## Homework 3

(1) (*well approximable vectors* $\leftrightarrow$ *excursions to the cusp*) A vector $x$ in $\mathbb{R}^d$ is called well approximable if for some $\epsilon > 0$, the inequality $\|x - p/q\| < q^{-1-1/d-\epsilon}$ has infinitely many solutions $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. Let

$$\Lambda_x = \mathbb{Z}^{d+1} \begin{pmatrix} id & 0 \\ x & 1 \end{pmatrix}, \quad x \in \mathbb{R}^d$$
$$a_t = \mathrm{diag}(e^t, \dots, e^t, e^{-dt}) \in \mathrm{SL}_{d+1}(\mathbb{R}),$$
$$\Delta(\Lambda) = \min\{\|v\| : v \in \Lambda, v \neq 0\}, \quad \Lambda \in \mathcal{L}^1_{d+1}.$$

Prove that a vector $x$ is well approximable if and only if there exists $\delta > 0$ such that the inequality $\Delta(\Lambda_x a_t) \leq e^{-\delta t}$ has solutions $t \to \infty$ (i.e., the flow visits exponentially shrinking neighbourhoods of $\infty$).

(2) Show that if $\alpha\beta < 2\alpha - 1$, then in the Schmidt game, Bob has a strategy to force the end point of the game to be any preassigned point.

(3) Let $x_1, x_2, x_3 \dots$ denote the sequence of the top digits of the numbers $2, 2^2, 2^3, \dots$ in the decimal expansion. Compute the frequences

$$\lim_{N\to\infty} \frac{1}{N} |\{i = 1, \dots, N : x_i = d\}|.$$

(Hint: use rotations on the circle.) Which of the digits is most likely to appear?

(4) Show that every closed set invariant under the horocycle flow is either the whole space or consists of periodic orbits.

(5) Consider a sequence $P_i = x_i U$ of periodic orbits of the horocycle flow $U = \{u_s\}$ with the periods going to $\infty$. Prove that this sequence becomes dense as $i \to \infty$. (Hint: imitate the "thickening" argument from the class and use mixing.)

(6) Give an example of a sequence of probability measures $\mu_n$ on $[0, 1]$ and a set $B \subset [0, 1]$ such that $\mu_n \to \mu$, $\mu_n(B) = 1$, and $\mu(B) = 0$.

(7) Let $L(x, y) = \alpha x + \beta y$ with $\alpha, \beta \in \mathbb{R}$, $\alpha/\beta \in \mathbb{R}\backslash\mathbb{Q}$. Using the properties of the horocycle flow, show that the set

$$\{L(p, q) : p, q \in \mathbb{Z}, \gcd(p, q) = 1\}$$

is dense in $\mathbb{R}$.