$\boxed{\text{Lecture 7}}$

Ergodic Theorems.

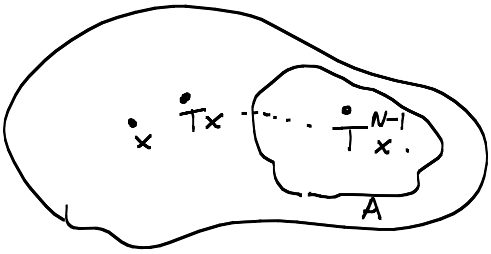
Let (X, μ) be a space with measure μ such that $\mu(X) = 1$, and $T: X \longrightarrow X$ a (measurable) map.

We assume that T is <u>measure-preserving</u> (i.e., $\mu(T^{-1}A) = \mu(A)$ for $A \subset X$).

<u>ex.</u>
- rotation of a circle,
- doubling map,
- $\Lambda \longmapsto \Lambda g$ on the space of lattices $\mathcal{L}_d$.

<u>Distribution of orbits:</u> for $A \subset X$,

$$\#\{i = 0, \dots N-1 : T^i x \in A\} \underset{n \to \infty}{\approx} ?$$



OR

for $f: X \to \mathbb{C}$,

$$\sum_{i=0}^{N-1} f(T^i x) \underset{n \to \infty}{\approx} ?$$

Let $\mathcal{H} = L^2(X) = \{ f: X \to \mathbb{C} : \int_X |f|^2 d\mu < \infty \}$.

$$\begin{aligned} \mathcal{U} : \mathcal{H} &\longrightarrow \mathcal{H} \\ f &\longmapsto f(T \cdot x) \end{aligned}$$

T - measure-preserving $\Rightarrow$ $\|\mathcal{U}f\| = \|f\|$, $f \in \mathcal{H}$.

von Neumann Mean Ergodic Thm.

$$\forall f \in L^2(X): \quad \frac{1}{N} \cdot \sum_{i=0}^{N-1} f \circ T^i \xrightarrow[N\to\infty]{L^2} P(f)$$

where P is the orthogonal projection on
$\mathcal{H}_0 = \{ f \in L^2(X) : f \circ T = f \}$.

$\ulcorner$ 1) $\forall f \in \mathcal{H}_0$: $\frac{1}{N}\sum_{i=0}^{N-1} f\circ T^i = f = P(f)$

2) $\forall f = g\circ T - g$ with $g \in L^2(X)$:

$$\frac{1}{N}\sum_{i=0}^{N-1} f\circ T^i = \frac{1}{N}\sum_{i=0}^{N-1} \left(g\circ T^{i+1} - g\circ T^i \right) = \frac{g\circ T^N - g}{N},$$

$$\left\| \frac{1}{N}\sum_{i=0}^{N-1} f\circ T^i \right\|_2 \leq \frac{2\|g\|_2}{N} \xrightarrow[N\to\infty]{} 0.$$

For $h \in \mathcal{H}_0$, $\langle f, h\rangle = \langle Ug - g, h\rangle = \langle g, U^*h - h\rangle$,

$$\|U^*h - h\|^2 = \langle U^*h, U^*h\rangle - 2\langle U^*h, h\rangle + \langle h, h\rangle$$
$$= \langle h, h\rangle - 2\langle Uh, h\rangle + \langle h, h\rangle = 0.$$

Hence, $P(f) = 0$.

3) If $\|f - g\|_2 < \varepsilon$, then $\left\| \frac{1}{N}\sum_{n=0}^{N-1} (f-g)\circ T^i - P(f-g) \right\|_2 \leq 2\varepsilon$.

Hence, it is suficient to prove the claim for a dense subset of functions.

4) $\langle g\circ T - g : g \in L^2(X) \rangle + \mathcal{H}_0$ is dense in $L^2(X)$.
Indeed, suppose that $f \in L^2(X)$ is orthoganal to this subspace. Then $\langle f, Ug - g\rangle = \langle U^*f - f, g\rangle$ for all $g \in L^2(X)$, and hence $U^*f = f$ and $Uf = f$, i.e., $f \in \mathcal{H}_0$ and
$\langle f, f\rangle = 0 \Rightarrow f = 0$. $\lrcorner$

Birkhoff Pointwise Ergodic Thm.

For every $f \in L^2(X)$ and almost every $x \in X$,

$$\frac{1}{N}\sum_{i=0}^{N-1} f(T^i x) \longrightarrow (Pf)(x).$$

Cor. For a.e. $x \in [0,1]$, the decimal expansion $x = 0.x_0 x_1 \ldots$ satisfies: for $d = 0, \ldots 9$

$$\frac{\#\{i = 0,\ldots N-1,\ x_i = d\}}{N} \xrightarrow[N\to\infty]{} \frac{1}{10}.$$

Proof. Consider $T : [0,1) \to [0,1) : x \longmapsto 10x \pmod 1$, T is measure-preserving (check) and mixing (Lecture 1).

If $f \in L^2([0,1])$ is T-invariant,

$$\underset{\|f - \int f\|_2^2}{\underset{\shortparallel}{\langle f\circ T^n - \int f,\ f - \int f\rangle}} \xrightarrow[n\to\infty]{} 0$$

Hence, $\{f : f\circ T = f\} = \mathbb{C}\cdot 1$, and

$$P(f) = \langle f, 1\rangle = \int_{[0,1)} f(x)\,dx.$$

Apply the pointwise ergodic thm to $f = \chi_{[0,\frac{1}{10})}$...

Let $S_N(f) = \sum_{i=0}^{N-1} f\circ T^i$.

Maximal Inequality. For $f \in L^1(X)$, $f \geq 0$, and $\delta > 0$,

$$\mu\Big(\underbrace{\Big\{x \in X : \sup_{N\geq 1} \frac{S_N(f)(x)}{N} > \delta\Big\}}_{\Omega_\delta}\Big) \leq \frac{1}{\delta}\int_X f\,d\mu.$$

Let $g = f - \delta$ and $M_N(g) = \max\{0, S_1(g), \dots S_N(g)\}$.

Then $\Omega_\delta = \bigcup_{N\geq 1} \{S_N(g) > 0\} = \bigcup_{N \geq 1} \{M_N(g) > 0\}$.

Clearly, $M_N(g) \geq S_n(f)$ for $n = 0, \dots N$, where $S_0(f) = 0$.

Hence, $M_N(g)\circ T + g \geq S_n(g)\circ T + g = S_{n+1}(g)$,

and $M_N(g)\circ T + g \geq \max_{n=1,\dots N} S_n(g) \underset{\uparrow}{=} M_N(g)$.

when $M_N(g) > 0$

We checked that $g \geq M_N(g) - M_N(g)\circ T$ when $M_N(g) > 0$,

$$\int_{\{M_N(g)>0\}} g \, d\mu \geq \int_{\{M_N(g)>0\}} M_N(g)\, d\mu - \int_{\{M_N(g)>0\}} M_N(g)\circ T \, d\mu$$

$$\geq \int_X M_N(g)\, d\mu - \int_X M_N(g)\circ T \, d\mu = 0.$$

Since $\{M_N(g) > 0\}$ is increasing with N, we can take limit $N \to \infty$, and get $\int_{\Omega_\delta} g \, d\mu \geq 0$,

$$\int_{\Omega_\delta} g \, d\mu = \int_{\Omega_\delta} f \, d\mu - \delta \cdot \mu(\Omega_\delta) \geq 0,$$

$$\mu(\Omega_\delta) \leq \frac{1}{\delta} \cdot \int_{\Omega_\delta} f \, d\mu \leq \frac{1}{\delta} \int_X f \, d\mu.$$

Proof of Pointwise ergodic Thm.

For simplicity, let's assume that $P(f) = 0$.

We need to show that $\mu(\{x : \overline{\lim_{N\to\infty}} |\frac{1}{N} S_N(f)(x)| > 0\}) = 0$

$$\Updownarrow$$

$\mu(\{x : \overline{\lim_{N\to\infty}} |\frac{1}{N} S_N(f)(x)| > \delta\}) = 0$ for all $\delta > 0$.

Suppose that $\|f-\tilde{f}\| < \varepsilon$ and $\lim\limits_{N\to\infty} \frac{1}{N} S_N(\tilde{f})(x) = 0$ a.e.

Then $\mu(\{x : \overline{\lim\limits_{N\to\infty}} |\frac{1}{N} S_N(f)(x)| > \delta\})$

$$\wedge\!\!\setminus$$

$$\mu(\{x : \underbrace{\overline{\lim_{N\to\infty}} |\tfrac{1}{N} S_N(\tilde{f})(x)|}_{=0} + \sup_{N\geq 1} \tfrac{1}{N} S_N(|f-\tilde{f}|) > \delta\})$$

$\wedge\!\!\setminus$ ← (maximal inequality)

$$\frac{1}{\delta} \cdot \int_X |f-\tilde{f}|\, d\mu$$

$\wedge\!\!\setminus$ ← (Cauchy–Schwartz inequality)

$$\frac{1}{\delta} \|f-\tilde{f}\|_2 < \frac{1}{\delta} \varepsilon.$$

Hence, it is sufficient to show that $\lim\limits_{N\to\infty} \frac{1}{N} S_N(f)(x) = 0$ a.e. for a dense family of functions.

Now the thm follows from

<u>Claim</u>: $\langle g\circ T - g : g - \text{bounded} \rangle$ is dense in $\{f : P(f) = 0\}$.

Indeed, for $f = g\circ T - g$,

$$\frac{1}{N} S_N(f) = \frac{g\circ T^N - g}{N} \xrightarrow[N\to\infty]{} 0.$$

$\square$