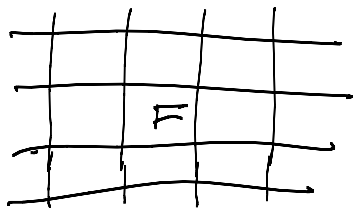
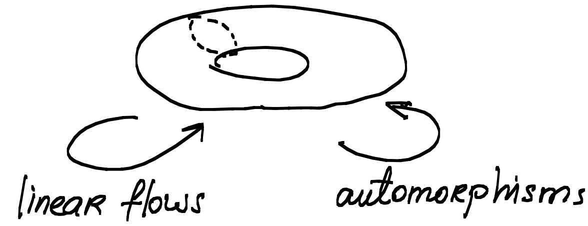
Lecture V: Lattices in Lie groups.

$\mathbb{Z}^d \subset \mathbb{R}^d$: $\quad \mathbb{R}^d = \bigsqcup_{z \in \mathbb{Z}^d} (F + z)$ $\qquad \mathbb{R}^d/\mathbb{Z}^d$ – torus





Let G be a Lie group and Γ a discrete subgroup of G.

Def. $F \subset G$ is called a <u>fundamental set</u> for Γ if

1) $G = \Gamma \cdot F$,
2) $\gamma_1 F \cap \gamma_2 F \neq \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma \Rightarrow \gamma_1 = \gamma_2$.

Note that 1), 2) $\Rightarrow$ $G = \bigsqcup_{\gamma \in \Gamma} \gamma F$ is a teselation of G.

Lem. There exists a Borel fundamental domain.

Proof. Since Γ is discrete, there exists a nbhd $\mathcal{U}$ of e in G such that $\mathcal{U} \cdot \mathcal{U}^{-1} \cap \Gamma = \{e\}$. Then
$G = \bigcup_{n=1}^{\infty} \mathcal{U} g_n$ for some $g_n \in G$. We set

$$F = \bigcup_{n=1}^{\infty} \left(\mathcal{U} g_n \setminus \left(\bigcup_{i=1}^{n-1} \Gamma \mathcal{U} g_i \right) \right).$$

It is easy to check that $G = \Gamma \cdot F$ and
$\gamma_1 F = \gamma_2 F \Rightarrow \gamma_1 = \gamma_2$. $\square$

Let $\pi: G \longrightarrow X = \Gamma\backslash G$ be the factor-map, and m a left-inv. measure on G.

We define a measure μ on X by:

$$\mu(B) = m(\pi^{-1}(B) \cap F) \quad \text{for Borel } B \subset G.$$

Lem. 1) The definition of μ does not depend on the choice of F.

2) If $m(F) < \infty$, then μ is right G-invariant.

Proof. Let $F_1, F_2 \subset G$ be Borel fundamental sets. Then

Since $G = \bigsqcup_{\gamma \in \Gamma} \gamma F_1 = \bigsqcup_{\gamma\in\Gamma} \gamma F_2$, we have:

$$m(\pi^{-1}(B) \cap F_1) = \sum_{\gamma \in \Gamma} m(\pi^{-1}(B) \cap F_1 \cap \gamma F_2)$$

$$= \sum_{\gamma \in \Gamma} m(\pi^{-1}(B) \cap \gamma^{-1} F_1 \cap F_2) = m(\pi^{-1}(B) \cap F_2).$$

For $g \in G$, consider the measure $\tilde{m}$ on G defined by:

$$\tilde{m}(A) = m(Ag).$$

By uniqueness of left-inv. measure on G, $\tilde{m} = c_g \cdot m$ for some $c_g > 0$. Then

$$m(\pi^{-1}(Bg) \cap F) = m((\pi^{-1}(B) \cap F g^{-1}) g) = c_g \cdot m(\pi^{-1}(B) \cap F g^{-1})$$

$$= c_g \cdot m(\pi^{-1}(B) \cap F),$$

where we used that Fg^{-1} is also a fundamental set.

Hence, $\mu(B \cdot g) = c_g \cdot \mu(B)$.

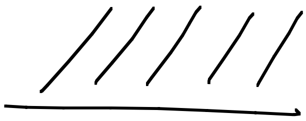
In particular, if $m(F) < \infty$,

$$\mu(X) = c_g \cdot \mu(X) \Longrightarrow m(F) = c_g \cdot m(F) \Rightarrow \underline{c_g = 1}.$$

Def. A discrete subgroup Γ of G is called a <u>lattice</u> if $\mu({}_{\Gamma}\backslash G) < \infty$ ($\Leftrightarrow m(F) < \infty$ for a fundamental set F).

Hyperbolic plane.

$\mathbb{H} = \{ x+iy : x \in \mathbb{R}, y > 0 \}$



For a curve $c : [0,1] \to \mathbb{H}$, the length of c is defined by $L(c) = \int_0^1 \frac{\|c'(t)\|}{Im(c(t))} dt$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL_2(\mathbb{R})$, define

$$T_g : \mathbb{H} \longrightarrow \mathbb{H} : z \longmapsto \frac{az+b}{cz+d}.$$

Properties:
1) $T_g = Id \Leftrightarrow g = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
2) $T_{g_1} \cdot T_{g_2} = T_{g_1 g_2}$,
3) $Im(T_g(z)) = \frac{Im(z)}{|cz+d|^2}$ $\quad (\Rightarrow T_g(\mathbb{H}) \subset \mathbb{H})$.
4) $Stab_G(i) = \{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2+b^2=1 \} = SO_2(\mathbb{R})$
5) $L(T_g \circ c) = L(c)$.

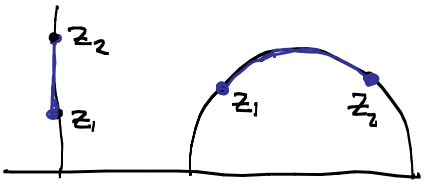
Lem. $SL_2(\mathbb{R}) = \{ \underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}_{u(x)} \underbrace{\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}}_{a(y)} k : x \in \mathbb{R}, y > 0, k \in SO_2(\mathbb{R}) \}$ (Iwasawa decomposition)

Proof. Observe that $u(x)a(y)k \cdot i = x+iy$.
In particular, <u>G acts transitively on $\mathbb{H}$.</u>

Hence, for $g \in G$, $g \cdot i = u(x) \cdot a(y) \cdot i$ for some $x \in \mathbb{R}, y > 0$, and $g \in u(x)a(y)K$.

Cor. $\mathbb{H} \simeq SL_2(\mathbb{R}) / SO_2(\mathbb{R})$.

Lem. For $z_1, z_2 \in \mathbb{H}$, the shortest path from z_1 to z_2 is either a vertical line or a half-circle (geodesics).



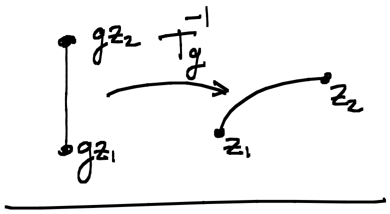
Proof. Suppose that $Re(z_1) = Re(z_2)$. Then for every path $c : [0,1] \to \mathbb{H}$ from z_1 to z_2,

$$L(c) = \int_0^1 \frac{\sqrt{c_1'(t)^2 + c_2'(t)^2}}{c_2(t)}\, dt \geq \int_0^1 \frac{|c_2'(t)|}{c_2(t)}\, dt$$

"$=$" $\Leftrightarrow$ $c_1'(t) = 0$.

Hence, the shortest path is a vertical line.

For general $z_1, z_2 \in \mathbb{H}$, $\exists g \in G$: $Re(gz_1) = Re(gz_2) = x_0$.
(check)



Since T_g preserves distances, the shortest path between z_1 and z_2 is

$$T_g^{-1}(\{x = x_0\}) = \{\text{half-circle}\}.$$

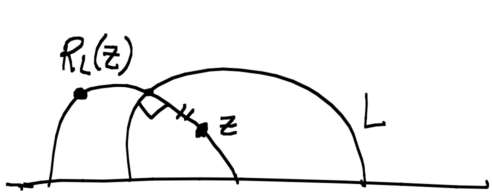
check.

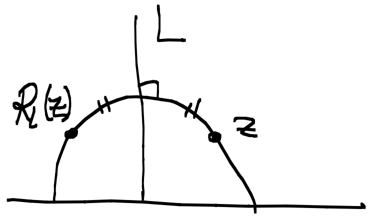
For a geodesic $L \subset \mathbb{H}$, we define a reflection

$$R_L : \mathbb{H} \to \mathbb{H}.$$

for $L = \{x = 0\}$, $R_L : z \to -\bar{z}$.

In general,





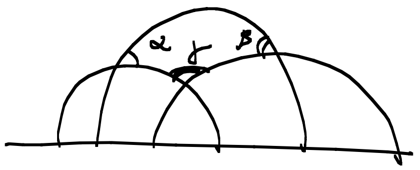
Reflections also preserve distances:

$$L(R_L \circ c) = L(c).$$

Poincare construction.

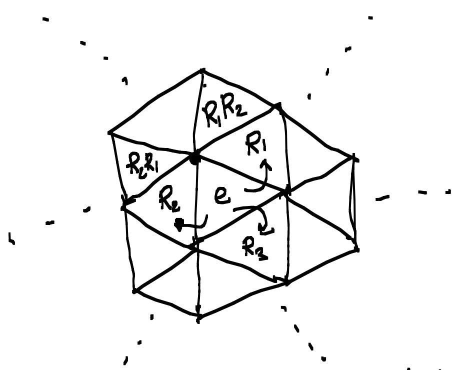
Let $n_1, n_2, n_3 \in \mathbb{N}$, with $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3} < 1$.
Take a hyperbolic triangle C with angles $\frac{\pi}{n_1}, \frac{\pi}{n_2}, \frac{\pi}{n_3}$.

ex. For every $\alpha, \beta, \gamma > 0$, $\exists$ a hyperbolic triangle with angles α, β, γ.



Let Λ be the group of maps $\mathbb{H} \to \mathbb{H}$ generated by R_1, R_2, R_3.

For $\lambda \in \Lambda$, $\lambda \cdot C$ is another triangle with the same angles.



Since $n_1, n_2, n_3 \in \mathbb{N}$, taking products of reflections R_1, R_2, R_3 we can "perfectly" tile a nbhd of each vertex of C. Continuing this process, we obtain $\mathbb{H} = \bigcup_{\lambda \in \Lambda} \lambda C$. Moreover, since all tiles fit "perfectly", if $\lambda_1 C^\circ \cap \lambda_2 C^\circ \neq \emptyset$, then $\lambda_1 C = \lambda_2 C$ and $\lambda_1 = \lambda_2$.

Let $\Lambda_0 \subset \Lambda$ be the subgroup consisting of products of even number of reflections.

ex. $\forall \lambda \in \Lambda_0 \ \exists \gamma \in SL_2(\mathbb{R}) : T_\gamma = \lambda$.

Let $\Gamma = T^{-1}(\Lambda_0) \subset SL_2(\mathbb{R})$.

Thm. Γ is a a cocompact lattice in $SL_2(\mathbb{R})$.

Proof. Let $p: G \longrightarrow \mathbb{H}: g \longmapsto g\cdot i$. Then $p(g\cdot h) = T_g \cdot p(h)$.

We set $F = p^{-1}(C \cup R_1(C))$.

Since $\Lambda = \Lambda_0 \cup \Lambda_0 \lambda$, we have

$$\mathbb{H} = \bigcup_{\lambda \in \Lambda_0} \lambda\cdot(C \cup R_1(C)) \Longrightarrow G = \Gamma\cdot F.$$

We note that F compact.

ex. Check that if $\Omega \subset \mathbb{H}$ is compact, then $p^{-1}(\Omega)$ is compact too.

To check discreteness, we observe that for every compact $\Omega \subset \mathbb{H}$, $\#\{\lambda \in \Lambda : \lambda\cdot\Omega \cap \Omega\} < \infty$.

This implies that for every compact $\tilde{\Omega} \subset G$, $\#\{\gamma \in \Gamma : \gamma\cdot\tilde{\Omega} \cap \tilde{\Omega} \neq \emptyset\} < \infty$. Hence, for every compact $\tilde{\Omega}$, $|\Gamma \cap \tilde{\Omega}| < \infty$, and Γ is discrete.

$\blacksquare$