Lecture 3: Finite-dimentional representations

G = a connected Lie group

Def. A representation is a continuous homomorphism
$$\rho: G \to GL_d(\mathbb{C}).$$

Aim: Find a "canonical form" for ρ
(e.g., Jordan canonical form)

Situation is quite different for $\begin{cases} \text{solvable groups} \\ \text{semisimple groups} \end{cases}$

For a Lie algebra $\mathfrak{g}$, we set
$$\mathfrak{g}^{(1)} = \langle [x,y] : x,y \in \mathfrak{g} \rangle, \quad \mathfrak{g}^{(2)} = \langle [x,y] : x,y \in \mathfrak{g}^{(1)} \rangle, \ldots$$

Def. G is solvable if $\mathfrak{g}^{(n)} = \{0\}$ for some n.

example: $\mathfrak{g} = \left\{ \underbrace{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}}_{d} \right\}$. Then $\mathfrak{g}^{(d)} = 0$.

Thm (Lie-Kolchin) Let $\rho: G \to GL_d(\mathbb{C})$ be a representation of a connected solvable group G. Then
$$\exists g \in GL_d(\mathbb{C}): \quad g \cdot \rho(G) \cdot g^{-1} \subset \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}.$$

Lem. For open $U \subset G$, $G = \langle U \rangle$.

Proof. The group $H = \langle U \rangle$ is open in G.
Indeed, $\forall h \in H$: $\underbrace{h \cdot U}_{\text{open}} \subset H$.

$$G = \bigsqcup_{g \in G/H} \underbrace{gH}_{\text{open}} \Rightarrow G = H \text{ by connectedness.}$$

<u>Proof.</u> We have $D\rho : \underset{\mathfrak{g}}{\overset{\|}{Lie(G)}} \longrightarrow M_d(\mathbb{R})$ — Lie-algebra homomorphism

such that $\rho(\exp(X)) = \exp(D\rho(X))$. Let $\mathfrak{h} = D\rho(\mathfrak{g})$.

If $\mathfrak{h} < \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$, then $\rho(U) < \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ for open $U \subset G$,

and by Lemma, $\rho(G) \subset \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$. Hence, it remains

to show that $D\rho(\mathfrak{g}) \subset \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$, up to conjugation.

Moreover, if $\exists \underset{\neq 0}{v} \in \mathbb{C}^d : D\rho(\mathfrak{g}) \cdot v \subset \mathbb{C} \cdot v$, then

we finish by induction on d.

Let $\mathfrak{h}_0$ be a codim. 1 subspace $\supset \mathfrak{h}^{(1)}$.

Then $\mathfrak{h} = \langle x, \mathfrak{h}_0 \rangle$ for some $x \in \mathfrak{h}$.

By induction on $\dim(\mathfrak{h})$, $\exists v \neq 0 : \mathfrak{h}_0 \cdot v = \mathbb{C} \cdot v$.

For $y \in \mathfrak{h}_0$, $y \cdot v = \lambda(y) \cdot v$. Let $v_i = x^i \cdot v$. Then

$$y \cdot v_i = yx \cdot v_{i-1} = xy \cdot v_{i-1} + \underset{\mathfrak{h}_0}{\underset{\cap}{[y,x]}} \cdot v_{i-1} \,. \quad (*)$$

By induction on i, $y \cdot v_i - \lambda(y) v_i \in \langle v_0, \dots v_{i-1} \rangle$.

Let $V = \langle v_0, v_1, \dots \rangle$. In this basis,

$$[y,x]|_V = \begin{pmatrix} \lambda([y,x]) & & * \\ & \ddots & \\ 0 & & \lambda([y,x]) \end{pmatrix}$$

$$0 = Tr([y,x]|_V) = \dim(V) \cdot \lambda([y,x]) \Rightarrow \lambda([y,x]) = 0.$$

Now using $(*)$, we prove by induction on i that $y \cdot v_i = \lambda(y) \cdot v_i$ for all $y \in \mathfrak{h}_0$. Hence, $y|_V = \lambda(y) \cdot I$, and every eigenvector of x in V is also an eigenvector of $\mathfrak{h} = \langle x, \mathfrak{h}_0 \rangle$, and hence $\mathfrak{h}$ has an eigenvector in V, which completes the proof.

<u>Def.</u> A connected Lie group is called <u>semisimple</u> if it contains no normal connected closed solvable subgroups $\neq \{e\}$.

<u>Thm</u> If G is semisimple, and $\rho: G \to GL_d(\mathbb{C})$ is a representation. Then every G-inv. $V \subset \mathbb{C}^d$ has a G-inv. complement.

<u>Cor.</u> $\mathbb{C}^d = V_1 \oplus \ldots \oplus V_s$, where V_i's are G-inv. and irreducible.

<u>Proof (Weyl unitary trick)</u>

1) Assume that G is compact.

Fix a positive-definite Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^d$, and define a new Hermitian form by:

$$\langle v_1, v_2 \rangle_G = \int_G \langle \rho(g^{-1}) v_1, \rho(g^{-1}) v_2 \rangle \, dm(g)$$

where m is a left inv. measure on G.

Note that m is finite because G is compact.
Then $\langle\cdot,\cdot\rangle_G$ is a positive-def. Hermitian too.
Moreover, it is G-invariant:

$$\langle \rho(h)v_1, \rho(h)v_2\rangle_G = \int_G \langle \underbrace{\rho(g^{-1})\rho(h)}_{\rho((h^{-1}g)^{-1})} v_1, \underbrace{\rho(g^{-1})\rho(h)}_{\rho((h^{-1}g)^{-1})} v_2\rangle \, dm(g)$$

$$= \int_G \langle \rho(g^{-1})v_1, \rho(g^{-1})v_2\rangle \, dm(g) = \langle v_1, v_2\rangle_G .$$

Now $\mathbb{C}^d = V \oplus V^\perp$ where $V^\perp = \{v : \langle v, V\rangle_G = 0\}$.
For $v \in V^\perp$, $\langle \rho(g)v, V\rangle = \langle v, \rho(g)^{-1} V\rangle = 0$.
Hence, $V^\perp$ is G-invariant.

2) Assuming $G = SL_2(\mathbb{R})$ (this can be extended to general semisimple G).

We have $D\rho : \underset{\mathfrak{g}}{\underset{\shortparallel}{\mathrm{Lie}(G)}} \longrightarrow M_d(\mathbb{C})$ – Lie-algebra homomorphism.

Note that $W \subset \mathbb{C}^d$ is G-Inv. $\Longleftrightarrow$ W is $\mathfrak{g}$-inv.
(because G is generated by a nbhd of e).
It remains to show that V has $\mathfrak{g}$-inv. complement

$$\begin{array}{ccc} \mathfrak{g} = \{X \in M_2(\mathbb{R}) : \mathrm{Tr}(X) = 0\} & \xrightarrow{D\rho} & M_d(\mathbb{C}) \\ \cap & \nearrow_{(D\rho)_{\mathbb{C}}} & \\ \mathfrak{g}\otimes\mathbb{C} = \{X \in M_2(\mathbb{C}) : \mathrm{Tr}(X) = 0\} & & \end{array}$$

Linear extension of $D\rho$.

Let $H = SU(2) = \left\{ g \in Gl_2(\mathbb{C}) : \begin{array}{l} \bar{g}^T g = I \\ \det(g) = 1 \end{array} \right\}$

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} : |a|^2 + |b|^2 = 1 \right\}.$$

$$\mathfrak{h} = Lie(H) = \left\{ X \in M_2(\mathbb{C}) : \begin{array}{l} \bar{X}^T + X = 0 \\ Tr(X) = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} u & v \\ -\bar{v} & -u \end{pmatrix} : u \in \mathbb{R},\ v \in \mathbb{C} \right\}.$$

Since $\mathfrak{h} \otimes \mathbb{C} = \{ X \in M_2(\mathbb{C}) : Tr(X) = 0 \} = \mathfrak{g} \otimes \mathbb{C}$,

we have the map:

$$\begin{tikzcd} \mathfrak{h} \otimes \mathbb{C} \arrow[r, "D\rho_{\mathbb{C}}"] & M_d(\mathbb{C}) \\ \mathfrak{h} \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

Because H is simply connected (in fact, $H \simeq$ 3-sphere)

$\exists$ representation $\tilde{\rho} : H \to Gl_d(\mathbb{C})$ such that $D\tilde{\rho} = D\rho_{\mathbb{C}}|_{\mathfrak{h}}$.

Note that V is $\mathfrak{h}$-inv. and H-inv.

Since H is compact, V has H-inv. complement which is

$\mathfrak{h}$-inv. $\Rightarrow$ $(\mathfrak{h} \otimes \mathbb{C})$-inv. $\Rightarrow$ $(\mathfrak{g} \otimes \mathbb{C})$-inv. $\Rightarrow$ $\mathfrak{g}$-inv. $\Rightarrow$ G-inv.