Homework 4

(1) Consider the sequence of linear functionals ℓ_n on $L^\infty(\mathbb{R})$ defined by

$$\ell_n(f) = \frac{1}{2n} \int_{-n}^{n} f(x)\, dx$$

(a) Show that $\|\ell_n\| = 1$.

(b) Show that every limit point of ℓ_n in the weak* topology is not of the form

$$\ell(f) = \int_{\mathbb{R}} h(x) f(x)\, dx, \quad h \in L^1(\mathbb{R}).$$

In particular, $L^\infty(\mathbb{R})^* \neq L^1(\mathbb{R})$.

(2) Let X be a compact metric space. Show that the set consisting of linear combinations of Dirac measures is dense in weak* topology in the space of all finite Borel measures, and deduce that the dual space $C(X)^*$ is separable (i.e., it contains a countable dense set).

(3) Let X and Y be compact metric spaces and $f : X \to Y$ a continuous surjection. Given a finite Borel measure ν on Y such that there exists a finite Borel measure μ on X such that $\nu(B) = \mu(f^{-1}(B))$ for all Borel sets B of Y. (Hint: Use the Hahn-Banach theorem.).

(4) Let μ_n be sequence of finite Borel measures on a compact metric space. Assume that μ_n converges to μ in the weak* topology.

(a) Prove that for every closed $F \subset X$,

$$\mu(F) \geq \limsup_{n\to\infty} \mu_n(F).$$

(b) Prove that for every open $U \subset X$,

$$\mu(U) \leq \liminf_{n\to\infty} \mu_n(U).$$

(c) Give examples where these inequalities are strict.

(5) Let I be a continuous linear functional on the space $C(X)$ of continuous functions. Show that I can be written as a difference of two positive continuous linear functional following the following steps. For a nonnegative function f, define

$$I^+(f) = \sup\{I(g) : 0 \leq g \leq f\}$$

and $I^- = I^+ - I$.

(a) Show that $I^+(f) \geq 0$ and $I^-(f) \geq 0$.

(b) Show that $I^+(f) \leq \|I\|\|f\|$.

(c) Show that I^+ are linear on the cone of positive functions.

(d) Define $I^+(f)$ for all functions by setting $I^+(f) = I^+(g) - I^+(h)$ where $f = g - h$. Show that this definition I^+ does not depend on a choice of the decomposition for f and gives a continuous linear functional.

(6) Let $\{\mu_n\}$ be a sequence of probability Borel measures on a compact metric space. We say that μ_n converges to μ in total variation if

$$\sup\{|\mu_n(B) - \mu(B)| : \text{Borel subset } B \subset X\} \to 0.$$

Compare convergence in variation to the weak* convergence. Does one implies the other?