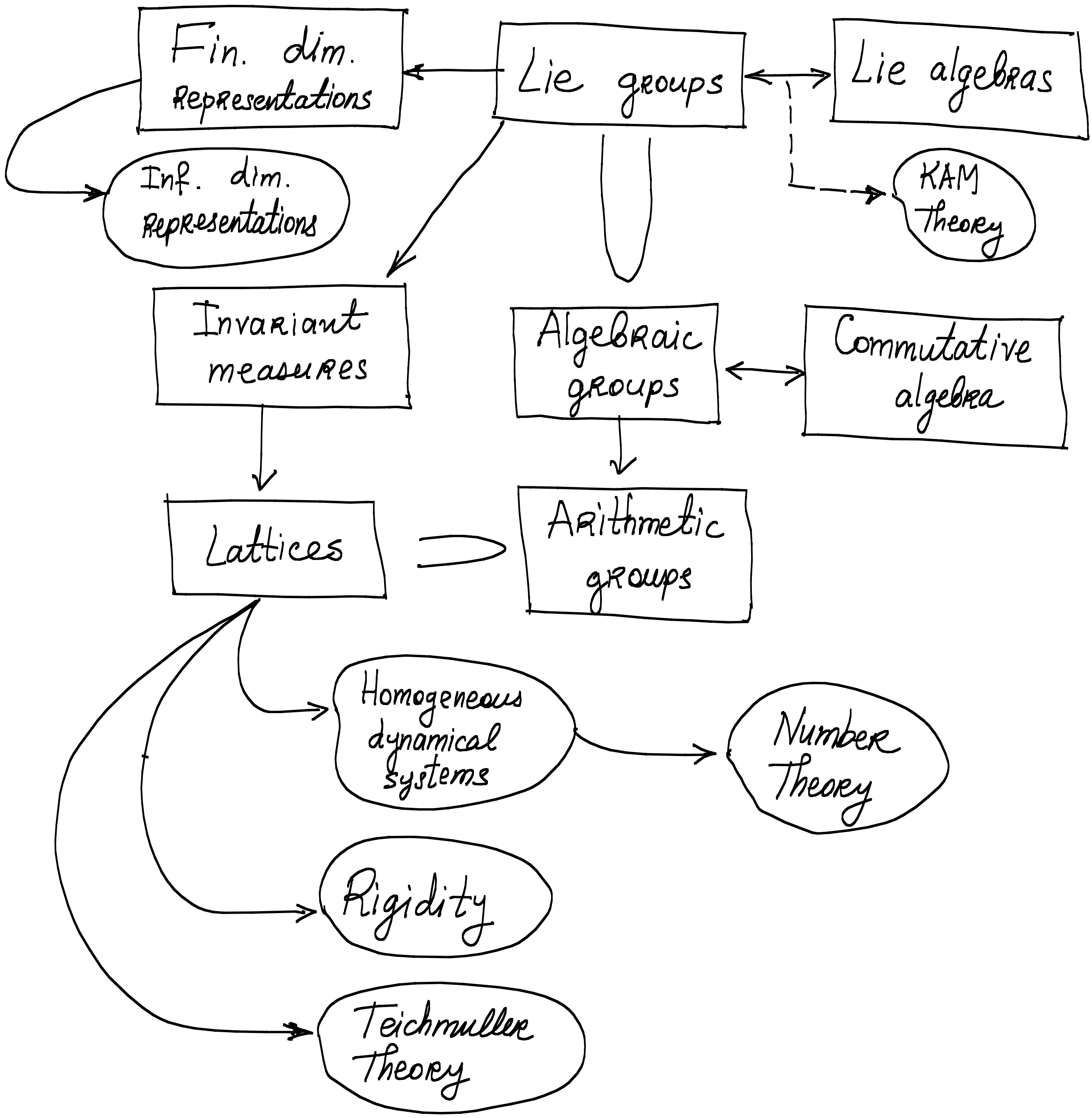
Plan.



Fin. dim. representations
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Lecture I: Lie groups and Lie algebras.

Def: A (matrix) Lie group is a closed subgroup of $GL_d(\mathbb{R})$ (or of $GL_d(\mathbb{C})$).

the group of invertable matrices.

Topology on $GL_d(\mathbb{R})$ is defined by the norm:

$$\|X\| = \sqrt{\sum_{i,j} |x_{ij}|^2}.$$

examples:
1) $SL_d(\mathbb{R}) = \{g : \det(g) = 1\}$,
2) $SO_d(\mathbb{R}) = \left\{ g : \begin{array}{l} g^T \cdot g = I \\ \det(g) = 1 \end{array} \right\}$,
3) $SL_d(\mathbb{Z})$.

Def A one-parameter group is a continuous homomorphism $S : \mathbb{R} \to GL_d(\mathbb{R})$.

For $A \in M_d(\mathbb{R})$, define $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$.
Since $\|A^n\| \leq \|A\|^n$, this series converges uniformly on compact sets, and $\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is an analytic map.

Properties: 1) $\exp(A^T) = \exp(A)^T$;

2) For $g \in GL_d(\mathbb{R})$, $\exp(g \cdot A \cdot g^{-1}) = g \cdot \exp(A) \cdot g^{-1}$.

3) If $AB = BA$, then $\exp(A+B) = \exp(A) \cdot \exp(B)$.

4) $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$.

Proof. (1) & (2) are easy.

(3): $$\exp(A)\cdot\exp(B) = \sum_{n,m=0}^{\infty} \frac{A^n B^m}{n!\cdot m!} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\sum_{m+n=\ell} \frac{\ell!}{n!\,m!} A^n B^m\right)$$

$$\underset{\text{by binomial formula}}{=} \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(A+B)^{\ell} = \exp(A+B)$$

(4) Using the Jordan normal form,

$$A = g\,(A_1 + A_2)\,g^{-1}, \quad g \in GL_d(\mathbb{R}), \quad A_1 \cdot A_2 = A_2 \cdot A_1,$$

$$A_1 = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

Because of (2)&(3), it remains to check the equality for A_1 and A_2:

$$\det(\exp(A_1)) = \det\begin{pmatrix} e^{a_1} & & 0 \\ & \ddots & \\ 0 & & e^{a_d} \end{pmatrix} = \exp(\mathrm{Tr}(A_1)),$$

$$\det(\exp(A_2)) = \det\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1 = \exp(\mathrm{Tr}(A_2)).$$

$\square$

Therefore, $S(t) = \exp(tA)$ defines a 1-par. group.

Note that $S(t) = I + t\cdot A + O(t^2)$ for $t \approx 0$.

Hence, $\left.\frac{dS}{dt}\right|_{t=0} = A$.

This implies that $(D\exp)_0 = I$, and by the inverse function Thm, exp gives a bijection from a small nbhd of 0 to a nbhd of I.

<u>Thm.</u> Every 1-par. group is of the form $S(t) = \exp(tA)$.

<u>Claim:</u> $\exists r > 0$: $\begin{cases} \|Y_1 - I\| < r \\ \|Y_2 - I\| < r \end{cases}$, $Y_1^2 = Y_2^2 \implies Y_1 = Y_2$

Write $Y_i = I + A_i$. Then

$(I + A_1)^2 = (I + A_2)^2 \implies 2A_1 - 2A_2 = A_2^2 - A_1^2 = A_2(A_2 - A_1) + (A_2 - A_1)A_1$

Hence, $2\|A_1 - A_2\| \leq (\|A_1\| + \|A_2\|) \cdot \|A_1 - A_2\|$.

If $\|A_1\| + \|A_2\| < 2$, then $A_1 = A_2$.

<u>Proof of Thm:</u> By continuity, $\exists \varepsilon, \delta > 0$:

$$S([-\varepsilon, \varepsilon]) \subset \exp(\{\|X\| < \delta\}) \subset \{\|Y - I\| < r\}.$$

Then $S(\varepsilon) = \exp(\varepsilon A)$ with $\|A\| < \frac{\delta}{\varepsilon}$.

Now $S(\frac{\varepsilon}{2})^2 = \exp(\frac{\varepsilon}{2} A)^2 \implies S(\frac{\varepsilon}{2}) = \exp(\frac{\varepsilon}{2} A)$.

This implies that $S(\frac{m}{2^n}\varepsilon) = \exp(\frac{m}{2^n}\varepsilon A)$ for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

Hence, by continuity, $S(t\cdot\varepsilon) = \exp(t\cdot\varepsilon)$ for all $t \in \mathbb{R}$.

Lie algebras.

We define:

$$Lie(G) = \{ X \in M_d(\mathbb{R}) : \ exp(tX) \in G \text{ for all } t \in \mathbb{R} \}.$$

<u>Def.</u> A subspace $\mathfrak{g}$ of $M_d(\mathbb{R})$ is called <u>Lie algebra</u> if $\forall X, Y \in \mathfrak{g}$: $[X,Y] \overset{def}{=} XY - YX \in \mathfrak{g}$

Lie bracket

<u>Thm:</u> Lie (G) is a Lie algebra.

If $A, B \in M_d(\mathbb{R})$ with $\|A\|, \|B\| < r$ (r is small), then $exp(A) \cdot exp(B) \approx I$ and $exp(A) \cdot exp(B) = exp(C)$ for a unique $C \in M_d(\mathbb{R})$.

<u>Lem.</u> $C = A + B + \frac{1}{2}[A,B] + O(r^3)$.

<u>Proof.</u>

- $exp(A) = I + O(r)$ and $exp(B) = I + O(r)$.
  Hence, $C = exp^{-1}(I + O(r)) = O(r)$.
- $exp(C) = I + C + O(r^2)$.
  $$exp(A) \cdot exp(B) = (I + A + O(r^2)) \cdot (I + B + O(r^2))$$
  $$= I + A + B + O(r^2)$$
  Hence, $C = A + B + O(r^2)$.

- Let $C = A+B+S$ for some $S \in M_d(\mathbb{R})$.
  We know that $S = O(r^2)$.
  $$\exp(C) = I + A + B + S + \tfrac{1}{2}(A+B+S)^2 + O(r^3)$$
  $$= I + A + B + S + \tfrac{1}{2}(A+B)^2 + O(r^3)$$
  $$\exp(A)\cdot\exp(B) = \left(I + A + \tfrac{A^2}{2} + O(r^3)\right)\left(I + B + \tfrac{B^2}{2} + O(r^3)\right)$$
  $$= I + A + B + AB + \tfrac{A^2}{2} + \tfrac{B^2}{2}.$$
  Hence, $S = \left(A + B + AB + \frac{A^2}{2} + \frac{B^2}{2}\right) - \left(A + B + \frac{1}{2}(A+B)^2\right) + O(r^3)$
  $$= \tfrac{1}{2}[A, B] + O(r^3).$$

In fact, we have

<u>Campbell – Baker – Hausdorff Formula</u>:
$$C(A,B) = \sum_{n=0}^{N} C_n(A,B) + O(r^{N+1})$$
where C_n = homogeneous polynomial of $\deg = n$ expressed in terms of Lie brackets.

<u>Cor.</u> 1) $\exp(A+B) = \lim_{n\to\infty} \left(\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right)\right)^n.$

2) $\exp([A,B]) = \lim_{n\to\infty} \left[\exp\left(\frac{A}{n}\right), \exp\left(\frac{B}{n}\right)\right]^{n^2}.$

(here: $[g_1, g_2] = g_1 \cdot g_2 \cdot g_1^{-1} g_2^{-1}$)

<u>Proof.</u> $\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right) = \exp(C_n)$,
where $C_n = \frac{A+B}{n} + O\left(\frac{1}{n^2}\right).$

Hence, $\left(\exp\left(\frac{A}{n}\right)\cdot\exp\left(\frac{B}{n}\right)\right)^n = \exp\left(A + B + O\left(\frac{1}{n}\right)\right) \longrightarrow \exp(A+B).$
The proof of (2) is similar.

We begin the proof of Thm by

Observation: If $C_n \in \exp^{-1}(G)$, $C_n \to 0$, $s_n C_n \to D$ for $s_n \in \mathbb{R}$, then $D \in Lie(G)$.

Indeed, set $m_n = \lfloor t s_n \rfloor$ with $t \in \mathbb{R}$, then $m_n C_n \to tD$. Since $m_n C_n \in \exp^{-1}(G)$, $tD \in \exp^{-1}(G)$ and $D \in Lie(G)$.

Proof of Thm.

Suppose that $A, B \in Lie(G)$

- By Lemma, $\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right) = \exp(C_n) \Rightarrow C_n \in \exp^{-1}(G)$, where $n \cdot C_n \longrightarrow A+B$.

By Observation, $A+B \in Lie(G)$.

- By Lemma, $\left[\exp\left(\frac{A}{n}\right), \exp\left(\frac{B}{n}\right)\right] = \exp(C_n) \Longrightarrow C_n \in \exp^{-1}(G)$ where $n^2 \cdot C_n \to [A,B]$. By Observation, $[A,B] \in Lie(G)$.

Thm. exp maps (bijectively) a nbhd of 0 in $\mathfrak{g}$ to a nbhd of I in G.

Proof. We write $M_d(\mathbb{R}) = Lie(G) \oplus V$ for a subspace $V \subset M_d(\mathbb{R})$.

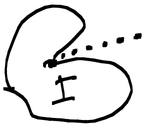
$$\begin{tikzcd} & M_d(\mathbb{R}) = Lie(G) \oplus V \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\pi}"] & \\ Lie(G) & & V \end{tikzcd}$$

Let $F: M_d(\mathbb{R}) \longrightarrow M_d(\mathbb{R}): A \longmapsto \exp(\pi(A))\exp(\tilde{\pi}(A))$

We have $\frac{d}{dt}\left(\exp(\pi(tA))\exp(\tilde{\pi}(tA))\right)\Big|_{t=0} = \pi(A) + \tilde{\pi}(A) = A$.

Hence, $(DF)_0 = I$, and $F: \mathcal{O} \to F(\mathcal{O})$ is a

Bijection for a sufficiently small nbhd $\mathcal{O}$ of 0.
Suppose that $\exp(\mathcal{O} \cap \mathrm{Lie}(G)) \subset G$ is not a nbhd of I in G. Then $\exists B_n \to 0$: $\exp(B_n) \in G$, $B_n \notin \mathrm{Lie}(G)$



We can write: $\exp(B_n) = F(A_n)$ for $A_n \to 0$
Since $B_n \in \mathrm{Lie}(G)$, $\tilde{\pi}(A_n) \neq 0$.
We have $\exp(\tilde{\pi}(A_n)) = \exp(\pi(A_n))^{-1} \exp(B_n) \in G$.
After passing to a subsequence,

$$\frac{\tilde{\pi}(A_n)}{\|\tilde{\pi}(A_n)\|} \longrightarrow C. \quad \text{Clearly, } C \in V.$$

On the other hand, by the observation $C \in \mathrm{Lie}(G)$, which is a contradiction.

<u>Rmk</u> This proof, in fact, shows that in a nbhd of I. G is the zero locus of $\tilde{\pi} \circ F^{-1} = 0$.
Moreover, $\mathrm{Lie}(G)$ can be identified with the tangent space of G at I.

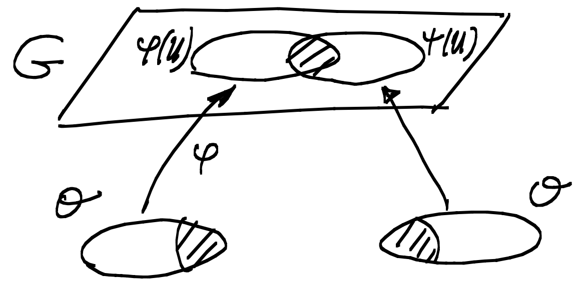
<u>Manifold structure:</u>

Fix an open nbhd V of 0 in $M_d(\mathbb{R})$ such that $V \underset{\exp^{-1}}{\overset{\exp}{\rightleftarrows}} \exp(V)$ is smooth, and set

$$\mathcal{O} = \mathrm{Lie}(G) \cap V.$$

For $g \in G$, define a coordinate chart around g

$$\varphi : \mathcal{O} \longrightarrow G : X \longmapsto g\exp(X).$$



Compatibility: if $\psi : \mathcal{O} \longrightarrow G$ is another coordinate chart, then the map $\psi^{-1}\varphi : \varphi^{-1}(\varphi(\mathcal{O}) \cap \psi(\mathcal{O})) \longrightarrow \psi^{-1}(\varphi(\mathcal{O}) \cap \psi(\mathcal{O}))$ is smooth.

Def. A map $f : G \to \mathbb{R}^k$ is smooth if $f \circ \varphi$ is smooth for all coordinate charts.

Rmk. The product map $G \times G \longrightarrow G : (g_1, g_2) \longmapsto g_1 \cdot g_2$, and the inverse map $G \to G : g \mapsto g^{-1}$ are smooth. This can be used to define a notion of Lie group abstractly as manifold with smooth group operations.

Thm. Let $f : G_1 \to G_2$ be continuous homomorphism of Lie groups G_1 and G_2. Then there exists a Lie-algebra homomorphism $Df : \mathrm{Lie}(G_1) \longrightarrow \mathrm{Lie}(G_2)$ such that $\exp((Df)(X)) = f(\exp(X))$, $X \in \mathrm{Lie}(G_1)$.

Proof: The map $t \longmapsto f(\exp(tX))$ defines a 1-par. group. Hence, $f(\exp(tX)) = \exp(tY)$ for some $Y \in \mathrm{Lie}(G_2)$. We define $Df(X) = Y$. It follows from the definition

that $Df(s\cdot X) = s\cdot Df(X)$ for all $s \in \mathbb{R}$.
Now it remains to show that Df is a Lie-algebra homomorphism. Namely,
$$Df(X+Y) = Df(X) + Df(Y),$$
$$Df([X,Y]) = [Df(X), Df(Y)].$$

We use the formula $\exp(A+B) = \lim\limits_{n\to\infty} \left(\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right)\right)^n$

$$\exp(Df(X_1+X_2)) = f(\exp(X_1+X_2)) = \lim_{n\to\infty} f\left(\left(\exp\left(\tfrac{X_1}{n}\right)\exp\left(\tfrac{X_2}{n}\right)\right)^n\right)$$
$$= \lim_{n\to\infty}\left(f\left(\exp\left(\tfrac{X_1}{n}\right)\right) f\left(\exp\left(\tfrac{X_2}{n}\right)\right)\right)^n = \lim_{n\to\infty}\left(\exp\left(\frac{Df(X_1)}{n}\right)\exp\left(\frac{Df(X_2)}{n}\right)\right)^n$$
$$= \exp(Df(X_1) + Df(X_2)).$$

The proof that $Df([X_1,X_2]) = [Df(X_1), Df(X_2)]$ is similar using the formula $\exp([A,B]) = \lim\limits_{n\to\infty}\left[\exp\left(\frac{A}{n}\right), \exp\left(\frac{B}{n}\right)\right]^{n^2}$.

<u>Cor.</u> Every continuous homomorphism $f: G_1 \to G_2$ is analytic.

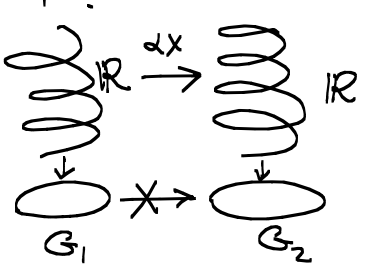
<u>Proof.</u> $f(g\cdot\exp(X)) = f(g)\exp(Df(X))$ – analytic in X.

$$F: \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2) \xrightarrow{\ ?\ } f: G_1 \longrightarrow G_2$$
$$Df = F.$$

<u>No</u>: $G_1 = G_2 = SO_2(\mathbb{R}) = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} : \theta \in \mathbb{R}\right\}$,

$\mathrm{Lie}(G_1) = \mathrm{Lie}(G_2) = \left\{\begin{pmatrix} 0 & x \\ -x & 0\end{pmatrix} : x \in \mathbb{R}\right\}$,

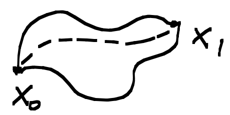
$F: x \longrightarrow \alpha\cdot x$, $\alpha \in \mathbb{R}\setminus\mathbb{Z}$.



Then F does not correspond to a homomorphism $G_1 \to G_2$.

The Lie algebra does not see the global picture.

Def. A space X is called <u>simply connected</u> if X is path-connected and any 2 paths between $x_0, x_1 \in X$ can be continuously deformed into each other, namely, for $\alpha_0, \alpha_1 : [0,1] \to X$ - continuous, $\alpha_i(0) = x_i$,



$\exists\ \alpha : [0,1]\times[0,1] \longrightarrow X$ - continuous,

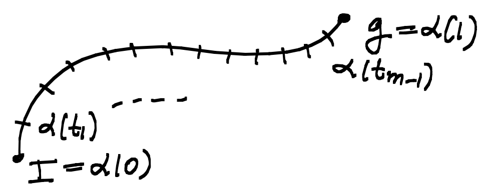
$$\alpha(0,*) = \alpha_0, \quad \alpha(1,*) = \alpha_1, \quad \alpha(*,0) = x_0, \quad \alpha(*,1) = x_1.$$

Thm: If G_1 is simply connected and $F : Lie(G_1) \to Lie(G_2)$ is a Lie-algebra homomorphism, then $\exists f : G_1 \longrightarrow G_2$ homomorphism such that $F = Df$.

Proof. Fix a small nbhd $\mathcal{U}$ of identity in G_1.

For $g \in G_1$, define $f(g) = \exp(F(\exp^{-1}(g)))$.

For general $g \in G_1$, take a continuous path $\alpha : [0,1] \to G_1$ from I to g.

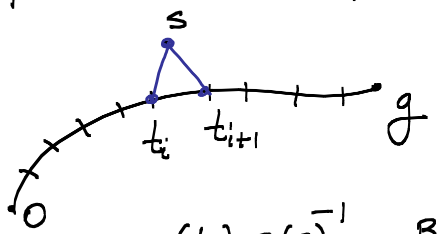


We pick a partition $\{t_i\}$ of $[0,1]$ so that $\alpha(t_{i+1}) = \mathcal{U}\cdot\alpha(t_i)$.

Then $g = \alpha(1) = \underbrace{\alpha(t_m)\alpha(t_{m-1})^{-1}}_{\in\mathcal{U}}\alpha(t_{m-1}) \ldots \quad \underbrace{\alpha(t_1)\alpha(t_0)^{-1}}_{\in\mathcal{U}}$

We define $f(g) = f(\alpha(t_m)\alpha(t_{m-1}))\ldots \quad f(\alpha(t_1)\alpha(t_0)^{-1})$.

Suppose that $\beta : [0,1] \to G_1$ be a small perturbation of α defined as follows:



$$\beta(t_i)\,\beta(s)^{-1},\quad \beta(s)\,\alpha(t_{i+1})^{-1} \in \mathcal{U}.$$

The path β gives the same $f(g)$ if

$$f\big(\underbrace{\alpha(t_i)\alpha(s)^{-1}}_{\exp(X)}\big)\cdot f\big(\underbrace{\alpha(s)\alpha(t_{i+1})}_{\exp(Y)}\big) = f\big(\underbrace{\alpha(t_i)\alpha(t_{i+1})^{-1}}_{\exp(X)\exp(Y)}\big).$$

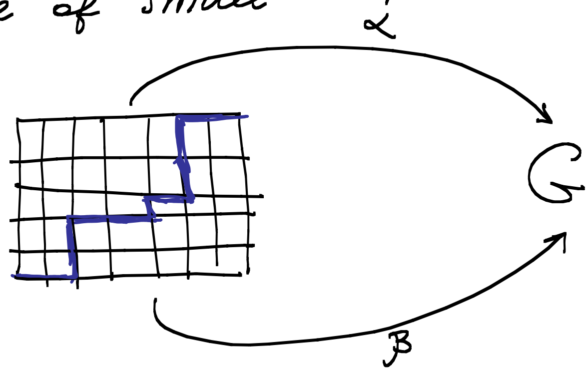
By the Baker-Campbell-Hausdorff formula,

$$f(\exp(X)\exp(Y)) = f(\exp(C(X,Y))) = \exp(F(C(X,Y)))$$

by the definition of f

$$= \exp(C(F(X), F(Y))) = \exp(F(X))\cdot\exp(F(Y)) = f(\exp(X))\,f(\exp(Y)).$$

This argument implies that $f(g)$ is independent of partition $\{t_i\}$. If $\beta : [0,1] \to G_1$ from I to g, then by simple connectedness, it can be deformed to α continuously. Moreover, this deformation can be realised as a sequence of "small" deformations as above:



This shows that f is well defined, i.e., Independent of the path.

Now we show that f is a homomorphism.

Let $g, h \in G$, and $\alpha, \beta : [0,1] \to G$ be path from I to g & h respectively. Define

$$\gamma(t) = \begin{cases} \beta(2t), & t \in [0, \tfrac{1}{2}], \\ \alpha(2t-1)h, & t \in [\tfrac{1}{2}, 1]. \end{cases}$$

Note that γ is a path from I to gh, and

$$f(gh) = \prod_i f(\gamma(t_i)\gamma(t_{i-1})^{-1}) = \left(\prod_i f(\alpha(t_i)\alpha(t_{i-1})^{-1})\right) \cdot \left(\prod_i f(\beta(t_i)\beta(t_{i-1})^{-1})\right)$$

$$= f(g) \cdot f(h).$$

$\blacksquare$