Lecture 4

Diophantine approximation & flows.

For $x \in \mathbb{R}^d$, we would like to find a rational approximation:

$$x \simeq \frac{p}{q} \in \mathbb{Q}^d.$$

Thm (Dirichlet) $\forall x \in \mathbb{R}^d\ \forall R > 1\ \exists p \in \mathbb{Z}^d, q \in \mathbb{N}$:

$$\begin{cases} \|x - \frac{p}{q}\| \leq \frac{R^{-1/d}}{q} \\ |q| \leq R. \end{cases}$$

Can one improve Dirichlet's thm?

ex. ($x = \sqrt{2}$) We know that $|\sqrt{2} - \frac{p}{q}| \leq \frac{1}{q^2}$ has infinitely many solutions. Suppose that for some $c \in (0,1)$, $|\sqrt{2} - \frac{p}{q}| \leq \frac{c}{q^2}$ also has infinitely many solutions.

Then

$$\frac{1}{q^2} \leq \frac{|2q^2 - p^2|}{q^2} = |\sqrt{2} - \frac{p}{q}| \cdot |\sqrt{2} + \frac{p}{q}| \leq \frac{c}{q^2} \cdot \left(\frac{c}{q^2} + 2\sqrt{2}\right).$$

Hence, $1 \leq c \cdot (\frac{c}{q^2} + 2\sqrt{2})$, and taking $q \to \infty$, we conclude that $c \geq \frac{1}{2\sqrt{2}}$.

Def $x \in \mathbb{R}^d$ is called badly approximable if $\exists c > 0$: $\forall p \in \mathbb{Z}^d\ \forall q \in \mathbb{N}$: $\|x - \frac{p}{q}\| > \frac{c}{q^{1+\frac{1}{d}}}$.

Open question: Is $\sqrt[3]{2}$ badly approximable?

Topology on the space of lattices:

We say that $\Lambda_n \to \Lambda$ for lattices $\Lambda_n, \Lambda \subset \mathbb{R}^d$ if for some (every) basis $\{v_n^{(i)}\}$ of Λ_n, $\{v_n^{(i)}\}$ converges to a basis of Λ.

Rmk: The space $\mathcal{L}_d^1$ of unimodular lattices is not compact (e.g., $\Lambda_n = \langle \frac{1}{n} e_1, n \cdot e_2, e_3, \dots e_d \rangle$ has no convergent subsequence).

Mahler compactness criterion:

$\Omega \subset \mathcal{L}_d^1$ is precompact (bounded) $\Leftrightarrow$ $\exists \delta > 0 : \forall \Lambda \in \Omega \ \forall v \in \Lambda \setminus \{0\} : \|v\| \geq \delta$.

$\Rightarrow$ Suppose that Ω is precompact, but $\exists \Lambda_n \in \Omega$, $\exists x_n \in \Lambda_n \setminus \{0\} : x_n \to 0$

After passing to a subsequence, we may assume that $\Lambda_n \to \Lambda$. Let $\Lambda_n = \langle v_n^{(i)} \rangle$ and $\Lambda = \langle v^{(i)} \rangle$ with $v_n^{(i)} \to v^{(i)}$.

Then $x_n = \sum_i \underbrace{\alpha_n^{(i)}}_{\in \mathbb{Z}} \cdot v_n^{(i)} = \sum_i \underbrace{\beta_n^{(i)}} v^{(i)}$ and $\beta_n^{(i)} \to 0$.

This implies that $\alpha_n^{(i)} \to 0$, and since $\alpha_n^{(i)} \in \mathbb{Z}$, $\alpha_n^{(i)} = 0$. contradiction

$\Leftarrow$ Recall that $SL_d(\mathbb{R}) = SL_d(\mathbb{Z}) \cdot \Sigma_{t,v}$ where $\Sigma_{t,v} = U_v A_t K$ is a Siegel set.

Hence, $\mathcal{L}_d^1 = \mathbb{Z}^d \cdot \Sigma_{t,v}$.

Let $\Omega \subset \mathcal{L}_d^1$ be such that $\forall \Lambda \in \Omega \ \forall x \in \Lambda \setminus \{0\} : \|x\| \geq \delta > 0$.

Then $\Omega = \mathbb{Z}^d \cdot \Sigma$ for some $\Sigma \subset \Sigma_{t,v}$.

For $g \in \Sigma$, $\frac{a(g)_i}{a(g)_{i+1}} \leq t$ and $\|e_1 g\| = a(g)_1 \geq \delta$.

Hence, $a(g)_i \geq t^{-1} \cdot a(g)_{i-1} \geq \ldots \geq t^{-(i-1)} \cdot \delta$.

On the other hand, $\det(g) = a(g)_1 \ldots a(g)_d = 1$,
and each $a(g)_i$ is uniformly bounded from above.
We have shown that $\exists c, C > 0 : \forall g \in \Sigma : c \leq a(g)_i \leq C$,
so that Σ is bounded in $SL_d(\mathbb{R})$, and hence precompact.

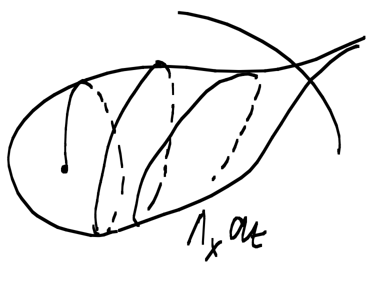
$\lrcorner$

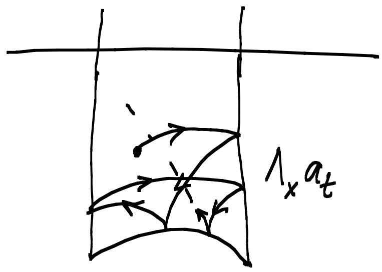
Dani correspondence.

For $x \in \mathbb{R}^d$, set $u(x) = \left(\begin{array}{c|c} Id & 0 \\ \hline x & 1 \end{array}\right)$ and $\Lambda_x = \mathbb{Z}^{d} \cdot u(x) \in \mathcal{L}'_{d+1}$.

Consider the flow $a_t : \mathcal{L}'_{d+1} \longrightarrow \mathcal{L}'_{d+1}$, $\Lambda \longmapsto \Lambda \cdot a_t$, where $a_t = \left(\begin{array}{c|c} e^t Id & 0 \\ \hline 0 & e^{-dt} \end{array}\right)$

Thm (Dani) For every $x \in \mathbb{R}^d$,
x is badly approximable $\Leftrightarrow$ the orbit $\{\Lambda_x a_t\}_{t \geq 0}$ is bounded in $\mathcal{L}'_{d+1}$.





$\ulcorner$ Note that $\Lambda_x = \mathbb{Z}^{d+1} u(x) = \{(p + qx, q) : p \in \mathbb{Z}^d, q \in \mathbb{Z}\}$.
By Mahler compactness criterion,
$\{\Lambda_x a_t\}_{t \geq 0}$ is bounded in $\mathcal{L}'_{d+1} \Leftrightarrow \exists \delta \in (0,1) : \max\{e^t \|p + qx\|, e^{-dt} |q|\} \geq \delta$ for all $(p,q) \in \mathbb{Z}^{d+1} \setminus \{0\}$ and $t \geq 0$.

$$\Updownarrow$$

$\exists \delta \in (0,1) : \max\{e^t \|p - qx\|, e^{-dt} |q|\} \geq \delta$
for all $p \in \mathbb{Z}^d$, $q \in \mathbb{N}$, $t \geq 0$.

Suppose that x is badly approximable, i.e.,

$\|x - \frac{p}{q}\| \geq \frac{c}{q^{1+\frac{1}{d}}}$ for all $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$.

Then $q^{1/d} \cdot \|p - qx\| = \left(e^{-dt} q\right)^{1/d} \cdot \left(e^t \cdot \|p - qx\|\right) \geq c$,

and $\max\{ e^t \|p - qx\|, e^{-dt} q \} \geq c^{(1+\frac{1}{d})^{-1}}$.

Hence, $\{\Lambda_x a_t\}_{t \geq 0}$ is bounded in $\mathcal{L}'_{d+1}$.

Conversely, suppose that $\{\Lambda_x a_t\}_{t\geq 0}$ is bounded in $\mathcal{L}'_{d+1}$, i.e.

for some $\delta \in (0,1)$: $\max\{ e^t \|p-qx\|, e^{-dt} q \} \geq \delta$

for all $p \in \mathbb{Z}^d$, $q \in \mathbb{N}$, $t \geq 0$.

We pick $t > 0$ such that $e^{-dt} q = \delta/2$.

Then $\left(\frac{2}{\delta} q\right)^{1/d} \cdot \|p - qx\| \geq \delta$ for all $p \in \mathbb{Z}^d$, $q \in \mathbb{N}$.

$$\Updownarrow$$

$$\|x - \tfrac{p}{q}\| \geq \frac{\delta \cdot (\delta/2)^{1/d}}{q^{1/d}} \Rightarrow x \text{ is badly approximable.}$$

<u>Cor.</u> The set of badly approximable vectors in $\mathbb{R}^d$ has measure zero.

It follows from Howe–Moore Thm that for a set of full measure in $SL_{d+1}(\mathbb{R})$, the orbit $\{\mathbb{Z}^{d+1} g a_t\}_{t \geq 0}$ is dense in $\mathcal{L}'_{d+1}$ (exercise).

In particular, the set $\{g \in SL_{d+1}(\mathbb{R}) : \{\mathbb{Z}^{d+1} g \cdot a_t\}_{t\geq 0} \text{ is bounded}\}$ has measure zero.

Now $\{g \in SL_d(\mathbb{R}) : \underbrace{M_{dd}}_{(d-1)\text{-minor}} \neq 0\} = \underbrace{\left(\begin{array}{c|c} Id & 0 \\ \hline \mathbb{R}^d & 1 \end{array}\right)}_{U} \cdot \underbrace{\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)}_{B}$

Invariant measure on $SL_{d+1}(\mathbb{R})$:

$$\int_{SL_{d+1}(\mathbb{R})} f(g)\, d\mu(g) = \int_{\mathbb{R}^d \times B} f(u(x)b)\, \underbrace{dx}\, d\rho(b)$$

Lebesgue measure on $\mathbb{R}^d$!

We have $\mathbb{Z}^{d+1} u(x) \cdot b\, a_t = (\mathbb{Z}^{d+1} u(x)\, a_t) \cdot (a_t^{-1} b\, a_t)$.

$$a_t^{-1} \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline 0 & B_{22} \end{array} \right) a_t = \left(\begin{array}{c|c} B_{11} & e^{-(d+1)t} B_{12} \\ \hline 0 & B_{22} \end{array} \right) \text{ - bounded for } t \geq 0$$

Hence, $\{\mathbb{Z}^{d+1} u(x) b a_t\}_{t \geq 0}$ is bounded $\Leftrightarrow$ $\{\mathbb{Z}^{d+1} u(x) a_t\}_{t \geq 0}$ is bounded,

and $\{ g \in u(\mathbb{R}^d) B : \{\mathbb{Z}^{d+1} g\, a_t\}_{t \geq 0} \text{ is bounded} \}$

$$\|$$

$\{ u(x) : \{\mathbb{Z}^{d+1} u(x)\, a_t\}_{t \geq 0} \text{ is bounded} \} \cdot B$

$\| \longleftarrow$ Dani's Thm.

$\{ u(x) : x \in \mathbb{R}^d \text{ - badly approximable} \} \cdot B$.

Since this set has measure zero, this implies the claim.

$\square$