Homework 4

(1) Let X be a compact metric space and $\mathcal{P}_1$ be the family of positive continuous linear functionals ℓ on $C(X)$ such that $\ell(1) = 1$. Show that the extreme points of $\mathcal{P}_1$ are given by the functionals $\ell_x(f) = f(x)$, $x \in X$.

(2) Let $c_0(\mathbb{N})$ denote the space of sequences convergent to zero equipped with the maximum norm. Prove that $c_0(\mathbb{N})$ cannot be isomorphic to the dual of any Banach space. (Hint: Use the Krein-Milman theorem.)

(3) Let $T : X \to X$ be a homeomorphism of a compact metric space. We denote by $\mathcal{M}_1(X, T)$ the collection of probability Borel measures on X which are T-invariant. A measure $\mu \in \mathcal{M}_1(X, T)$ is called ergodic if for every T-invariant Borel subset $B \subset X$, either $\mu(B) = 0$ or $\mu(B) = 1$.

   (a) Check that $\mathcal{M}_1(X, T)$ is a compact (in weak* topology) convex nonempty set.

   (b) Show that if μ is an extreme point of $\mathcal{M}_1(X, T)$, then it is ergodic.

   (c) Deduce that there always exists an ergodic T-invariant measure.

(4) Let X and Y be compact metric spaces. Prove that the algebra of functions generated by $f(x, y) = g(x)h(y)$, with $g \in C(X)$ and $h \in C(Y)$, is dense in $C(X \times Y)$.

(5) Let X be compact metric space. Show that the space $C(X)$ of continuous functions of X contains a countable dense set. (Hint: use the functions $f_y(x) = d(x, y)$ and the Stone-Weierstrass theorem.)

(6) Let

$$K = \{f \in C([0, 1]) : f(0) = 0, f \text{ is 1-Lipschitz}\}.$$

Show that

   (a) K is convex and compact in the norm topology.

   (b) Show that any $f \in K$ which is piecewise linear with slopes ± 1 is extremal in K.

   (c) Show that the extreme points of K are dense in K.