Lecture 4: Algebraic groups

Def. A subgroup G of $GL_d(\mathbb{C})$ is called algebraic if $G = \{g \in M_d(\mathbb{C}) : P(g)=0, P \in I\}$ for $I \subset \mathbb{C}[x_{11}, \dots x_{dd}]$.

Thm. $f: G_1 \to G_2$ – polynomial homomorphism of alg. groups. Then $f(G_1)$ is algebraic group.

ex. This is false for Lie groups: consider the map $f: \mathbb{R} \longrightarrow GL_2(\mathbb{C}): t \longmapsto \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix}$.
When $f(\mathbb{R})$ is closed?

We define $V(I) = \{x \in \mathbb{C}^d : f(x)=0 \text{ for } f \in I\}$.

Properties:
1) $V(\{1\}) = \emptyset$, $V(\{0\}) = \mathbb{C}^d$,
2) $\bigcap_\alpha V(I_\alpha) = V(\bigcup_\alpha I_\alpha)$,
3) $V(I_1) \cup V(I_2) = V(I_1 \cdot I_2)$,
4) If $f: \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ is a polynomial, then $f^{-1}(V(I)) = V(\{P \circ f : P \in I\})$.

The collection $\{V(I)\}_{I \subset \mathbb{C}[x_1, \dots x_d]}$ satisfies the axioms of closed sets defining the Zariski topology on $\mathbb{C}^d$.

By (4), polynomial maps $\mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ are continuous with respect to this topology.

Rmk
- every infinite subset of $\mathbb{C}$ is dense,
- Zariski topology is not Hausdorff.

For $X \subset \mathbb{C}^d$, define $I(X) = \{P \in \mathbb{C}[x_1, \ldots x_d] : P|_X = 0\}$.

ex. Show that $V(I(X))$ is equal to the closure of X.

Def. A closed set X is called irreducible if
$X \neq X_1 \cup X_2$ for closed $X_1, X_2 \subsetneq X$.

ex. $\{P_1 \cdot P_2 = 0\} = \{P_1 = 0\} \cup \{P_2 = 0\}$.

Thm. Every closed $X = X_1 \cup \ldots \cup X_\ell$ where X_i's are irreducible.

Proof. If this fails, then $\exists$ a chain:
$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \ldots \supsetneq X_n \supsetneq \ldots$$
where X_i's are closed reducible sets. We get
$$I(X) \subset I(X_1) \subset \ldots \subset I(X_n) \subset \ldots$$
the chain of ideals in $\mathbb{C}[x_1, \ldots x_d]$.

[Hilbert basis thm: Every ideal in $\mathbb{C}[x_1, \ldots x_d]$ is finitely generated.]

Hence, $\bigcup_{n \geq 1} I(X_n)$ is fin. generated and hence
$I(X_n) = I(X_{n+1}) = \ldots \implies X_n = V(I(X_n)) = V(I(X_{n+1})) = X_{n+1}$, which is a contradiction.

For closed $X \subset \mathbb{C}^d$, we define the <u>coordinate ring of X</u>:

$$\mathcal{A}_X = \mathbb{C}[x_1, \dots x_d] / I(X).$$

	Geometry	Commutative algebra
(1)	points of X	algebra homomorphisms $\mathcal{A}_X \to \mathbb{C}$.
(2)	X is irreducible	$\mathcal{A}_X$ has no divisors of zero
(3)	Polynomial map $f: X \to Y$	Algebra homomorphism $f_*: \mathcal{A}_Y \to \mathcal{A}_X$
(4)	$\overline{f(X)} = Y$	f_* is injective.

(1): Every $x_0 \in \mathbb{C}^d$ defines an algebra homomorphism

$$\begin{tikzcd}
\alpha_{x_0}: \mathbb{C}[x_1, \dots x_d] \arrow[r] \arrow[d] & \mathbb{C} : P \mapsto P(x_0) \\
\mathcal{A}_X = \mathbb{C}[x_1, \dots x_d]/I(I) \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

$$I(X) \subset \operatorname{Ker}(\alpha_{x_0}) \iff x_0 \in X$$

(3) For a polynomial map $f: \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$,

$$
\begin{tikzcd}
f_*: \mathbb{C}[x_1,\dots x_{d_2}] \arrow[r] \arrow[d] & \mathbb{C}[x_1,\dots x_{d_1}] \arrow[d] : P \longmapsto P\circ f \\
\mathbb{C}[x_1,\dots x_{d_2}]/I(Y) \arrow[r, dashed, "?"] & \mathbb{C}[x_1,\dots x_{d_1}]/I(X)
\end{tikzcd}
$$

$$f_*(I(Y)) \subset I(X) \iff \forall P \in I(Y):\ P\circ f \in I(X)$$
$$\Updownarrow$$
$$f(X) \subset Y$$

(4) $\overline{f(X)} = Y \iff \forall P \in I(Y):\ P(f(X)) \neq 0.$
$$\Updownarrow$$
$$\forall P \in I(Y):\ f_*(P) \notin I(X)$$
$$\Updownarrow$$
$$f_* \text{ is injective}$$

<u>Thm.</u> Let $f: X \to Y$ be a polynomial map between closed sets. Then $f(X) \supset$ open subset of $\overline{f(X)}$.

<u>Ex.</u> Show that this is false for algebraic maps between algebraic sets in $\mathbb{R}^d$!

<u>Proof.</u> We reduce proof to the case when X and Y are irreducible and $Y = \overline{f(X)}$. Then we have an injective algebra homomorphism $f_*: a_Y \to a_X$.

$$\underset{\substack{\parallel \\ A}}{f_*(a_Y)} \subset \underset{\substack{\parallel \\ B}}{a_X}.$$

$$\{\text{points of } f(X)\} \longleftrightarrow \{\text{homomorphisms } B \to \mathbb{C}\}$$
$$\cap \qquad\qquad\qquad\qquad \cap$$
$$\{\text{points of } Y\} \longleftrightarrow \{\text{homomorphisms } A \to \mathbb{C}\}$$

We need to show that:

$$\boxed{\exists\, P \in a_Y: \underbrace{\{P \neq 0\} \cap Y}_{\text{open in } Y} \subset f(X)} \iff \boxed{\begin{array}{l}\exists\, p \in A: \\ \text{every } \varphi: A \to \mathbb{C} \text{ with } \varphi(p) \neq 0 \\ \text{extends to } B\end{array}}$$

The later statement is proved in Commutative algebra (see Lang's book).

<u>Thm</u> Let $f: G_1 \to G_2$ be a polynomial homomorphism of algebraic groups. Then $f(G)$ is an algebraic group.

<u>Proof.</u> Let $L = f(G_1)$. We use that multiplication and inverse are continuous in Zariski topology.

Since $L^{-1} \cdot L \subset L \Rightarrow \overline{L}^{-1} \cdot \overline{L} \subset \overline{L^{-1} \cdot L} \subset \overline{L}$
continuity

This shows that $\overline{L}$ is a group.

By Thm, $L \supset$ open subset of $\overline{L}$, so that L is an open subgroup of $\overline{L}$.

Now $\overline{L} = \bigsqcup_{\ell \in \overline{L}/L} \ell \cdot L$. Hence, L is closed.
open

<u>Def</u> An algebraic set $X \subset \mathbb{C}^d$ is defined over K (for a subfied $K \subset \mathbb{C}$) if $I(X)$ is generated by elements of $K[x_1, \dots x_d]$.

<u>ex.</u> $Sl_d(\mathbb{C})$ is defined over $\mathbb{Q}$.

<u>Thm:</u> Let G be an algebraic group defined over $\mathbb{R}$. Assume that G is connected (as a Lie group). Then $G(\mathbb{R}) = G \cap M_d(\mathbb{R})$ is Zariski dense in G.

<u>ex.</u> If $G = \{P_1 = \dots = P_s = 0\}$, then

$$\mathrm{Lie}(G) = \{X \in M_d(\mathbb{C}) : (DP_1)_e \cdot X = \dots = (DP_s)_e \cdot X = 0\};$$

$$\mathrm{Lie}(G(\mathbb{R})) = \{X \in M_d(\mathbb{R}) : \text{——} \,''\, \text{——}\}.$$

<u>Proof.</u> Let L be the Zariski closure of $G(\mathbb{R})$.
Then $\mathrm{Lie}(L) \supset \mathrm{Lie}(G(\mathbb{R})) \otimes \mathbb{C} = \mathrm{Lie}(G)$.
This implies that $L \supset$ (Euclidean) nbhd of e in G (indeed, use the exponential map).
Since G is connected (as a Lie group), it follows that $L = G$.

<u>Thm.</u> Let G be an irreducible algebraic group defined over $\mathbb{R}$,
$X \subset \mathbb{C}^d$ an algebraic set defined over $\mathbb{R}$,
$x \in X(\mathbb{R}) = X \cap \mathbb{R}^d$,
$G \times X \to X$ is polynomial action defined over $\mathbb{R}$.

Then the map $G(\mathbb{R}) \longrightarrow \overline{G \cdot x} \cap \mathbb{R}^d$, $g \longmapsto g \cdot x$ is open in $\mathbb{R}$-topology.

Proof. We may assume that $X = \overline{G\cdot x}$, and X is irreducible. We know that $G\cdot x$ contains a Zariski-open subset of X. Since G acts transitively on $G\cdot x$, $G\cdot x$ is, in fact, Zariski open in X.

Let X_0 be the set of smooth points in X. This set is also Zariski open in X, and G-inv. Since X is irreducible, $G\cdot x \cap X_0 \neq \emptyset$.

Hence, $G\cdot x \subset X_0$.

Consider the map $F: g \to g\cdot x$ and its derivative $(DF)_g: T_g(G) \to T_{gx}(X)$, where $T_g(G)$ and $T_{gx}(X)$ denote the tangent planes to surfaces G and X.

Since $G\cdot x$ is Zariski open in X, $(DF)_g$ is onto.

Then $(DF)_g: T_g(G(\mathbb{R})) \to T_{gx}(X(\mathbb{R}))$ is onto.

Hence, by the Implicit function theorem, the map $F: G(\mathbb{R}) \to X(\mathbb{R})$ is open. $\square$

Def. Let $\{s(t)\}$ be a 1-par. group acting on a space X. A point $x \in X$ is called recurrent if
$$u(t_n)x \to x \quad \text{for some } t_n \to \infty.$$

Cor. Let $S = \{s(t)\}_{t\in\mathbb{C}}$ be a 1-par. algebraic group defined over $\mathbb{R}$, $X \subset \mathbb{C}^d$ an algebraic set, and $S \times X \to X$ a polynomial action defined over $\mathbb{R}$. Then all $S(\mathbb{R})$-recurrent points in $X(\mathbb{R})$

are fixed by S.

Proof. By the previous thm, $s((-\varepsilon,\varepsilon))\cdot x$ is open in $\overline{S(\mathbb{R})x}$. If $s(t_n)x \to x$, then $s(t_n)x \in s((-\varepsilon,\varepsilon))x$ for all sufficiently large n. Hence, $Stab_S(x)$ is infinite, so that it is Zariski dense in S. On the other hand, $Stab_S(x)$ is an algebraic subgroup. Hence, $Stab_S(x) = S$. $\square$

Semisimple and unipotent elements.

Def. 1) $g \in GL_d(\mathbb{C})$ is called semisimple if it is diagonalisable over $\mathbb{C}$.

2) $g \in G$ is called unipotent if all eigenvalues of g are $= 1$.

Jordan canonical form $\Rightarrow$ $\forall g \in GL_d(\mathbb{C})$: $g = g_s g_u = g_u g_s$ where g_s is semisimple, g_u is unipotent.

Thm. Let $\rho: G \to GL_N(\mathbb{C})$ be a polynomial homomorphism. Then

1) g - semisimple $\Rightarrow$ $\rho(g)$ - semisimple;
2) g - unipotent $\Rightarrow$ $\rho(g)$ - unipotent.

Proof. 1) Let V_λ be a Jordan subspace for $\rho(g)$. Then $\lambda^{-n}\rho(g)^n|_{V_\lambda}$ is polynomial in n. On the other hand, $\lambda^{-n}\rho(g^n)|_{V_\lambda}$ is polynomial in $\lambda^{-n}, \lambda_1^n, \dots \lambda_s^n$ where λ_i's are eigenvalues of g. Hence, $\lambda^{-n} \rho(g)^n|_{V_\lambda} = \text{const}$, and $\rho(g)$ is semisimple.

2) Let v be an eigenvector of $\rho(g)$ with eigenvalue λ. Then $\rho(g)^n \cdot v = \lambda^n v$, but $\rho(g^n) \cdot v$ is polynomial in n. Hence, $\lambda = 1$.