Ex. $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - \sqrt{2}\, x_3^2$.

Conj (Oppenheim) If $d \geq 3$, then $Q(\mathbb{Z}^d)$ is dense in $\mathbb{R}$.

This conjecture was proved by Margulis.

Ratner Thm $\Rightarrow$ Oppenheim Conj. (sketch)

Let $SO(Q, \mathbb{R}) = \{ g \in SL_d(\mathbb{R}) : Q(x \cdot g) = Q(x) \text{ for all } x \}$.

Let $H = \left[\begin{array}{l}\text{the connected component of } SO(Q,\mathbb{R}),\\ \text{containing identity.}\end{array}\right]$

<u>Algebraic properties:</u> when $d \geq 3$,

1) H is generated by unipotent subgroups.
2) H is maximal connected subgroup of $SL_d(\mathbb{R})$.

By the Ratner Thm, we have two cases:

$$\boxed{\overline{\mathbb{Z}^d H} = \{\text{whole space}\}} \quad \text{or} \quad \boxed{\mathbb{Z}^d H \text{ is closed}}$$

$\Updownarrow$

$SL_d(\mathbb{Z})H$ is dense in $SL_d(\mathbb{R})$

$$Q(\mathbb{Z}^d) = Q(\mathbb{Z}^d \cdot SL_d(\mathbb{Z})H)$$
$$\cap \text{ dense}$$
$$Q(\mathbb{Z}^d SL_d(\mathbb{R}))$$
$$\|$$
$$Q(\mathbb{R})$$

$\Updownarrow$

$SL_d(\mathbb{Z})H$ is closed in $SL_d(\mathbb{R})$

Let $\Gamma = H \cap SL_d(\mathbb{Z})$.
Then $vol(\Gamma \backslash H) < \infty$.
(Γ is "large")

Consider the system of linear equations:

$$\{ {}^tA = A,\ {}^t h A h = A : h \in H\}. \quad (*)$$

One can check that the set of solutions is $\langle A_Q \rangle$ where A_Q is the matrix of the quadratic form Q.

We also consider the system of linear equations:

$$\{ {}^tA = A,\ {}^t h A h = A : h \in \Gamma\}, \quad (**)$$

which has <u>integral</u> coefficients. Hence, its set of solutions is a <u>rational</u> subspace of $Mat_d(\mathbb{R})$.

Using that Γ is "large", one can show that $(*)$ and $(**)$ have the same sets of solutions.

This implies that A_Q is a multiple of a rational matrix.

Sprindzuk Conjecture.

A vector $x \in \mathbb{R}^d$ is called <u>well approximable</u> if

$$\left\| x - \frac{p}{q} \right\| < \frac{1}{q^{1+\frac{1}{d}+\varepsilon}}, \quad p \in \mathbb{Z}^d,\ q \in \mathbb{N},$$

has infinitely many solutions for some $\varepsilon > 0$.

By the Borel-Cantelli lemma, the set WA_d of well approximable vectors in $\mathbb{R}^d$ has measure 0.

Let $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^k$ and $f : \mathcal{U} \to \mathbb{R}^d$ be polynomial map.

Conj. (Sprindzuk) Assume that $f(\mathcal{U})$ is not contain in a proper affine subspace of $\mathbb{R}^d$. Then $\{x \in \mathcal{U} : f(x) \in WA_d\}$ has measure zero.

This conjecture was proved by Kleinbock & Margulis using "nondivergence" properties of unipotent flows.

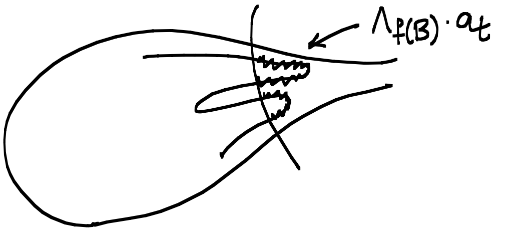
Let $\Lambda_x = \mathbb{Z}^{d+1}\left(\begin{array}{c|c} I_d & 0 \\ \hline x & 1 \end{array}\right)$, $a_t = \left(\begin{array}{c|c} e^t I_d & 0 \\ \hline 0 & e^{-dt} \end{array}\right)$, $\Delta(\Lambda) = \min\{\|v\| : v \in \Lambda \setminus \{0\}\}$.

Lemma. $x \in VA_d \iff \exists \delta > 0 : \exists n_i \in \mathbb{N} : n_i \to \infty : \Delta(\Lambda_x a_{n_i}) \le e^{-\delta n_i}$.

(excursions to exponentially shrinking nbhds of infinity).

Prop. $\forall x \in \mathcal{U}$ $\exists$ a ball $x \in B \subset \mathcal{U}$: such that $\forall \varepsilon, t > 0$:

$$\bar{\mu}\left(\{x \in B : \Delta(\Lambda_{f(x)} a_t) < \varepsilon\}\right) \leq C \cdot \varepsilon^{\alpha} \cdot \mathrm{vol}(B).$$



The percentage of time spend near infinity can be controled.

Prop. $\Rightarrow$ Conj. $\{x \in B : f(x) \in VA_d\} = \bigcup_{\delta > 0} \underbrace{\{x \in B : \Delta(\Lambda_{f(x)} a_n) \leq e^{-\delta n} \text{ for inf. many } n \in \mathbb{N}\}}_{= \Omega_\delta}$

It is sufficient to show that $\mathrm{vol}(\Omega_\delta) = 0$ for all $\delta > 0$.

Let $\Omega_\delta(n) = \{x \in B : \Delta(\Lambda_{f(x)} a_n) \leq e^{-\delta n}\}$.

Then $\Omega_\delta = \overline{\lim}\, \Omega_\delta(n) = \bigcap_{N \geq 1} \bigcup_{n \geq N} \Omega_\delta(n)$, and

$$\mathrm{vol}(\Omega_\delta) \leq \sum_{n \geq N} \mathrm{vol}(\Omega_\delta(n)) \leq \underbrace{\sum_{n \geq N} C \cdot (e^{-\delta n})^{\alpha} \mathrm{vol}(B)}_{\text{tail of convergent series.}} \xrightarrow[N \to \infty]{} 0.$$

$\square$

Def. A function $f: \mathcal{U} \to \mathbb{R}$ is called $\underline{(C,\alpha)\text{-good}}$ if for any ball $B \subset \mathcal{U}$ and any $\varepsilon > 0$,

$$\mathrm{vol}\left(x \in B : |f(x)| < \varepsilon \cdot \sup_{x \in B} |f(x)|\right) \leq C \cdot \varepsilon^{\alpha} \mathrm{vol}(B).$$

Lem. $\forall d \geq 1$ $\exists (C, \alpha)$: $\forall$ polynomial of degree d $f: \mathbb{R} \to \mathbb{R}$ is (C,α)-good.

Without loss of generality, $\sup_{x\in B}|f(x)|=1$.

Let $B_\varepsilon=\{x\in B:\ |f(x)|<\varepsilon\}$ and $\ell=|B_\varepsilon|$.

Since B_ε cannot be covered by $(d-1)$ intervals of length ℓ/d, one can find $x_1,\dots x_{d+1}\in B_\varepsilon$: $|x_i-x_j|\geq\frac{\ell}{2d}$ for $i\neq j$.

By the Lagrange Interpolation,

$$f(x)=\sum_{i=1}^{d+1}f(x_i)\cdot\frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}.$$

$$\forall x\in B:\quad |f(x)|\leq(d+1)\cdot\varepsilon\cdot\frac{|B|^d}{(\ell/2d)^d}.$$

Hence, $1\leq(d+1)\varepsilon\cdot\frac{|B|^d}{(\ell/2d)^d}\Rightarrow \ell\leq C_d\cdot\varepsilon^{1/d}\cdot|B|.$

To prove Proposition, we need to estimate:

$$\mathrm{vol}\left(\{x\in B:\ \Delta(\Lambda_{f(x)}a_t)<\varepsilon\}\right).$$

For $v\in\Lambda$, set $P_{v,t}(x)=\left\|v\cdot\left(\begin{array}{c|c}Id&0\\\hline f(x)&1\end{array}\right)a_t\right\|$, $x\in B$.

By the (C,α)-good property,

$$\mathrm{vol}\left(\{x\in B:\ P_{v,t}(x)<\varepsilon\}\right)\leq C\cdot\left(\frac{\varepsilon}{\sup_B(P_{v,t})}\right)^\alpha\cdot|B|.$$

This is the first step towards the proof.

However, $\{x\in B:\ \Delta(\Lambda_{f(x)}a_t)<\varepsilon\}=\bigcup_{0\neq v\in\Lambda}\{x\in B:\ P_{v,t}(x)<\varepsilon\}$,

infinite union

so that one needs additional combinatorial argument to deduce the estimate.