Homework 2

(1) Let X be an infinite-dimensional Banach space.
  (a) Prove that every finite-dimensional subspace of X is closed and has empty interior in X.
  (b) Prove that X does not have a countable Hamel basis. (A Hamel basis is a linearly independent subset such that every element of X can be written as a finite linear combination of elements of S.)

(2) Let (c_n) be a sequence of real numbers such that for every $f \in L^1(\mathbb{N})$ the limit

$$\lim_{N\to\infty} \sum_{n=1}^{N} c_n f(n)$$

exists. Prove that the sequence (c_n) must be bounded.

(3) Prove that $L^2([0,1])$ is a meager subset of $L^1([0,1])$.

(4) (a) Show that if (s_n) is sequence of continuous functions such that $\limsup |s_n(x)| = \infty$ on a dense set of x. Then the set $\{x : \limsup |s_n(x)| = \infty\}$ is comeager.
  (b) For a function $f \in C(\mathbb{R}/\mathbb{Z})$, we denote by

$$s_n(f,x) = \sum_{m=-n}^{n} a_m e^{2\pi i m}$$

its Fourier series. Show that there exists a function $f \in C(\mathbb{R}/\mathbb{Z})$ such that $\limsup |s_n(f,x)| = \infty$ on a dense subset of $\mathbb{R}/\mathbb{Z}$.
  (c) Deduce that there exists a function $f \in C(\mathbb{R}/\mathbb{Z})$ such that $\{x \in \mathbb{R}/\mathbb{Z} : \limsup |s_n(f,x)| = \infty\}$ is comeager.

(5) In this exercise you will show that there exist continuous nowhere differentiable functions. In fact, most functions in the sense of Baire category have this property.
  (a) For $c, \epsilon > 0$, consider sets

$$S_{c,\epsilon} = \{f \in C([0,1]) : \exists x \in [0,1] : \ |f(x)-f(y)| \le c|x-y| \ \text{ for all } y \in B_\epsilon(x)\}.$$

Show that $S_{c,\epsilon}$ is closed in $C([0,1])$.
  (b) Show that piecewise linear functions are dense in $C([0,1])$.
  (c) Using (b) show that $S_{c,\epsilon}$ has empty interior in $C([0,1])$. (Hint: check that every open ball in $C([0,1])$ contains piecewise-linear functions with arbitrary large slopes.)
  (d) Prove that the set $f \in C([0,1])$ such that f is differentiable at least one point is a meager subset of $C([0,1])$.