Lecture 5

Exponential mixing and Khinchin-Groshev Thm

Let $\psi : [1,\infty) \to (0,1)$ be a continuous nonincreasing function.

Def. $x \in \mathbb{R}^d$ is called <u>ψ-approximable</u> if

$$\begin{cases} \|x - \frac{p}{q}\| \le \frac{\psi(q)}{q}, \\ p \in \mathbb{Z}^d,\ q \in \mathbb{N}. \end{cases}$$

has infinitely many solutions.

$W_d(\psi) = \{$set of ψ-approx. vectors in $\mathbb{R}^d\}$.

Thm (Khinchin – Groshev)

1) $\sum_{q \ge 1} \psi(q)^d < \infty \Rightarrow W_d(\psi)$ has measure zero. $\}$ easy

2) $\sum_{q \ge 1} \psi(q)^d = \infty \Rightarrow W_d(\psi)$ has full measure.

ex.
- every $x \in \mathbb{R}^d$ is $x^{-1/d}$ – approximable (Dirichlet)
- a.e. $x \in \mathbb{R}^d$ is $(x \log x)^{-1/d}$ – approximable, but not $(x (\log x)^{1+\varepsilon})^{-1/d}$ – approximable (Khinchin).
- many $x \in \mathbb{R}^d$ are not $(x \log x)^{-1/d}$ – approximable (e.g., badly approximable numbers).

$$W_d(\psi) = \underbrace{\{x \in \mathbb{R}^d : x \in B(\tfrac{p}{q}, \tfrac{\psi(q)}{q}) \text{ infinitely often}\}}_{\overline{\lim}\, B(\frac{p}{q}, \frac{\psi(q)}{q})}$$

Borel-Cantelli Lemma $A_n \subset (X,\mu)$ – probability space.

1) $\sum_{n\geq 1} \mu(A_n) < \infty \Rightarrow \overline{\lim} A_n$ has measure zero.

2) $\left.\begin{array}{l} \sum_{n\geq 1} \mu(A_n) = \infty, \\ \mu(A_{n_1} \cap \ldots \cap A_{n_k}) = \mu(A_{n_1}) \ldots \mu(A_{n_k}) \\ \text{for all } n_1 < \ldots < n_k \end{array}\right] \Rightarrow \overline{\lim} A_n$ has full measure.

Lem. 1 Given $\overbrace{\text{functions}}^{\text{nonnegative}}$ φ_n on (X,μ) such that

$$\sum_{n\geq 1} \int_X \varphi_n \, d\mu = \infty,$$

$$\sum_{m,n=1}^{N} \int_X \varphi_m \varphi_n \, d\mu = \left(\sum_{n=1}^{N} \int_X \varphi_n \, d\mu\right)^2 + C \cdot \left(\sum_{n=1}^{N} \int_X \varphi_n \, d\mu\right),$$

we have $\sum_{n\geq 1} \varphi_n(x) = \infty$ for a.e. $x \in X$.

Let $S_N(x) = \sum_{n=1}^{N} \varphi_n(x)$ and $E_N = \sum_{n=1}^{N} \int_X \varphi_n \, d\mu \longrightarrow \infty$.

Then $\| S_N(x) - E_N \|_2^2 = \left\langle \sum_{n=1}^{N} \left(\varphi_n - \int_X \varphi_n\right), \sum_{m=1}^{N} \left(\varphi_m - \int_X \varphi_m\right) \right\rangle$

$$= \sum_{n,m=1}^{N} \int_X \varphi_n \varphi_m \, d\mu - \left(\sum_{n=1}^{N} \int_X \varphi_n\right)^2 \leq C \cdot E_N.$$

Hence, $\left\| \frac{S_N}{E_N} - 1 \right\|_2 \leq \frac{C}{E_N} \xrightarrow[N\to\infty]{} 0$, and $\frac{S_N}{E_N} \to 1$ in L^2.

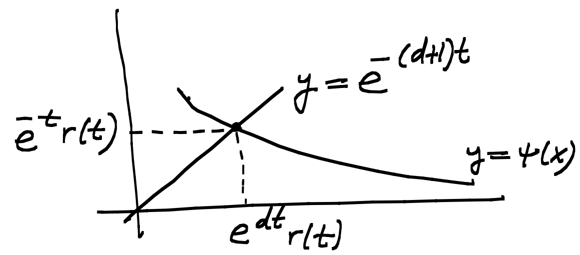
Fact: If $F_N \to F$ in L^2, then $\exists$ subsequence N_i : $F_{N_i}(x) \xrightarrow{a.e.} F(x)$.

Hence, $S_{N_i}(x) \xrightarrow{a.e.} \infty$ and $S_N(x) \xrightarrow{a.e.} \infty$ because $S_N(x)$ is monotone.

Lem 2. Given a continuous nonincreasing function $\psi : [1,\infty) \to (0,1)$ there exists unique $r : [t_0, \infty) \to (0,1)$ such that:

- $t \mapsto e^{dt} r(t)$ is strictly increasing,
- $t \mapsto e^{-t} r(t)$ is nonincreasing,
- $\psi(e^{dt} r(t)) = e^{-t} r(t)$
- $\sum_{q \geq 1} \psi(q)^d = \infty \iff \sum_{n \geq t_0} r(n)^{d+1} = \infty$

Proof of the Lemma follows from the picture:

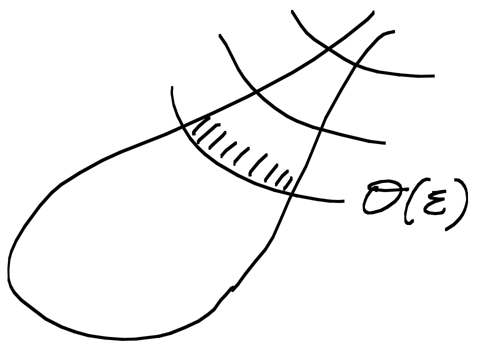


Notation: $\Delta : \mathcal{L}^1_{d+1} \to (0,\infty) : \Delta(\Lambda) = \min_{0 \neq v \in \Lambda} \|v\|.$

$$\Theta(\varepsilon) = \{\Lambda \in \mathcal{L}^1_{d+1} : \Delta(\Lambda) \leq \varepsilon\}.$$

By Mahler compactness criterion,

$$\Lambda_n \to \infty \text{ in } \mathcal{L}^1_{d+1} \iff \Delta(\Lambda_n) \to 0.$$



Hence, $\Theta(\varepsilon)$, $\varepsilon > 0$, gives a basis of nbhds of ∞.

Recall: $\Lambda_x = \mathbb{Z}^{d+1} \left(\begin{array}{c|c} Id & 0 \\ \hline x & 1 \end{array} \right)$, $a_t = \left(\begin{array}{c|c} e^t \cdot Id & 0 \\ \hline 0 & e^{-dt} \end{array} \right)$

Lem.3 $x \in \mathbb{R}^d$ is ψ-approximable $\Leftrightarrow$ $\Lambda_x a_{t_n} \in \mathcal{O}(r(t_n))$ for a sequence $t_n \to \infty$.

$\Rightarrow$ Suppose that $\|x - \frac{p_n}{q_n}\| \leq \frac{\psi(q_n)}{q_n}$ for $p_n \in \mathbb{Z}^d$, $q_n \in \mathbb{N}$, $q_n \to \infty$.

We pick t_n such that $q_n = e^{dt_n} r(t_n)$.

Then $t_n \to \infty$, and $\psi(q_n) = e^{-t_n} \cdot r(t_n)$, so that

$$\max\{ e^{t_n} \| q_n x - p_n \|, e^{-dt_n} |q_n| \} \leq r(t_n).$$

This shows that $\Delta(\Lambda_x a_{t_n}) \leq r(t_n)$ and $\Lambda_x a_{t_n} \in \mathcal{O}(r(t_n))$.

$\Leftarrow$ Similar. $\square$

Lem.4 $\mathrm{vol}(\mathcal{O}(\varepsilon)) \asymp \varepsilon^{d+1}$ (up to a constant) as $\varepsilon \to 0^+$.

Recall that $\mathcal{L}'_{d+1} = \mathbb{Z}^{d+1} \cdot \Sigma_{t,v}$ where $\Sigma_{t,v} = \mathcal{U}_v A_t K$ is the Siegel set.

We also use that the map
$$\Sigma_{t,v} \longrightarrow \mathcal{L}'_{d+1}$$
$$g \longmapsto \mathbb{Z}^{d+1} g$$
is finite-to-one.

For $g = uak \in \Sigma_{t,v}$, $\Delta(\mathbb{Z}^{d+1} uak) = \Delta(\mathbb{Z}^{d+1} a \cdot (a^{-1} u a) k)$,

$$a^{-1} u a = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \frac{a_j}{a_i} u_{ij} & & 1 \end{pmatrix}.$$

Since $\frac{a_i}{a_{i+1}} \leq t$, $|u_{ij}| \leq v$, $(a^{-1} u a) k$ is uniformly bounded.

Hence, $\Delta(\mathbb{Z}^{d+1} g) \asymp \Delta(\mathbb{Z}^{d+1} a) = \min\{a_i\}$

$$= a_1 \cdot \min\left\{\frac{a_i}{a_1}\right\} \asymp a_1.$$

We have $\mathrm{vol}(\mathcal{O}(\varepsilon)) \asymp \mathrm{vol}(\{ uak \in \Sigma_{t,v} : a_1 \leq \varepsilon \})$.

Recall that invariant measure on $SL_d(\mathbb{R})$:

$$\int_{U\times A\times K} f(uak)\cdot \prod_{i=1}^{d-1} b_i^{\,i(d-i)} \Big(\prod_{i>j} du_{ij}\Big)\Big(\prod_{i=1}^{d-1}\frac{db_i}{b_i}\Big)\, d\nu_1(k),$$

where $b_i = \frac{a_i}{a_{i+1}}$. Note that $a_1 = \prod_{i=1}^{d-1} b_i^{(d-i)/d}$.

Now $\mathrm{vol}(\Theta_\varepsilon) \asymp \displaystyle\int_{\substack{b_i \le t \\ \prod_i b_i^{(d-i)/d} \le \varepsilon}} \Big(\prod_{i=1}^{d-1} b_i^{\,i(d-i)-1}\Big)\, db_1 \ldots db_{d-1}$

$= \text{computation} \ldots \asymp \varepsilon^{d+1}$. $\square$

<u>Thm</u> (quantitative Howe – Moore) $\exists \delta>0$:
for all smooth functions $f_1, f_2 : \mathcal{L}'_{d+1} \to \mathbb{C}$ with $\int_{\mathcal{L}'_{d+1}} f_i\, d\mu = 0$,

$$|\langle V_{a_t} f_1, f_2\rangle| \le c\cdot S(f_1) S(f_2)\cdot e^{-\delta t}.$$

<u>Proof of Khinchin – Groshev Thm (2)</u>

Let $r(t)$ be as Lem. 2. Since $\sum_{q\ge 1} \psi(q)^d = \infty$,
we have $\sum_{n\ge t_0} r(n) = \infty$, and by Lem. 4, $\sum_{n\ge t_0} \mathrm{vol}(\Theta(r(n))) = \infty$.
By Lem. 3, it is sufficient to show that
for a.e. $x \in \mathbb{R}^d$, $\Lambda_x a_n \in \Theta(r(n))$ infinitely often.

We "approximate" the characteristic function of $\Theta(r(n))$
by nonnegative smooth functions f_n so that:
$\{f_n \ne 0\} \subset \Theta(r(n))$, $\int_{\mathcal{L}'_{d+1}} f_n \asymp \mathrm{vol}(\Theta(r(n)))$, $S(f_n) \le \text{const}\cdot \int_{\mathcal{L}'_{d+1}} f_n$.

Then $\sum_{n\geq 1}\left(\int_{\mathcal{L}'_{d+1}} f_n\right) = \infty$, and we also claim that

$$\sum_{n,m=1}^{N} \langle U_{a_n} f_n , U_{a_m} f_m \rangle \leq \left(\sum_{n=1}^{N} \int_{\mathcal{L}'_{d+1}} f_n\right)^2 + C\cdot\left(\sum_{n=1}^{N}\int_{\mathcal{L}'_{d+1}} f_n\right).$$

Let $\tilde{f}_n = f_n - \int_{\mathcal{L}'_{d+1}} f_n$. Then

$$\sum_{n,m=1}^{N}\langle U_{a_n}f_n, U_{a_m}f_m\rangle - \left(\sum_{n=1}^{N}\int_{\mathcal{L}'_{d+1}} f_n\right)^2 = \sum_{n,m=1}^{N}\langle U_{a_n}\tilde{f}_n , U_{a_m}\tilde{f}_m\rangle$$

$$= \sum_{n,m=1}^{N}\langle U_{a_{n-m}}\tilde{f}_n , \tilde{f}_m\rangle \ll \sum_{n,m=1}^{N} e^{-\delta|n-m|} S(\tilde{f}_n) S(\tilde{f}_m)$$

$\uparrow$ quantitative Howe–Moore Thm.

$$\ll \sum_{1\leq m\leq n\leq N} e^{-\delta(n-m)}\left(\int_{\mathcal{L}'_{d+1}} f_n\right)\left(\int_{\mathcal{L}'_{d+1}} f_m\right)$$

$$\ll \sum_{1\leq m\leq n\leq N} e^{-\delta(n-m)}\left(\int_{\mathcal{L}'_{d+1}} f_n\right) = \sum_{n=1}^{N}\underbrace{\left(\sum_{m=1}^{n} e^{-\delta(n-m)}\right)}_{\text{uniformly bounded}}\left(\int_{\mathcal{L}'_{d+1}} f_n\right).$$

$\ll \sum_{n=1}^{N}\int_{\mathcal{L}'_{d+1}} f_n$, which proves the claim.

Now we apply Lem. 1 to $\varphi_n = U_{a_n} f_n$.

It follows that for a.e. $\Lambda \in \mathcal{L}'_{d+1}$,

$$\sum_{n\geq 1}\varphi_n(\Lambda) = \sum_{n\geq 1} f_n(\Lambda\cdot a_n) = \infty.$$

Since $\{f_n \neq 0\} \subset \mathcal{O}(r(n))$, this shows that $\Lambda a_n \in \mathcal{O}(r(n))$ infinitely often.

We have $\mathbb{Z}^{d+1} g a_n \in \mathcal{O}(r(n))$ inf. often, for a.e. $g \in SL_{d+1}(\mathbb{R})$.

For $g = \left(\begin{array}{c|c} Id & 0 \\ \hline x & 1\end{array}\right)\left(\begin{array}{c|c} A & b \\ \hline 0 & c\end{array}\right)$, we have

$$g a_n = \left(\begin{array}{c|c} Id & 0 \\ \hline x & 1 \end{array}\right) a_n \cdot \underbrace{\left(\begin{array}{c|c} A & e^{-(d+1)n} b \\ \hline & c \end{array}\right)}_{\text{uniformly bounded.}}$$

Hence, $\exists \alpha > 0 : \Lambda_x a_n \in \mathcal{O}(\alpha \cdot r(n))$ infinitely often.

We conclude that for a.e. $x \in \mathbb{R}^d$, $\Lambda_x a_n \in \mathcal{O}(\alpha r(n))$ infinitely often.

Applying above argument to $\alpha^{-1} \cdot r(t)$,
we deduce that for a.e. $x \in \mathbb{R}^d$, $\Lambda_x a_n \in \mathcal{O}(r(n))$ infinitely often.