Homework 3

(1) Let X be a normed vector space.
  (a) Prove that the closed unit ball in X is closed with respect to the weak topology.
  (b) Assuming that X is infinite-dimensional, prove that every nontrivial weakly open subset of X is unbounded. In particular, the open unit ball in X is not weakly open.

(2) Let X be an infinite-dimensional normed vector space. Show that the weak closure of the set $\{\|x\| = 1\}$ is equal to $\{\|x\| \leq 1\}$.

(3) Let $c_0(\mathbb{N})$ be the space of sequences convergent to zero equipped with the max norm.
  (a) Show that every $u = (u_k) \in L^1(\mathbb{N})$ defines a continuous linear functional

$$\ell_u(x) = \sum_{k \geq 1} u_k x_k$$

  on $c_0(\mathbb{N})$ and in fact $c_0(\mathbb{N})^* = L^1(\mathbb{N})$.
  (b) Show that a sequence $x^{(n)}$ converges in $c_0(\mathbb{N})$ weakly if and only if it is bounded and each of the coordinates of $x^{(n)}$ converges.
  (c) Give an example of bounded sequence in $c_0(\mathbb{N})$ which does not have a weakly convergent subsequence.

(4) Use the Banach-Alaouglu theorem to construct an invariant mean on $L^\infty(\mathbb{Z})$.

(5) Let $1 \leq p < \infty$. Show that $L^p(\mathbb{N})^* = L^q(\mathbb{N})$ where $1/p + 1/q = 1$.

(6) Using the previous exercise, show that
  (a) For $1 < p < \infty$ there exists a sequence in $L^p(\mathbb{N})$ which converges weakly, but not strongly.
  (b) Every sequence in $L^1(\mathbb{N})$ which converges weakly also converges strongly.