Homework 1

In the following $\mathcal{L}_d^1 \simeq \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ denotes the space of unimodular lattices in $\mathbb{R}^d$.

(1) (a concrete realisation of $\mathcal{L}_2^1$)
For $g \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, we set

$$g \cdot z = \frac{g_{11}z + g_{12}}{g_{21}z + g_{22}}.$$

(a) Show that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}$ and the stabiliser of i is equal to $\mathrm{SO}(2,\mathbb{R})$. Hence,

$$\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \simeq \mathbb{H},$$

and

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H},$$

(b) Show that the region

$$\begin{aligned} D =&\{z \in \mathbb{H} : |z| > 1, -1/2 < \mathrm{Re}(z) \le 1/2\} \\ &\cup \{z \in \mathbb{H} : |z| = 1, 0 \le \mathrm{Re}(z) \le 1/2\} \end{aligned}$$

is a fundamental domain for $\mathrm{SL}(2,\mathbb{Z})/\{\pm id\}$ acting on $\mathbb{H}$. (Hint: Imitate the proof from the lecture — try to maximise $\mathrm{Im}(g \cdot z)$.)

(c) Show that the space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is homeomorphic to a punctured sphere.

(2) A discriminant of a quadratic form $Q(x_1,x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ is defined by $d(Q) = b^2 - 4ac$. We say that two quadratic forms are equivalent if one can transform one of them to the other by an integral change of coordinates. Prove that there are only finitely many equivalence classes of integral quadratic forms in two variables with given discriminant.

(3) Let $Q(x) = \sum_{i,j=1}^d a_{ij}x_ix_j$ be a nondegenerate positive definite quadratic form with real coefficients. Using the theory of Siegel sets, prove that there exists $x \in \mathbb{Z}^d$ such that $x \neq 0$ and

$$Q(x) \le (4/3)^{(d-1)/2} \det(a_{ij})^{1/d}.$$

(4) Prove that an odd prime number p can be written as a sum of two squares if and only if $p = 1 \mod 4$. (Hint: apply the previous exercise to the quadratic form

$$Q(x_1,x_2) = (a^2+1)x_1^2 + 2apx_1x_2 + p^2x_2^2$$

with suitably chosen a.)

(5) Use Exercise 3 to prove the Lagrange theorem: every positive integer is a sum of four squares. You may wish to follow the following steps:
   (a) Prove that if integers m and n are sums of four squares that so is $m \cdot n$ (hint: introduce a "norm" on the field of quaternions).
   (b) Show that for a lattice

$$\Lambda = \begin{pmatrix} p & 0 & a & b \\ 0 & p & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^4,$$

   there exists $(x_1, x_2, x_3, x_4) \in \Lambda$ such that $0 < x_1^2 + x_2^2 + x_3^2 + x_4^2 < 2p$.
   (c) Prove the Lagrange theorem for prime number (hint: choose the parameters $a, b, c, d \in \mathbb{Z}$ so that for all $(x_1, x_2, x_3, x_4) \in \Lambda$, we have $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \mod p$).

(6) Recall that $\mathcal{L}_d \simeq \mathcal{B}_d/\sim$ where $\mathcal{B}_d$ is the space of bases in $\mathbb{R}^d$. Since $\mathcal{B}_d$ is an open subset of $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$, it is equipped with a natural topology. We equip $\mathcal{L}_d$ with the factor-space topology. Namely, a subset $U \subset \mathcal{L}_d$ is open if the preimage of U in $\mathcal{B}_d$ is open.
   (a) Prove that for $L_n, L \in \mathcal{L}_d$, we have $L_n \to L$ as $n \to \infty$ if and only if there exist bases $\{e_i^{(n)}\}$ and $\{e_i\}$ of L_n and L respectively such that $e_i^{(n)} \to e_i$ as $n \to \infty$.
   (b) Show that the space $\mathcal{L}_d^1$ is not compact.

(7) A subgroup Γ of $G = \mathrm{SL}(d, \mathbb{R})$ is called a *lattice* if it is discrete and there exists a (measurable) set $F \subset G$ such that $\mathrm{vol}(F) < \infty$ and $G = F\Gamma$ (for example, $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ is a lattice in G).
   (a) Show that if $\Gamma_1 \subset \Gamma_2$ are lattices in G, then Γ_1 has finite index in Γ_2.
   (b) Let Γ be a lattice in G. Prove that the space G/Γ is compact if and only if e is not an accumulation point of $\{g\gamma g^{-1} : g \in G, \gamma \in \Gamma\}$.

(8) (a) Give a formula for left and right invariant measures for the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+,\ b \in \mathbb{R} \right\}$$

   (b) Show that the group G has no lattice subgroups.

(9) Let $G = \mathrm{SO}(1, n)(\mathbb{R})$ be the orthogonal group.
   (a) Construct the Iwasawa decomposition for G.

(b) Construct Siegel sets of $G = \mathrm{SO}(1,n)(\mathbb{R})$ and $\Gamma = \mathrm{SO}(1,n)(\mathbb{Z})$.

(c) Show that Γ is a lattice in G.