$\boxed{\text{Lecture 3}}$

Mixing on the space of lattices.

$\left.\begin{array}{l}\mathcal{L}_d' \simeq SL_d(\mathbb{Z})\backslash SL_d(\mathbb{R}) \\ \text{equipped with finite} \\ \text{invariant measure}\end{array}\right] \longrightarrow$ a dynamical system:

For $g \in SL_d(\mathbb{R})$,

$$\begin{array}{ccc} x & \longrightarrow & x \cdot g \\ \Psi & & \Psi \\ \mathcal{L}_d' & \longmapsto & \mathcal{L}_d' \end{array}$$

$$\mathcal{H} = L^2(\mathcal{L}_d') = \left\{ \varphi : \mathcal{L}_d' \to \mathbb{C} : \int_{\mathcal{L}_d'} |\varphi|^2 d\mu < \infty \right\}.$$

Scalar product: for $\varphi_1, \varphi_2 \in L^2(\mathcal{L}_d')$:

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathcal{L}_d'} \varphi_1 \cdot \overline{\varphi_2} \, d\mu$$

(Then $\mathcal{H}$ is a Hilbert space.)

For $g \in SL_d(\mathbb{R})$, define a linear map:

$$U_g : \mathcal{H} \longrightarrow \mathcal{H} : \varphi \longmapsto \varphi(x \cdot g)$$

Properties:
- $U_{g_1 \cdot g_2} = U_{g_1} \cdot U_{g_2}$ for $g_1, g_2 \in SL_d(\mathbb{R})$
- $\|U_g \varphi\| = \|\varphi\|$ for $g \in SL_d(\mathbb{R})$, $\varphi \in \mathcal{H}$ ($\Leftarrow$ invariance of the measure)

__Thm__ (Howe-Moore) $\forall \varphi, \psi \in \mathcal{H}$:

$$\langle U_g \varphi, \psi \rangle \xrightarrow[\|g\| \to \infty]{} \left(\int_{\mathcal{L}_d'} \varphi \right) \cdot \left(\int_{\mathcal{L}_d'} \overline{\psi} \right)$$

(i.e., states $U_g \varphi$ and ψ become "asymptotically independent").

Lem (Cartan decomposition)

$$SL_d(\mathbb{R}) = K \cdot A^+ \cdot K \quad \text{where} \quad K = SO_d(\mathbb{R}),$$
$$A^+ = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix} : a_1 \geq a_2 \geq \dots \geq a_d > 0 \right\}.$$

This corresponds to the fact from linear algebra: every quadratic form can be reduced to diagonal shape using orthogonal transformation.

Proof of Thm.

Step 1: Cartan decomposition.

Let $\mathcal{H}_0 = \{\varphi : \int_{\mathcal{L}_d'} \varphi \, d\mu = 0\}$.

$$\varphi = \underbrace{\left(\varphi - \int_{\mathcal{L}_d'} \varphi \, d\mu\right)}_{\varphi_1 \in \mathcal{H}_0} + \left(\int_{\mathcal{L}_d'} \varphi \, d\mu\right).$$

$$\langle U_g \varphi, \psi \rangle = \langle U_g \varphi_1, \psi_1 \rangle + \left(\int_{\mathcal{L}_d'} \varphi\right)\left(\int_{\mathcal{L}_d'} \overline{\psi}\right).$$

It will be sufficient to show that

$$\langle U_g \varphi, \psi \rangle \xrightarrow[\|g\| \to \infty]{} 0 \quad \text{for all } \varphi, \psi \in \mathcal{H}_0.$$

Suppose that for some $\varphi, \psi \in \mathcal{H}_0$ and $g_n : \|g_n\| \to \infty$, we have $\langle U_{g_n} \varphi, \psi \rangle \not\to 0$.

By Cartan decomposition, $g_n = k_n a_n \ell_n$ with $k_n, \ell_n \in K$ and $a_n \in A^+$, $\|a_n\| \to \infty$.

Since $K = SO_d(\mathbb{R})$ is compact, passing to a subsequence, we may assume that $U_{k_n^{-1}} \psi \longrightarrow \tilde{\psi} \in \mathcal{H}_0$.

$U_{\ell_n} \varphi \longrightarrow \tilde{\varphi} \in \mathcal{H}_0$

Then $\langle U_{k_n a_n \ell_n} \varphi, \psi \rangle - \langle U_{a_n} \tilde{\varphi}, \tilde{\psi} \rangle$

$$= \langle U_{a_n}(U_{\ell_n}\varphi), U_{k_n}^{-1}\psi \rangle - \langle U_{a_n}\tilde{\varphi}, \tilde{\psi} \rangle$$

$$= \langle U_{a_n} (\underbrace{U_{\ell_n}\varphi - \tilde{\varphi}}_{\to 0}), U_{k_n}^{-1}\psi \rangle + \langle U_{a_n}\tilde{\varphi}, \underbrace{U_{k_n}^{-1}\psi - \tilde{\psi}}_{\to 0} \rangle$$

By Cauchy-Schwartz inequality, this expression $\longrightarrow 0$ (here we used that $\|U_g\varphi\| = \|\varphi\|$).
Hence, we conclude that $\langle U_{a_n}\tilde{\varphi}, \tilde{\psi} \rangle \not\to 0$.

Weak convergence: a sequence $v_n \in \mathcal{H}$ converges to v weakly if $\langle v_n, w \rangle \to \langle v, w \rangle$ for all $w \in \mathcal{H}$.

ex. $\mathcal{H} = L^2(S^1)$, $\varphi_n = z^n \xrightarrow[n\to\infty]{} 0$ weakly, but not L^2.

Alaoglu Thm: Closed bounded sets in $\mathcal{H}$ are compact in weak topology.

Step 2: the case of $SL_2(\mathbb{R})$.

Consider $a_n = \begin{pmatrix} t_n & 0 \\ 0 & t_n \end{pmatrix}$ with $t_n \to +\infty$ and suppose that
$$\langle U_{a_n}\varphi, \psi \rangle \not\to 0 \text{ for some } \varphi, \psi \in \mathcal{H}_0.$$
Since $\|U_{a_n}\varphi\| = \|\varphi\|$, by Alaoglu Thm, we can pass to a subsequence and arange that $U_{a_n}\varphi \to \tilde{\varphi} \in \mathcal{H}_0$ weakly (i.e., $\langle U_{a_n}\varphi, \psi \rangle \to \langle \tilde{\varphi}, \psi \rangle$ for all $\psi \in \mathcal{H}_0$).
Note that $\tilde{\varphi} \neq 0$.

Let $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. Note that $a_n^{-1} u_s a_n = u_{s/t_n^2} \longrightarrow Id$.

$$U_{u_s} \tilde{\varphi} = \operatorname*{w-lim}_{n\to\infty} U_{u_s} U_{a_n} \varphi = \operatorname*{w-lim}_{n\to\infty} U_{a_n} \cdot U_{u_{s/t_n^2}} \cdot \varphi$$

(Since $\| U_{a_n} \cdot U_{u_{s/t_n^2}} \varphi - U_{a_n} \varphi \| = \| U_{u_{s/t_n^2}} \varphi - \varphi \| \longrightarrow 0$

$$= \operatorname*{w-lim}_{n\to\infty} U_{a_n} \varphi = \tilde{\varphi}$$

Hence, $\tilde{\varphi}$ is fixed by $N = \{u_s\}$.

Consider the function $F(g) = \langle U_g \tilde{\varphi}, \tilde{\varphi} \rangle$, $g \in SL_2(\mathbb{R})$.

Since $\tilde{\varphi}$ is N-invariant, $F(NgN) = F(g)$.

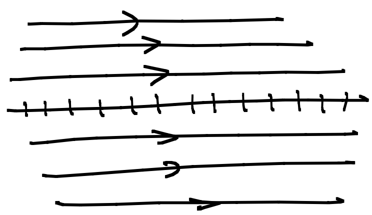
We use identification: $SL_2(\mathbb{R})/N \simeq \mathbb{R}^2 \setminus \{0\}$

(indeed, $SL_2(\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$ and $\mathrm{Stab}(e_1) = N$).

Now F is N-inv. function on $\mathbb{R}^2 \setminus \{0\}$.

Orbits of N: $\begin{cases} \text{lines } \{y = c\},\ c \neq 0, \\ \text{points } \{(x, 0)\}. \end{cases}$

$\mathbb{R}^2$



Hence, $F = $ const on $y = c$, $c \neq 0$, and by continuity, $\varphi = $ const on $y = 0$.

From this we obtain: $\langle U_{a_t} \tilde{\varphi}, \tilde{\varphi} \rangle = F(a_t \cdot e_1) = F(t e_1) = F(e_1) = \|\tilde{\varphi}\|^2$

This gives equality in the Cauchy-Schwartz inequality, so that $U_{a_t} \tilde{\varphi} = \tilde{\varphi}$.

We conclude that F is (AN)-biinvariant.

Since $(AN) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (AN)$ is dense in $SL_2(\mathbb{R})$, $F = $ const.

Hence, $\langle U_g \tilde{\varphi}, \tilde{\varphi} \rangle = \|\tilde{\varphi}\|^2$ (equality in Cauchy-Schwartz inequality), and $U_g \tilde{\varphi} = \tilde{\varphi}$ for all $g \in SL_2(\mathbb{R})$.

Finally, $\tilde{\varphi} = $ const, and $\tilde{\varphi} = 0$ because $\int_{\mathcal{L}_d^1} \varphi \, d\mu = 0$.

Step 3: general case.

Let $\alpha_i(a) = \frac{a_i}{a_{i+1}}$ for $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix} \in A^+$.

If $\|a_n\| \to \infty$, then after passing to a subsequence,
$\alpha_i(a_n) \to \infty$ for some i.

(This is because the set
$$\begin{cases} a_1 \le t a_2 \le \dots \le t a_d \\ a_1 \ge \dots \ge a_d > 0 \\ a_1 \dots a_d = 1 \end{cases}$$
is bounded.)

We argue as in $SL_2(\mathbb{R})$-case.

Suppose that $\langle U_{a_n}\varphi, \varphi\rangle \not\to 0$ and $U_{a_n}\varphi \to \tilde{\varphi}$ weakly.
$\alpha_i(a_n) \to \infty$

Let $N_i = \left(\begin{array}{c|c} id & * \\ \hline 0 & id \end{array}\right)\}i$. We have
$$a^{-1}\left(\begin{array}{c|c} id & s_{ek} \\ \hline 0 & id \end{array}\right)a = \left(\begin{array}{c|c} id & \frac{a_k}{a_e}\cdot s_{ek} \\ \hline 0 & id \end{array}\right), \quad \frac{a_k}{a_e} \le \frac{a_{i+1}}{a_i}$$

As in SL_2-case, we conclude that $N_i \cdot \tilde{\varphi} = \tilde{\varphi}$.

Let $G_{ek} = \begin{array}{c} \\ \ell \\ k \end{array}\!\!\begin{array}{c} \begin{array}{cc} \ell & k \end{array} \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{array} \simeq SL_2(\mathbb{R})$

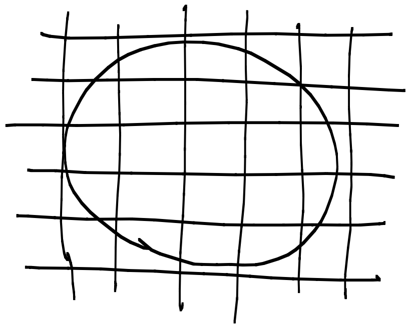
Next, we get that $\tilde{\varphi}$ is $G_{\ell k}$-invariant for $1 \le \ell \le i$ and $i+1 \le k \le d$.

Since these subgroups generate $SL_d(\mathbb{R})$, φ is $SL_d(\mathbb{R})$-invariant, which gives a contradiction. $\square$

Counting integral points.

Gauss circle problem:

$$\#\{(x,y)\in\mathbb{Z}^2 : x^2+y^2\le T^2\} = \pi T^2 + O(T)$$
as $T\to\infty$



(this is elementary...)

What is the best error term?

Conj: Error term $= O_\varepsilon(T^{\frac{1}{2}+\varepsilon})$ for every $\varepsilon>0$.

More generally, $f:\mathbb{R}^d\to\mathbb{R}$ – rational polynomial.

$$\#\{x\in\mathbb{Z}^d : f(x)=n,\ \|x\|<T\} \underset{T\to\infty}{\sim} ?$$

Thm (Duke–Rudnick–Sarnak)

$$\#\{x\in \mathrm{Mat}(d,\mathbb{Z}) : \det(x)=n,\ \|x\|\le T\} \underset{T\to\infty}{\sim} c(n,d)\cdot T^{d^2-d}.$$

For simplicity, let $n=1$.

Let $B_T=\{g\in SL(d,\mathbb{R}) : \|g\|\le T\}$.

We expect that $\#(SL(d,\mathbb{Z})\cap B_T)\sim \mathrm{vol}(B_T)$, where vol is the invariant volume normalised so that $\mathrm{vol}(SL_d(\mathbb{Z})\backslash SL_d(\mathbb{R}))=1$.

Let $\mathcal{O}_\varepsilon=\{g\in SL_d(\mathbb{R}) : \|g-I\|\le\varepsilon\}$

Lem: 1) $\mathcal{O}_\varepsilon B_T \mathcal{O}_\varepsilon \subset B_{(1+2\varepsilon)T}$ (easy: triangle inequality).

2) $vol(B_T) \sim c \cdot T^{d^2-d}$ as $T \to \infty$

Notation: $G = SL_d(\mathbb{R})$, $\Gamma = SL_d(\mathbb{Z})$.

Consider a function

$$F_T(g_1, g_2) = \sum_{\gamma \in \Gamma} \chi_{B_T}(g_1^{-1} \gamma g_2) \quad \text{for } g_1, g_2 \in G.$$

Note that $F_T(\gamma_1 g_1, \gamma_2 g_2) = F(g_1, g_2)$, so that

$$F_T : \Gamma\backslash G \times \Gamma\backslash G \longrightarrow \mathbb{R}.$$

We need to show that $F_T(e,e) \underset{T\to\infty}{\sim} vol(B_T)$.

Step 1: weak convergence

Let $\varphi_1, \varphi_2 : \Gamma\backslash G \to \mathbb{R}$ and $\varphi = \varphi_1 \otimes \varphi_2 : \Gamma\backslash G \times \Gamma\backslash G \longrightarrow \mathbb{R}$.

We claim that $\langle F_T, \varphi\rangle = vol(B_T) \cdot \int_{\Gamma\backslash G \times \Gamma\backslash G} \varphi + o(vol(B_T))$,

where $\langle F_T, \varphi\rangle = \int_{\Gamma\backslash G \times \Gamma\backslash G} F_T(g_1, g_2) \overline{\varphi_1(g_1)} \overline{\varphi_2(g_2)} \, d\bar{\mu}(g_1) d\bar{\mu}(g_2)$.

[Recall that $\int_{\Gamma\backslash G} f(g) \, d\bar{\mu}(g) = \int_F f(g) \, d\mu(g)$
where F is a fundamental domain.

$$\langle F_T, \varphi\rangle = \int_{\Gamma\backslash G} \left(\int_F \left(\sum_{\gamma \in \Gamma} \chi_{B_T}(g_1^{-1} \gamma g_2) \overline{\varphi_1(g_1)} \overline{\varphi_2(g_2)} \, d\mu(g_2) \right) \right) d\bar{\mu}(g_1)$$

$$= \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma} \left(\int_{\gamma F} \chi_{B_T}(g_1^{-1} g_2) \overline{\varphi_2(g_2)} \, d\mu(g_2) \right) \overline{\varphi_1(g_1)} \, d\bar{\mu}(g_1)$$

$$= \int_{\Gamma\backslash G} \left(\int_G \chi_{B_T}(g_1^{-1} g_2)\, \overline{\varphi_2(g_2)}\, d\mu(g_2) \right) \overline{\varphi_1(g_1)}\, d\bar{\mu}(g_1)$$

Change of variables $b = g_1^{-1} g_2 \in B_T$

$$= \int_{\Gamma\backslash G} \left(\int_{B_T} \overline{\varphi_2(g_1 b)}\, d\mu(g_2) \right) \overline{\varphi_1(g_1)}\, d\bar{\mu}(g_1)$$

$$= \int_{B_T} < U_b \overline{\varphi_2}, \varphi_1 > d\mu(b)$$

By Howe-Moore Thm, $<U_b\overline{\varphi_2}, \varphi_1> \xrightarrow[\|b\|\to\infty]{} \left(\int_{\Gamma\backslash G} \overline{\varphi_2}\right)\left(\int_{\Gamma\backslash G} \overline{\varphi_1}\right)$.

Hence, $\forall T \geq T_\varepsilon \quad \forall b \notin B_{T_\varepsilon}$: $\left| <U_b\varphi_2, \varphi_1> - \left(\int_{\Gamma\backslash G}\varphi_2\right)\left(\int_{\Gamma\backslash G}\varphi_1\right)\right| \leq \varepsilon$.

Now $\left| <F_T, \varphi> - vol(B_T)\left(\int_{\Gamma\backslash G}\overline{\varphi_1}\right)\left(\int_{\Gamma\backslash G}\overline{\varphi_2}\right)\right|$

$$\leq \int_{B_T} \left| <U_b\overline{\varphi_2}, \varphi_1> - \left(\int_{\Gamma\backslash G}\overline{\varphi_1}\right)\left(\int_{\Gamma\backslash G}\overline{\varphi_2}\right)\right| d\mu(b)$$

$$= \int_{B_{T_\varepsilon}} + \int_{B_T - B_{T_\varepsilon}} \leq c(\varphi_1, \varphi_2)\cdot B_{T_\varepsilon} + \varepsilon \cdot vol(B_T - B_{T_\varepsilon}).$$

This implies that

$$\overline{\lim_{T\to\infty}} \left| <F_T, \varphi> - vol(B_T)\left(\int_{\Gamma\backslash G}\overline{\varphi_1}\right)\left(\int_{\Gamma\backslash G}\overline{\varphi_2}\right)\right| \Big/ vol(B_T) \leq \varepsilon$$

for every $\varepsilon > 0$.

Step 2: pointwise convergence

We apply Step 1 to the "bump" functions:

$$\varphi_1(g) = \varphi_2(g) = \sum_{\gamma\in\Gamma} \frac{\chi_{\mathcal{O}_\varepsilon}(\gamma g)}{vol(\mathcal{O}_\varepsilon)}.$$

Note that $\int_{\Gamma\backslash G} \varphi_i \, d\mu = 1$.

We have

$$\langle F_T, \varphi \rangle = \int_{\Gamma\backslash G \times \Gamma\backslash G} F_T(g_1, g_2)\, \varphi_1(g_1)\, \varphi_2(g_2)\, d\bar{\mu}(g_1)\, d\bar{\mu}(g_2)$$

$$= \int_{F\times F} \sum_{\gamma_1,\gamma_2\in\Gamma} F_T(g_1,g_2)\, \frac{\chi_{\mathcal{O}_\varepsilon}(\gamma_1 g_1)}{vol(\mathcal{O}_\varepsilon)} \cdot \frac{\chi_{\mathcal{O}_\varepsilon}(\gamma_2 g_2)}{vol(\mathcal{O}_\varepsilon)}\, d\mu(g_1)\, d\mu(g_2)$$

$$= vol(\mathcal{O}_\varepsilon)^{-2} \cdot \int_{\mathcal{O}_\varepsilon \times \mathcal{O}_\varepsilon} F_T(g_1,g_2)\, d\mu(g_1)\, d\mu(g_2).$$

$$\int_{\mathcal{O}_\varepsilon\times\mathcal{O}_\varepsilon} F_T\, d(\mu\times\mu) = \sum_{\gamma\in\Gamma} \int_{\mathcal{O}_\varepsilon\times\mathcal{O}_\varepsilon} \chi_{B_T}(g_1^{-1}\gamma g_2)\, d\mu(g_1)\, d\mu(g_2)$$

$$= \sum_{\gamma\in\Gamma} \int_{\mathcal{O}_\varepsilon\times\mathcal{O}_\varepsilon} \chi_{g_1 B_T g_2^{-1}}(\gamma)\, d\mu(g_1)\, d\mu(g_2)$$

$$\leq \sum_{\gamma\in\Gamma} \int_{\mathcal{O}_\varepsilon\times\mathcal{O}_\varepsilon} \chi_{B_{(1+2\varepsilon)T}}(\gamma)\, d\mu(g_1)\, d\mu(g_2)$$

$$= F_{(1+2\varepsilon)T}(e,e) \cdot vol(\mathcal{O}_\varepsilon)^2.$$

Similarly, $\displaystyle\int_{\mathcal{O}_\varepsilon\times\mathcal{O}_\varepsilon} F_T\, d(\mu\times\mu) \geq F_{(1-2\varepsilon)T}(e,e)\cdot vol(\mathcal{O}_\varepsilon)^2.$

We have proved that

$$\langle F_{(1+2\varepsilon)^{-1}T}, \varphi \rangle \leq F_T(e,e) \leq \langle F_{(1-2\varepsilon)^{-1}T}, \varphi\rangle.$$

Now

$$\overline{\lim_{T\to\infty}} \frac{F_T(e,e)}{vol(B_T)} \leq \left(\overline{\lim_{T\to\infty}} \frac{vol(B_{(1-2\varepsilon)^{-1}T})}{vol(B_T)}\right)\cdot\left(\overline{\lim_{T\to\infty}} \frac{\langle F_{(1-2\varepsilon)^{-1}T}, \varphi\rangle}{vol(B_{(1-2\varepsilon)^{-1}T})}\right)$$

$$= \underbrace{(1-2\varepsilon)^{-(d^2-d)}}_{\text{by Lemma}} \cdot \underbrace{\left(\int_{\Gamma\backslash G\times\Gamma\backslash G} \varphi\right)}_{\text{by Step 1}} = (1-2\varepsilon)^{-(d^2-d)},$$

for every $\varepsilon > 0$.

The lower estimate is proved similarly.