$\boxed{\text{Lecture 10}}$

Furstenberg's ×2, ×3 theorem and Littlewood conjecture.

$X = \mathbb{R}/\mathbb{Z}$, $\quad D_2 : X \to X : \quad x \mapsto 2x \pmod 1$

$\qquad\qquad\quad D_3 : X \to X : \quad x \mapsto 3x \pmod 1$

a single transformation D_3 has many closed invariant sets (e.g. the Cantor set).
However, for the semigroup $\langle D_2, D_3 \rangle$, the picture is much more rigid.

Thm (Furstenberg) $\forall$ irrational $x \in X$,
$$\{2^m 3^n x\}_{m,n\geq 0} \text{ is dense.}$$

1) Let $S = \{2^m 3^n\}_{m,n\geq 0} = \{s_1 < s_2 < \ldots < s_n < \ldots\}$.
Then $\lim_{i\to\infty} \frac{s_{i+1}}{s_i} = 1$.

$\alpha = \frac{\log 2}{\log 3}$

Consider $T = \{m\alpha + n\}_{m,n\geq 1} = \{t_1 < t_2 < \ldots < t_n < \ldots\}$
We need to show that $\lim_{i\to\infty} (t_{i+1} - t_i) = 0$.
We know that $\{m\alpha \pmod 1\}_{m\geq 1}$ is dense in $[0,1)$.
Hence, $\forall \varepsilon > 0 : \exists m_0 = m_0(\varepsilon) : \quad m_0\alpha - \lfloor m_0\alpha \rfloor \in (0, \varepsilon)$
Then for $m \geq m_0$ and $n \geq 0$,

$(m\alpha + n) - ((m-m_0)\alpha + n + \lfloor m_0\alpha \rfloor) = m_0\alpha - \lfloor m_0 \alpha \rfloor \in (0,\varepsilon)$

This implies the claim.

[2) If a closed $<2,3>$-invariant set F has a rational limit point, then $F = X$.]

First, suppose that 0 is a limit point of F.

$\forall \varepsilon > 0 \quad \forall i \geq i_0(\varepsilon): \quad \frac{s_{i+1}}{s_i} < 1+\varepsilon.$

Since 0 is a limit point of F, $\exists x \in F: \ x \in (0, \varepsilon)$:

Then $\{s_i x\}_{i \geq i_0}$ forms ε-net in X.

Indeed, when $s_i \leq \frac{1}{x}$, $\quad s_{i+1}x - s_i x \leq s_i \cdot ((1+\varepsilon)x - x) < \varepsilon.$

This implies that $F = X$ (since ε is arbitrary).

In general, let $r = \frac{p}{q} \pmod 1$ be a limit point of F.

We may assume that q coprime to 2 and 3.

Then $2^u = 3^u = 1 \pmod q$. Then $2^u \cdot r = 3^u \cdot r = r$.

We apply the above argument to $F - r$ which is $<2^u, 3^u>$-invariant.

[3) Every closed $<2,3>$-invariant subset F contains a rational limit point.]

Suppose not. Take $\varepsilon > 0$, integer $N > \frac{1}{\varepsilon}$ coprime to 2,3, and integer u such that $2^u = 3^u = 1 \pmod N$.

Define inductively sequence of set:

$$F_0 = F' \supset F_1 \supset \ldots \supset F_{N-1}$$

by $F_0 = F'$ is the set of limit points of F,

$$F_{i+1} = \{x \in F_i : (x + \tfrac{1}{N}) \pmod 1 \in F_i\}.$$

- If F_i is closed, then F_{i+1} is also closed.
- If F_i is 2^u- and 3^u-invariant, then F_{i+1} is also 2^u- and 3^u-invariant.
- If F_i is finite, then $\exists u$: $2^u \cdot x = x$ for all $x \in F_i$, and F consists of rational points.

Now suppose that F_i is infinite. Then $F_i - F_i$ is closed $\langle 2^u, 3^u \rangle$-invariant set which has 0 as accumulation point. By step 2, $F_i - F_i = X$.

This implies that F_{i+1} is not empty.

Pick $x \in F_{N-1}$. Then $x_i = x + \frac{i}{N}$, $i = 0, \ldots N-1$ is in F and it is an ε-net in X.

Since $\varepsilon > 0$ is arbitrary, $F = X$. $\square$

<u>Open problem:</u> Is $\left\{\left(\frac{3}{2}\right)^n \pmod 1\right\}_{n \geq 0}$ is dense in $[0,1]$?

Conj (Furstenberg) If μ is $\langle 2,3\rangle$-ergodic probability measure on $\mathbb{R}/\mathbb{Z}$, then μ is either supported on a finite set or μ is Lebesgue.

Littlewood Conjecture.

Conj (Littlewood) $\forall \alpha,\beta\in\mathbb{R}$: $\boxed{\liminf_{n\to\infty} n\cdot \mathrm{dist}(n\alpha,\mathbb{Z})\,\mathrm{dist}(n\beta,\mathbb{Z})=0}$

If α is not badly approximable, $\liminf_{n\to\infty} n\cdot\mathrm{dist}(n\alpha,\mathbb{Z})=0$.
Hence, it is clear that Conj holds on a set of full measure.

Consider a cubic form $F(x_1,x_2,x_3)=(x_1+\alpha x_3)(x_2+\beta x_3)\,x_3$.
Note that $F(x)=F_0(x\cdot u_{\alpha,\beta})$ where $F_0(y_1,y_2,y_3)=y_1y_2y_3$,

$$u_{\alpha,\beta}=\begin{pmatrix}1&0&0\\0&1&0\\\alpha&\beta&1\end{pmatrix}.$$

The form F_0 is invariant under the diagonal group:

$$A=\left\{\begin{pmatrix}e^s&&0\\&e^t&\\0&&e^{-s-t}\end{pmatrix}: s,t\in\mathbb{R}\right\}.$$

Let

$$D=\left\{\begin{pmatrix}e^s&&0\\&e^t&\\0&&e^{-s-t}\end{pmatrix}: s,t\geq 0\right\}.$$

Prop. Littlewood Conj. $\Leftrightarrow$ $\mathbb{Z}^3 g_{\alpha,\beta} D$ is unbounded in the space of lattices $\mathcal{L}_3$!

$\Leftarrow$ If $\mathbb{Z}^3 g_{\alpha,\beta} D$ is unbounded, then by Mahler compactness criterion, $\forall \varepsilon \in (0,1): \exists v = (p_1, p_2, q) \in \mathbb{Z}^3,\ v \neq 0,\ \exists s,t \geq 0:$

$$\| \underbrace{v \cdot g_{\alpha,\beta} \cdot d_{s,t}}_{(e^s(p_1+q\alpha),\ e^t(p_2+q\beta),\ e^{-s-t} q)} \| < \varepsilon$$

$$\Downarrow$$

Since $\varepsilon < 1$, we have $q \neq 0$. It follows that $|p_1+q\alpha| \cdot |p_2+q\beta| \cdot |q| < \varepsilon^3$, which implies the claim.

$$\begin{cases} e^s |p_1+q\alpha| < \varepsilon \\ e^t |p_2+q\beta| < \varepsilon \\ e^{-s-t}|q| < \varepsilon \end{cases}$$

$\Rightarrow$ We consider the case when α, β are irrational (exercise: complete the proof when α or β is rational).

For every $\varepsilon \in (0,1)$: $\exists q \in \mathbb{N},\ (p_1,p_2) \in \mathbb{Z}^2:\ q\,|q\alpha+p_1| \cdot |q\beta+p_2| < \varepsilon.$

Suppose that $|q\alpha+p_1| \geq \varepsilon^{1/5}$. Then $q \cdot |q\beta+p_1| < \varepsilon^{4/5}$.

By the Dirichlet Thm, $\exists m \in \mathbb{Z}, n \in \mathbb{N}:$ $\begin{cases} |q\alpha - \frac{m}{n}| \leq \frac{\varepsilon^{1/5}}{n}, \\ n \leq \varepsilon^{-1/5}. \end{cases}$

Then $\begin{cases} nq \cdot |nq\beta + np_1| < \varepsilon^{2/5}, & \Rightarrow\ |nq\beta + np_1| < \varepsilon^{2/5} < \varepsilon^{1/5}. \\ |nq\alpha - m| \leq \varepsilon^{1/5}. & \Rightarrow\ nq \cdot |nq\alpha - m| \cdot |nq\beta + np_1| < \varepsilon^{3/5} \end{cases}$

We conclude that $\forall \varepsilon > 0$:

$\exists q \in \mathbb{N},\ (p_1,p_2) \in \mathbb{Z}^2:$ $\begin{cases} q\,|q\alpha+p_1| \cdot |q\beta+p_2| < \varepsilon^{3/5} \\ |q\alpha+p_1| \leq \varepsilon^{1/5} \\ |q\beta+p_2| \leq \varepsilon^{1/5} \end{cases}$

Take $s,t \geq 0$: $e^s \cdot |q\alpha+p_1| = \varepsilon^{1/5}$, $e^t\,|q\beta+p_2| = \varepsilon^{1/5}$.

Then $e^{-s-t} q = \varepsilon^{-1/5} |q\alpha+p_1| \cdot \varepsilon^{-1/5} |q\beta+p_2| \cdot q < \varepsilon^{1/5}$.

Hence, $\|(p_1,p_2,q)\, u_{\alpha,\beta}\, d_{s,t}\| < \varepsilon^{1/5}$, and $\mathbb{Z}^3 u_{\alpha\beta} d_{s,t}$ is <u>unbounded</u>.

Conj (Margulis) Every orbit of A (or D) in $\mathcal{L}_3'$ is either compact or unbounded.

Conj (Margulis) Every ergodic A-invariant probability measure on $\mathcal{L}_3'$ is either the measure on a compact orbit, or the invariant measure on $\mathcal{L}_3'$.

Thm. (Einsiedler-Katok-Lindenstrauss)
Littlewood Conj holds for all $(\alpha, \beta) \in \mathbb{R}^2$, but possiblly a set of Hausdorff dimension 0.