Homework 2

(1) An element v of a lattice $\Lambda \subset \mathbb{R}^d$ is called *primitive* if $v \neq n \cdot w$ for all integers $n \geq 2$ and $w \in \Lambda$. Show that there exists universal $\epsilon_0 > 0$ such that every $\Lambda \in \mathcal{L}_2^1$ contains at most one (up to sign) primitive v with $\|v\| \leq \epsilon_0$.

(2) *(exponential mixing)* Let $D_2 : S^1 \to S^1 : z \mapsto z^2$ be the doubling map on the circle. Prove that there exists $\theta \in (0,1)$ such that for every continuously differentiable functions $\phi, \psi \in C^1(S^1)$,

$$\left| \int_{S^1} \phi(D_2^n z)\psi(z)dz - \left(\int_{S^1} \phi\right)\left(\int_{S^1} \psi\right) \right| \leq c(\phi,\psi)\theta^n$$

(hint:use Fourier analysis).

(3) Let $T : X \to X$ be a continuous map of a topological space X. Assume that X has a countable basis for open sets, and X is equipped with a probability measure μ of full support (this means that $\mu(U) > 0$ for every open set $U \subset X$). Show that if T is mixing, then for a set of full measure in X, the orbit $\{T^n x\}_{n \geq 0}$ is dense.

(4) Prove that every number of the form $a + b\sqrt{d}$ where $a, b \in \mathbb{Z}$ and $d \in \mathbb{N} \backslash \mathbb{N}^2$ is badly approximable.

(5) Let $\psi : [1, \infty) \to (0,1)$ be any decreasing function. Show that there exist irrational numbers which are ψ-approximable.

(6) *(quadratic irrationals $\leftrightarrow$ periodic orbits)* Let $d \in \mathbb{N}$ and $(x, y) \in \mathbb{Z}^2$ be a solution of the Pell equation $x^2 - dy^2 = 1$. Show that every such solution gives rise to a periodic orbit of the flow $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ with period $\cosh^{-1}(x)$. Namely, construct $z \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ such that $za_{t_0} = z$ for $t_0 = \cosh^{-1}(x)$.

(7)
  (a) Show that for every $g \in \mathrm{SL}_2(\mathbb{Z})$ and $h \in \mathrm{SL}_2(\mathbb{Q})$, there exists $n \in \mathbb{N}$ such that $h^{-1}g^n h \in \mathrm{SL}_2(\mathbb{Z})$.
  (b) Show that if the orbit of $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ for the point $z_0 = \mathrm{SL}_2(\mathbb{Z})g_0 \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ is periodic, then so is the orbit for $z = \mathrm{SL}_2(\mathbb{Z})hg_0$ for every $h \in \mathrm{SL}_2(\mathbb{Q})$.
  (c) Deduce that the periodic orbits of the flow a_t in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ are dense.

(8) An element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is called *primitive* if it cannot be written as $\gamma = \gamma_0^m$ for some $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ and $m \in \mathbb{N}$, $m \geq 2$. Show that every element of infinite order in $\mathrm{SL}_2(\mathbb{Z})$ is a power of a primitive element.

(9) Prove that there is a one-to-one correspondence between periodic orbits of the flow $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ and conjugacy classes of primitive elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{Tr}(\gamma) > 2$. Show that this correspondence the periods of the orbits are given by $\cosh^{-1}(\mathrm{Tr}(\gamma)/2)$.

(10) *(singular vectors $\leftrightarrow$ divergent trajectories)* A vector $x \in \mathbb{R}^d$ is called *singular* if for every $\varepsilon > 0$ and $N \geq N_0(\epsilon)$, the system of inequalities

$$\left\| x - \frac{p}{q} \right\| < \frac{\epsilon N^{-1/d}}{q}, \quad 0 < q < N$$

has a solution $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. As in the lectures, we use notation:

$$\Lambda_x = \mathbb{Z}^{d+1} \begin{pmatrix} id & 0 \\ x & 1 \end{pmatrix} \in \mathcal{L}^1_{d+1},$$
$$a_t = \mathrm{diag}(e^t, \ldots, e^t, e^{-dt}) \in \mathrm{SL}_{d+1}(\mathbb{R}).$$

(a) Prove that a vector $x \in \mathbb{R}^d$ is singular if and only if the orbit $\{\Lambda_x a_t\}_{t \geq 0}$ is divergent (that is, $\Delta(\Lambda_x a_t) \to 0$ as $t \to \infty$).

(b) Deduce that the set of singular vectors in $\mathbb{R}^d$ has Lebesgue measure zero.