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1 Introduction: Topics of the Lecture

Goal of the lecture: the most simple model which are not “totally realistic”. Binomial model is not realistic (at least for simple num. values)

1.1 Example of Financial Assets in the Market

- **Bonds**: (evolve like “money put into a bank account”). The interest rate is fixed (or changes slowly). It is an unrisky asset (at least less risky than other assets).
- **Stocks**: A stock is a part for a company.
- **Commodities**: Based on a physical product (oil, gold, etc.).
- **Currencies**: Exchange €, $, CHF, etc.

*Remark.* Stock, commodities and currencies are risky assets (i.e. can evolve fastly). There are more complicated financial products (in particular: derivatives: products based on other financial products).

*Example.* A product which depends on the evolution of a stock price. [derivatives ≅ options]

1.2 Some Basic Example of Options

- **Call/Put options**: (the most non-trivial derivative). It is the right but not the obligation to buy/sell a financial product (typically a stock) at a given price, at a given date in the future.

*Example.* A call (put) option on UBS at strike price $K=10$ CHF and maturity $T=1.4.2010$ is the right to buy (sell) the stock UBS for 10 CHF at the date 1.4.2010. Call and put options are exchanged in the market: the price is known.

- **Other options**: The question is: “Can we determine a price for these options, which are not directly traded in the moment?” We need to model the stock prices to answer this question.

*Example.* 1. We suppose that the price of UBS is 5 CHF and never changes. No interest rate. If I have the option, I can buy the stock a the strike price 10 CHF, and immediately sell it in the market for 15 CHF. I earn 5 CHF at time T. No interest rate ⇒ Fair price: 5 CHF
2. Price of UBS is 15 CHF now, 8 CHF at T (a.s.). Call option is useless. Fair price is 0 CHF.

3. Let us suppose that the price is 15 CHF now and can be 5 CHF or 30 CHF at time T. No interest rate. It is a binomial model. In the first case, the option is useless. In the second case, you buy the stock for 10 CHF and sell it for 30 CHF. You earn 20 CHF. I borrow 4 CHF and buy 0.8 stocks (in real markets the number of stocks is an integer).

- Initial value of the portfolio: \(0.8 \times 15 - 4 = 8\) CHF
- Terminal value: In the first case: \(5 \times 0.8 = 4\) (by giving the borrowed 4 CHF back, the terminal value is 0 CHF). In the second case: \(30 \times 0.8 - 4 = 20\). The terminal value is equal to the payoff of the option. The fair price of the option should be equal to the initial value of the portfolio: 8 CHF.

**Conclusion.** The fair price of the option depends on the model.

### 1.3 How Realistic Can a Model for a Stock Price Be?

Model 2) is unrealistic. Imagine you have a stock at the origin. You can sell the stock at price 15 CHF and buy it for 8 CHF at time T. You have earned 7 CHF without taking any risk. In this case, a lot of trader do the same operation. A lot of people want to sell the stock at the origin. (Nobody wants to buy). So the price decreases before the sellers can sell until the possibility of earning money with no risk disappears (this possibility is called arbitrage opportunity). At this moment the price is 8 CHF. The model 2) is impossible.

One generally considers the viable markets: markets in which one cannot earn money without taking any risk (i.e. no arbitrage opportunity). Expression: “No free lunch”. (In real markets, the assumption is not perfectly true).

### 1.4 Binomial Model and Its Continuous-Time Limit

Let us consider a financial market with one bound and one stock. The price of the bond is constant (no interest rate). The price of the stock is described as follows:

- The initial value is 1 (time 0).
- For \(t \in \mathbb{N}\) the value is constant in the interval of time \([t, t + 1)\), at time \(t + 1\), it is multiplied by \(a > 1\) with probability \(1/2\), and by \(1/a < 1\) with probability \(1/2\), independently of the past. One can prove that there is no arbitrage opportunity in this model.
The stock price can be described by the following picture:

It is a particular case of the binomial model. The stock price at $t$ is given by: $S_t = \alpha^{\lfloor t \rfloor} \sum_{k=1}^{\lfloor t \rfloor} \zeta_k$, where $\lfloor t \rfloor = \sup\{n \in \mathbb{N} | n \leq t \}$ is the integer part of $t$ and $(\zeta_k)_{k \in \mathbb{N}}$ are i.i.d. random variables with probability $P[k \equiv -1] = P[3k = -1] = \frac{1}{2}$. This is a discrete model. The price does not change in $(n, n + 1)$.

**Question.** This model is not totally unrealistic if the frequency of steps is high. Does there exist a “continuous time” model which is the “limit of binomial model when the frequency of steps goes to infinity”?

**Answer.** Yes!

Suppose that the number of steps by unit of time is $N$. (The changes of price one traded at times $\frac{k}{N}$, $k \in \mathbb{N}^*$). Before time $t$, the number of steps is $\lfloor Nt \rfloor$. The stock price is given by $S_t = \alpha^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \zeta_k$

Can we take the limit, when $N$ goes to infinity?

**Problem.** If $\alpha$ is constant huge fluctuations occur.

At infinity nothing realistic happens. In order to make something realistic, make the fluctuations smaller when $N$ goes to infinity. Replace $\alpha$ by $\alpha_N$, where $\alpha_N \to 1$ ($N \to \infty$). We obtain:

$$S_t = \alpha_N^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \zeta_k = \exp \left( \beta_N \sum_{k=1}^{\lfloor Nt \rfloor} \zeta_k \right), \quad \beta_N := \log(\alpha_N), \beta_N \to 0 \ (N \to \infty)$$

**Question.** How can we compute a good value of $\beta_N$?

**Answer.** We need to have $\beta_N \sum_{k=1}^{\lfloor Nt \rfloor} \zeta_k$ convergent “in some sense”.

5
\[
E \left[ \beta_N \sum_{k=1}^{[Nt]} \zeta_k \right] = \beta_N \sum_{k=1}^{[Nt]} E[\zeta_k] = 0
\]

\[
Var \left( \beta_N \sum_{k=1}^{[Nt]} \zeta_k \right) = E \left[ \left( \beta_N \sum_{k=1}^{[Nt]} \zeta_k \right)^2 \right] - \left( \beta_N \sum_{k=1}^{[Nt]} E[\zeta_k] \right)^2
\]

\[
= \beta_N^2 \sum_{k_1=1}^{[Nt]} \sum_{k_2=1}^{[Nt]} E[\zeta_{k_1} \cdot \zeta_{k_2}]
\]

\[
= \beta_N^2 \sum_{k_1=1}^{[Nt]} \sum_{k_2=1}^{[Nt]} 1
\]

\[
= [Nt] \cdot \beta_N^2
\]

\[
\approx Nt \beta_N^2 \quad (t \to \infty)
\]

\[\!\!\! For \ k_1 = k_2: \ E[\zeta_{k_1} \cdot \zeta_{k_2}] = \frac{1}{\sqrt{N}}\]

\[\!\!\! For \ k_1 \neq k_2: \ E[\zeta_{k_1} \cdot \zeta_{k_2}] = E[\zeta_{k_1}] \cdot E[\zeta_{k_2}] = 0\]

\[\!\!\! For \ t \ fixed, \ one \ needs: \ N \beta_N^2 \ convergent \ for \ N \to \infty. \ One \ chooses \ naturally: \ \beta_N = \frac{1}{\sqrt{N}}\]

\[
S_t^{[N]} = \exp \left( \frac{1}{\sqrt{N}} \sum_{k=1}^{[Nt]} \zeta_k \right)
\]

One can prove that the process \((S_t^{[N]})_{t \geq 0}\) has the limit (in a sense which can made more precise): \((S_t)_{t \geq 0}, S_t = \exp(\beta_t)\). \((\beta_t)_{t \geq 0}\) is called Brownian motion.

Black-Scholes-model: Stock price is based on the exponential of a Brownian motion.

## 2 Brownian Motion and Stochastic Calculus

Brownian motion is a stochastic process, i.e. a family of random variables indexed by the time.

**Definition 1.** A continuous stochastic process with values in \(\mathbb{R}^d\) is a family of random variables with values in \(\mathbb{R}^d\), indexed by an interval of time \(I\) (generally \(I = [0, T]\) or \(I = \mathbb{R}^+\)).

\(\forall t \in I : X_t : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}^d\), such that \(\forall \omega \in \Omega\) the function \(t \mapsto X_t(\omega)\) is continuous.

**Example.** \(\Omega = \{1, 2, 3\}, \mathcal{F} = \mathcal{P}(\Omega), \mathbb{P}\) probability measure such that \(\mathbb{P}(\{1\}) = \frac{1}{4}, \mathbb{P}(\{2\}) = \frac{1}{3}, \mathbb{P}(\{3\}) = \frac{1}{6}\). We define for all \(t \in I := [0, 20]\), \(X_t\) such that: \(X_t(1) = 0, X_t(2) = \sin(t - 1), X_t(3) = (3t - 4)(t - 3)(2t^2 - 2t)\).

What is \(X_{10}, X_0, X_1?\) (the law)

- \(X_{10}: X_{10}(0) = 0, X_{10}(2) = \sin(9), X_{10}(3) = 360360. X_{10} \in \{0, \sin(9), 360360\}\) with probability 1.
\[ \mathbb{P}(X_{10} = 0) = \mathbb{P}(\{1\}) = \frac{1}{2} \]
\[ \mathbb{P}(X_{10} = \sin(9)) = \mathbb{P}(\{2\}) = \frac{1}{3} \]
\[ \mathbb{P}(X_{10} = 360360) = \mathbb{P}(\{3\}) = \frac{1}{6} \]

- \( X_0 \): \( X_0(1) = 0, X_0(2) = -\sin(1), X_0(3) = 0 \). \( X_0 \in \{0, -\sin(1)\} \) almost surely.
  \[ \mathbb{P}(X_0 = 0) = \mathbb{P}(\{1\}) + \mathbb{P}(\{3\}) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \]
  \[ \mathbb{P}(X_0 = -\sin(1)) = \mathbb{P}(\{2\}) = \frac{1}{3} \]

- \( X_1 \): \( X_1 \in \{0\} \) almost surely.
  \[ \mathbb{P}(X_1 = 0) = 1 \]

The process is continuous because the three functions are continuous:

- \( t \mapsto 0 \)
- \( t \mapsto \sin(t - 1) \)
- \( t \mapsto (3t - 4)(t - 3)(2t^3 - 2t) \)

are continuous. This example has not much interest because the probability space is too small.

### 2.1 Brownian Motion

**Definition 2.** A Brownian motion of dimension \( d \geq 1 \) is a continuous stochastic process:

\( (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}^d \) defined on an interval of time \( I \) and satisfying the following assumptions:

- For all \( t \in I \), the coordinates \( X_t^{(1)}, \ldots, X_t^{(d)} \) of the random variables \( X_t \) are independent Gaussian variables of variance \( t \) and expectation 0. In particular \( X_0 = 0 \) almost surely.

- For \( 0 < t_1 < \cdots < t_k \), the increments of \( (X_t)_{t \in I} \), i.e. \( X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots \) are independent.

- (condition implied by the two conditions above). For all \( s, t \in I, s < t \), \( X_t - X_s \) has the same law as \( X_{t-s} \).

\[ d = 1 \]

(One can prove that the set \( (X_t^{(1)}, X_t^{(2)})_{t \in \mathbb{R}^+} \) is dense in the plane, but not all the plane).

**Proposition 1.** Brownian motion exist (admitted)

**Remark.** We can also define left-continuous, right-continuous, etc process in the same way.
2.1.1 Interpretation of the Relation with Binomial Model

Let \((\zeta_k)_{k \geq 1}\) i.i.d. random variables such that \(\mathbb{P}[\zeta_k = 1] = \mathbb{P}[\zeta_k = -1] = \frac{1}{2}\). The stock price obtained before is an exponentially increasing function of the following quantity:

\[
W_t^{[N]} = \frac{1}{\sqrt{N}} \sum_{k=1}^{[Nt]} \zeta_k
\]

Let us consider the family of random variables \((W_t^{[N]})_{t \geq 0}\) for \(N \geq 1\) we know that

\[
\sum_{k=1}^{[Nt]} \zeta_k \cdot (\sqrt{[Nt]})^{-1}
\]

converges in law to a standard Gaussian variable, by central limit theorem. \((\zeta_k)_{k \geq 1}\) are i.i.d random variables, expectation 0, variance 1, in particular square-integrable \(\Rightarrow \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \zeta_k \rightarrow N(0,1)\) if \(n \rightarrow \infty\). Now

\[
W_t^{[N]} = \frac{\sqrt{[Nt]}}{\sqrt{N}} \sum_{k=1}^{[Nt]} \zeta_k \rightarrow \sqrt{t}N(0,1) = N(0,t) \ (N \rightarrow \infty)
\]

(Compare with the first property of the BM). Let \(0 < t_1 < \cdots < t_k\):

\[
W_{t_2}^{[N]} - W_{t_1}^{[N]} = \frac{1}{\sqrt{N}} \left[ \sum_{k=[Nt_2]}^{[Nt_1]} \zeta_k - \sum_{k=1}^{[Nt_1]} \zeta_k \right]
\]

depends only on \(\zeta_1, \ldots, \zeta_{[Nt_1]}\):

\[
W_{t_2}^{[N]} - W_{t_1}^{[N]} = \frac{1}{\sqrt{N}} \sum_{k=[Nt_1]+1}^{[Nt_2]} \zeta_k
\]

deepends only on \(\zeta_{[Nt_1]+1}, \ldots, \zeta_{[Nt_2]}\).

Since the variables \((\zeta_k)_{k \geq 1}\) are independent, a family of variables which depend on disjoint sets of \((\zeta_k)_{k \geq 1}\) are necessarily independent. Hence \(W_{t_1}^{[N]}, W_{t_2}^{[N]} - W_{t_1}^{[N]}, \ldots\) are independent.

The second assumption satisfied by the Brownian motion is satisfied by \((W_t^{[N]})_{t \geq 0}\). The jumps of \(W_t^{[N]}\) are of the size \(\frac{1}{\sqrt{N}}\), in some sense \((W_t^{[N]})_{t \geq 0}\) is “almost continuous”.

The properties of the Brownian motion are “almost satisfied”. It is natural to expect that \((W_t^{[N]})_{t \geq 0}\) “tends in law” to a Brownian motion. It is possible to give a rigorous definition of these heuristics.

That is why Brownian motion is used in finance and in other areas of application.

2.2 Filtrations and Stopping Times

Definition 3. Let \(I = \mathbb{R}^+\) or \(I = [0,T]\), and let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space. A filtration on this space is a family of sub-\(\sigma\)-algebras of \(\mathcal{A}\), indexed by \(I\) (generally denoted by \((\mathcal{F}_t)_{t \in I}\) and increasing, i.e. \(\mathcal{F}_s \subset \mathcal{F}_t \ \forall s \leq t\).
• A filtration \((F_t)_{t \in I}\) is right-continuous iff for all \(t \in I\), such that \(t < T\) if \(I = [0,T]\), \(F_t = \bigcap_{s \in I, s > t} F_s\).

• A filtration \((F_t)_{t \in I}\) is complete iff for all sets \(A \in \mathcal{A}, B \subset A\), such that \(P[A] = 0 \Rightarrow B \in F_t\) \(\forall t \in I\) (i.e. \(B \in F_0\)).

Remark. The sets \(B\) are called negligible.

• One say that a filtration \((F_t)_{t \in I}\) satisfies the usual conditions iff it is right-continuous and complete.

**Definition 4** (Natural filtration). Let \((X_t)_{t \in I}\) be a continuous process defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\). Let us define \(F_t := \sigma(X_s | s \leq t)\). \(F_t\) is increasing \(\Rightarrow\) it is a filtration. It is not complete or right-continuous in general. Let us define \(F_t^+ = \bigcap_{s \in I, s > t} F_s\) (and if \(I = [0,T]\) : \(F_T^+ = F_T\)), then \(\mathcal{N} := \{\text{nonsignificant subsets of } \Omega\}\) and define: \(\tilde{F}_t := \sigma(F_t^+ \cup \mathcal{N})\). Then \((\tilde{F}_t)_{t \in I}\) is again a filtration, \(F_t \subset \tilde{F}_t\) and it satisfies the usual assumptions. \((\tilde{F}_t)_{t \in I}\) is called **natural filtration of the process** \((X_t)_{t \in I}\).

Remark. For all \(t \in I\), \(F_t\) is, by construction, measurable with respect to \(\tilde{F}_t\) and hence, with respect to \(\tilde{F}_t\). One says that \((X_t)_{t \in I}\) is **adapted** with respect to the filtration \((\tilde{F}_t)_{t \in I}\).

Intuitive meaning: The \(\sigma\)-algebra \(F_t\) represents the information available at time \(t\). (typically, the evolution of the stock price before time \(t\)). In general, the stock price at time \(u > t\) is not in \(F_t\).

### 2.2.1 Natural Filtration of a Process

Canonical filtration \(\sigma(X_s | s \leq t)\) at time \(t\). Add the nonsignificant sets \(\rightarrow\) right continuous version \(\rightarrow\) we have defined the natural filtration of \(X\) (it satisfies the natural conditions).

When we consider a stochastic process and we don’t precise the filtration, it is a priori the natural filtration of the process.

### 2.2.2 Stopping Time

Let \((F_t)_{t \in I}\) be a filtration.

1. If \(I\) is a finite interval, a random variable \(\tau\) in \((\Omega, \mathcal{A}, \mathbb{P})\) is a \((F_t)_{t \in I}\)-stopping time iff \(\tau \in I\), and for all \(t \in I\) the event \(\{\tau \leq t\}\) is in \(F_t\).

2. If \(I = \mathbb{R}^+\), then a random variable \(\tau\) from \((\Omega, \mathcal{A}, \mathbb{P})\) to \(\mathbb{R}^+ \cup \{+\infty\}\) such that for all \(t \in \mathbb{R}\) : \(\{\tau \leq t\} \in (F_t)_{t \in I}\) is called \((F_t)_{t \in I}\)-stopping time. In this case, necessarily, \(\tau\) is measurable to \(F_\infty = \sigma(F_t | t \in \mathbb{R}^+)\), which is a priori smaller than \(\mathcal{A}\) (in any case, \(F_\infty \subset \mathcal{A}\)).
2.2.3 Financial Interpretation

An instance of trading (for example the time when the trader decides to buy a stock) should be known only by using the information available at this time. If $\tau$ is such an instant, the event $\{\tau \leq t\}$ needs to be in $\mathcal{F}_t \Rightarrow \tau$ has to be a stopping time.

**Definition 5.** If $\tau$ is a stopping time, then the $\sigma$-algebra $\mathcal{F}_\tau$ is defined as follows:

$$\mathcal{F}_\tau := \{A \in \mathcal{A} | \forall t \in I : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$$

**Interpretation.** $\mathcal{F}_\tau$ represents the information available at time $\tau$. If $\tau$ is deterministic, i.e. $\tau = t_0 \in I$, then $\tau$ is a stopping time and $\mathcal{F}_\tau = \mathcal{F}_{t_0}$.

2.2.4 Properties of Stopping Times

- If $S$ is a stopping time, finite almost surely, if $X$ is a continuous process defined on a filtered probability space satisfying the usual conditions, and if $X$ is adapted (i.e. $X_t$ is $\mathcal{F}_t$-measurable for all $t \in I$), then $X_s$ is $\mathcal{F}_s$-measurable.

- If $S$ and $T$ are two stopping times and if $S \leq T$, $\mathcal{F}_S \subset \mathcal{F}_T$.

- If $S$ and $T$ are two stopping times, then $\inf(S, T)$ and $\sup(S, T)$ are also stopping times. In particular, if $T$ is a stopping time and $t$ is deterministic, then $\inf(t, T)$ and $\sup(t, T)$ are stopping times.

2.2.5 Brownian Motion with Respect to a Filtration

**Definition 6.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_{t \in I}$ be a filtration. Then a process with values in $\mathbb{R}^d$, defined on $(\Omega, \mathcal{A}, \mathbb{P})$ is an $(\mathcal{F}_t)_{t \in I}$-Brownian motion iff:

- The process is continuous.

- $X_t$ is $\mathcal{F}_t$-measurable for all $t \in I$. ($X_t$ is $(\mathcal{F}_t)_{t \in I}$-adapted).

- $X_t - X_s$ is independent of $\mathcal{F}_s$ if $s < t$; for all $A \in \mathcal{F}_s$, $B \in \mathcal{B}(\mathbb{R}^d)$, $\mathbb{P}(A \cap \{X_t - X_s \in B\}) = \mathbb{P}(A)\mathbb{P}(\{X_t - X_s \in B\})$.

- For $s \leq t$, the $d$ coordinates of $X_t - X_s$ are independent, centered variables, with variance $t - s$.

**Properties.**

- By the second point $\mathcal{F}_t$ contains necessarily the $\sigma$-algebra generated by $X_s$ for $s \leq t$.

- A Brownian motion with respect to a filtration is a Brownian motion.

- A Brownian motion is a Brownian motion with respect to its neutral filtration.
2.3 Continuous-Time Martingales

**Definition 7.** Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space and \((\mathcal{F}_t)_{t \in I}\) a filtration. Let \((M_t)_{t \in I}\) be an adapted process \((M_t)\) is \(\mathcal{F}_t\)-measurable for all \(t \in I\) such that \(\mathbb{E}[|M_t|] < \infty\) for all \(t \in I\) (i.e. \(M_t\) is integrable). \((M_t)_{t \in I}\) is:

- A submartingale if \(\forall s, t \in I, s \leq t : \mathbb{E}[M_t | \mathcal{F}_s] \geq M_s\) a.s.
- A supermartingale if \(\forall s, t \in I, s \leq t : \mathbb{E}[M_t | \mathcal{F}_s] \leq M_s\) a.s.
- A martingale if \(\forall s, t \in I, s \leq t : \mathbb{E}[M_t | \mathcal{F}_s] = M_s\) a.s.

In particular \(\mathbb{E}[M_t]\) is:

- non-decreasing in \(t\), if \((M_t)_{t \in I}\) is a submartingale.
- non-increasing in \(t\), if \((M_t)_{t \in I}\) is a supermartingale.
- constant with respect to \(t\), if \((M_t)_{t \in I}\) is a martingale.

**Proposition 2.** If \((X_t)_{t \in I}\) is a \((\mathcal{F}_t)_{t \in I}\)-Brownian motion of dimension 1.

- \((X_t)_{t \in I}\) is a martingale.
- \((X_t^2 - t)_{t \in I}\) is a martingale.
- \((\exp(\lambda X_t - t\lambda^2/2))_{t \in I}\) is a martingale for all \(\lambda \in \mathbb{C}\).

One of the reasons of dealing with stopping time is the following: Let \((M_t)_{t \in I}\) be a [super,-,sub] martingale, let \(\tau_1\) and \(\tau_2\) be two stopping times with respect to the filtration on which \((M_t)_{t \in I}\) is defined, such that there exists \(K \in \mathbb{R}_+\) with \(\tau_1 \leq \tau_2 \leq K\). Then \(M_{\tau_2}\) is integrable and \(\mathbb{E}[M_{\tau_2} | \mathcal{F}_{\tau_1}]\) \([\leq, =, \geq]\) \(M_{\tau_1}\) a.s.

In particular \(\mathbb{E}[M_{\tau_2}]\) \([\leq, =, \geq]\) \(\mathbb{E}[M_0]\) if \(\tau_2\) is a bounded stopping time.

This result is an extension of the property which define [super,-,sub] martingale to random times.

**Application.**

Let \((X_t)_{t \in I}\) be a Brownian motion and let \(T_a := \inf\{s | X_s = a\}\). Then \(T_a\) is a.s. finite (i.e. almost surely there exists \(s \in \mathbb{R}_+ : X_s = a\)) and the law of the probability of \(T_a\) is given by the Laplace transform:

\[
\mathbb{E}[e^{-\lambda T_a}] = e^{-|a|\sqrt{2\lambda}}, \forall \lambda \geq 0
\]

**Remark.** the result above is called “optional stopping theorem”, or “optional stopping time”, or “stopping theorem”.

**Lemma 3** (Dook’s inequality). Let \((M_t)_{t \in I}\) be a continuous martingale and \(p > 1\). Then:

- \(\mathbb{E}[\sup_{0 \leq s \leq t} |M_s|^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_t|^p], \forall t \in I\)
\[ \mathbb{E}[\sup_{s \in I}|M_s|^p] \leq \left( \frac{p}{p-1} \right)^p \sup_{s \in I} \mathbb{E}[|M_s|^p] \]

This result (admitted) gives some control on the supremum of |M|, only from the control on each variable (M_t).

### 2.3.1 Stochastic Integral and Itô Calculus

**Discrete-time market**: We consider a self-financing portfolio with bonds (constant price = 1) and stocks (price \( S_t \) at time \( t \in \mathbb{N} \)). We suppose that, in the portfolio, there are \( H_\gamma \) stocks on the interval of time \((\gamma - 1, t], \gamma \in \mathbb{N}^* \).

If \( V_t \) is the value of the portfolio at time \( t \), then

\[ V_t = V_0 + \sum_{\gamma=1}^t H_\gamma (S_\gamma - S_{\gamma-1}) \]

In continuous-time, we expect to have:

\[ V_t = V_0 + \int_0^t H_s dS_s \]

The problem is that the integral is not well-defined in the usual sense (the increments of \( S \) should be "too irregular"). That’s why we define the so-called stochastic integral.

**Definition 8** (Simple processes). Let \( I = [0, T] \) be a finite interval, \((\Omega, \mathcal{A}, \mathbb{P})\) a probability space and \((\mathcal{F}_t)_{t \in I}\) a filtration on this space. A process \((H_t)_{t \in I}\) is simple with respect to the filtered probability space \((\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})\) iff there exists \( 0 = t_0 < t_1 < \cdots < t_p = T \) and \( \phi_1, \ldots, \phi_p \) such that \( \phi_i \) is \( \mathcal{F}_{t_i-1} \)-measurable and bounded for all \( i = 1, \ldots, p \) and such that: \( H_t = \sum_{i=1}^p \phi_i \mathbf{1}_{(t_{i-1}, t_i]}(t) \).

**Remark.** In particular, a simple process is adapted.

**Definition 9** (Stochastic integral of simple processes (with respect to a Brownian motion)). Let \( I = [0, T] \) be an \((\mathcal{F}_t)_{t \in [0,T]}\)-Brownian motion, and \( H \) a simple process. The stochastic integral of \( H \) with respect to \( W \) is the process \((I(H))_t\) such that:

- \( (I(H))_0 = 0 \)

- For \( t \in (t_k, t_{k+1}) \) for some \( k \in \{0, \ldots, p-1\} \): \( (I(H))_t = \sum_{i=1}^k \phi_i (W_{t_i} - W_{t_{i-1}}) + \phi_{k+1} (W_t - W_{t_k}) \).

Another possibility to define \((I(H))_t\): \( (I(H))_t = \sum_{1 \leq s \leq p} \phi_s (W_{t_s \wedge t} - W_{t_{s-1} \wedge t}) \).

It is the “intuitive” version of \( \int_0^t H_s \, dW_s \).
The following proposition is the main tool of the extension of stochastic integrals to more general processes $H$.

**Proposition 4.** If $(H_t)_{t \in I}$ is a simple process,

- $(I(H))_t \in [0,T]$ is an $(\mathcal{F}_t)$-martingale.
- $\mathbb{E} \left[ |(I(H))_t|^2 \right] = \mathbb{E} \left[ \int_0^t H_s^2 \, ds \right]$.
- $\mathbb{E} \left[ \sup_{t \leq T} |(I(H))_t|^2 \right] \leq 4 \mathbb{E} \left[ \int_0^T H_s^2 \, ds \right]$.

**Proof.** Omitted

$(W_t)_{t \in [0,T]}$ a Brownian motion, $(H_s)_{s \in [0,T]}$ simple function. Random process $(I(H))_t = \int_0^t H_s \, dW_s$

- $\left( \int_0^t H_s \, dW_s \right)_{t \leq T}$ continuous $(\mathcal{F}_t)_{t \leq T}$-martingale.
- $\mathbb{E} \left[ \left( \int_0^t H_s \, dW_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t H_s^2 \, ds \right]$.
- $\mathbb{E} \left[ \sup_{t \in I} \left| \int_0^t H_s \, dW_s \right|^2 \right] \leq 4 \mathbb{E} \left[ \int_0^T H_s^2 \, ds \right]$.

We first extend the stochastic integral to the space

$$\mathcal{H} := \left\{ (H_t)_{t \in I} \left| (\mathcal{F}_t)_{t \in I}-\text{adapted process, progressively measurable, such that } \mathbb{E} \left[ \int_0^T H_s^2 \, ds \right] < \infty \right. \right\}$$

**Definition 10.** Progressively measurable means that for all $t \in [0,T] = I$ the map $(s,w) \mapsto (H_s(w))$ from $[0,t] \times \Omega$ to $\mathbb{R}$ is measurable with respect to the $\sigma$-algebra $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$

**Remark.**

- It will not be needed explicitly in this lecture.
- This property is for example satisfied if $(H_t)_{t \in I}$ is adapted, and left or right continuous (or continuous).

$\mathcal{H}$ contains the simple processes. All the processes in $\mathcal{H}$ are adapted.

**Proposition 5.** Let $(W_t)_{t \leq T}$ be an $(\mathcal{F}_t)_{t \leq T}$-Brownian motion. Then there exist a map $J$ from $\mathcal{H}$ to the space of continuous $(\mathcal{F}_t)_{t \leq T}$-martingales such that:

- If it is a simple process, then $(J(H))_t = (I(H))_t$ for all $t \leq T$, $\mathbb{P}$-almost surely.
- $J$ is linear in $\mathcal{H}$. I.e. $J(H + \lambda K) = J(H) + \lambda J(K)$ $\mathbb{P}$-almost surely $\forall H,K \in \mathcal{H}, \lambda \in \mathbb{R}$.
• If \( t \in I, H \in \mathcal{H} \) then \( \mathbb{E}[(J(H))_t^2] = \mathbb{E}\left[\int_0^t H_s^2 \, ds\right] \).

\( J \) is unique in the following sense: If \( J_1 \) and \( J_2 \) satisfy the conditions above, then almost surely:

• \( (J_1(H))_t = (J_2(H))_t \) for all \( H \in \mathcal{H}, \forall t \leq T \).

• \( J(H) \) is consistently denoted by: \( \int_0^t H_s \, dW_s \)

**Properties.** If \( (H_t)_{t \leq T} \) is in \( \mathcal{H} \):

• \( \mathbb{E} \left[ \sup_{t \leq T} |\int_0^t H_s \, dW_s|^2 \right] \leq 4 \mathbb{E} \left[ \int_0^T H_s^2 \, ds \right] \)

• If \( \tau \) is an \( (\mathcal{F}_t)_{t \leq T} \)-stopping time, then \( \mathbb{P} \)-almost surely:

\[
\int_0^\tau H_s \, dW_s = \int_0^{\tau \wedge \tau} H_s \, dW_s
\]

(In particular for deterministic \( \tau \)).

One can extend the stochastic integral to a still larger set, but one looses some properties.

\( \tilde{\mathcal{H}} := \{ (H_t)_{t \leq T} \mid (\mathcal{F}_t)_{t \leq T} \text{-adapted, progressively measurable process, } \int_0^T H_s^2 \, ds < \infty, \mathbb{P} - a.s. \} \)

**Proposition 6.** There exist a map \( \tilde{J} \) from \( \tilde{\mathcal{H}} \) to the space of continuous processes such that:

• If \( (H_t)_{t \leq T} \) is in \( \mathcal{H} \), then \( \mathbb{P} \)-almost surely \( \tilde{J}(H) = J(H) \).

• If \( (H^{(n)})_{n \geq 0} \) is a sequence of processes in \( \tilde{\mathcal{H}} \) such that \( \int_0^T (H^{(n)}_s)^2 \, ds \to 0 \) \( (n \to \infty) \) in probability. (i.e. \( \mathbb{P} \left[ \left| \int_0^T (H^{(n)}_s)^2 \, ds \right| \geq \varepsilon \right] \to 0 \) \( (n \to \infty) \) \( \forall \varepsilon > 0 \)) then, \( \sup_{t \leq T} |(\tilde{J}(H^{(n)}))_t| \to 0 \) \( (n \to \infty) \) in probability.

• \( \tilde{J} \) is linear.

The map is unique in the sense explained above. We can again denote: \( (\tilde{J}(H))_t = \int_0^t H_s \, dW_s \) (consistent with the previous notation).

**2.3.2 Itô Calculus**

Once the stochastic integral is constructed, it is natural to ask how we can do some computation on them. For example:

• Can we compute the integral \( \int_0^t W_s \, dW_s \)?

• Can we write, for example, \( \exp(-W_t^4) \) as a stochastic integral?
Let $f$ be a continuous differentiable function from $\mathbb{R}$ to $\mathbb{R}$, we want to write $f(W_t)$ as a stochastic integral. (Second question above).

If $W$ were differentiable with respect to the time, we would expect the following computation:

$$f(0) + f(W_t) = \int_0^t \frac{d}{ds}(f(W_s)) \, ds$$

$$= \int_0^t W'_s f'(W_s) \, ds$$

$$= \int_0^t f'(W_s) \, dW_s$$

*: $W'_s \, ds = dW_S$

In fact, $W$ is not differentiable.

**Question.** Do we have $f(W_t) = f(0) + \int_0^t f'(W_s) \, dW_s$?

For $f : x \mapsto x^2$, we would have: $W_t^2 = 2 \int_0^t W_s \, dW_s$.

Since

$$E \left[ \int_0^t W_s^2 \, ds \right] = \int_0^t E[W_s^2] \, ds = \int_0^t s \, ds = \frac{t^2}{2} < \infty$$

the previous $W$ is in $\mathcal{H}$.

Then $(2 \int_0^t W_s \, dW_s)_{t \leq T}$ is a martingale.

**Problem.** $(W_t^2)_{t \leq T}$ is not a martingale. ($(W_t^2 - t)_{t \leq T}$ is a martingale). The formula above is wrong.

Suppose $f$ is analytic: one has Taylor formula: $f(x+h) = \sum_{k=0}^{\infty} f^{(k)}(x) \frac{h^k}{k!}$.

Let $n \geq 1$ integer:

$$f(W_t) - f(W_0) = \sum_{k=0}^{n-1} \left[ f(W_{\frac{t}{n}} + \frac{k}{n}) - f(W_{\frac{t}{n}}) \right]$$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{\infty} f^{(l)}(W_{\frac{t}{n}}) \frac{(W_{\frac{t}{n}} + \frac{k}{n} - W_{\frac{t}{n}})^l}{l!}$$

$$= \sum_{l=0}^{\infty} \sum_{k=0}^{n-1} \frac{f^{(l)}(W_{\frac{t}{n}})}{l!} (W_{\frac{t}{n}} + \frac{k}{n} - W_{\frac{t}{n}})^l$$

* $l=1$: $\sum_{k=0}^{n} f^{(1)}(W_{\frac{t}{n}})(W_{\frac{t}{n}} + \frac{k}{n} - W_{\frac{t}{n}}) \to \int_0^t f'(W_s) \, dW_s$ $(n \to \infty)$

(in a sense which has to be made precise). The sum is the stochastic integral $\int_0^t f'(W_{\frac{t}{n}}) \, dW_s$

* $l=2$: $\sum_{k=0}^{n-1} \frac{f^{(2)}(W_{\frac{t}{n}})}{2} (W_{\frac{t}{n}} + \frac{k}{n} - W_{\frac{t}{n}})^2$

The terms $W_{\frac{t}{n}} + \frac{k}{n} - W_{\frac{t}{n}}$ are Gaussian with expectation 0 and variance $\frac{t}{n}$, and are independent. Hence, their squares are independent, with expectation $\frac{t}{n}$. By “some” kind
of law of large numbers, we can replace \((W_{k+\frac{t}{n}} - W_{k\frac{t}{n}})^2\) by \(\frac{1}{n}\). In this case we obtain
\[
\frac{n-1}{n} \sum_{k=0}^{n-1} f''(W_{k\frac{t}{n}}) \cdot \frac{1}{2}.
\]
This (Riemann) sum converges to \(\frac{1}{2} \int_0^t f''(W_s) \, ds\)

\[\to \frac{1}{2} \int_0^t f''(W_s) \, ds \cdot \frac{n}{t} \to \frac{1}{2} (l/n) \int_0^t f''(W_s) \, ds \cdot \frac{n}{t} \to 0 \quad \text{as} \quad n \to \infty\]

One has the “first heuristic formula” + an additional term due to the “quadratic variation” of \((W_t)_{t \leq T}\).

A rigorous statement can be written:

**Proposition 7.** If \((W_t)_{t \geq 0}\) is on \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion, and if \(f\) is a function from \(\mathbb{R}\) to \(\mathbb{R}\), continuously twice differentiable, then almost surely for all \(t \leq T\):

\[
f(W_t) = f(0) + \int_0^t f'(W_s) \, dW_s + \frac{1}{2} \int_0^t f''(W_s) \, ds
\]

This formula, fundamental in stochastic calculus, is called **Itô formula**.

**Examples:**

1. \(W_t^2 = 2 \int_0^t W_s \, dW_s + \int_0^t ds = 2 \int_0^t W_s \, dW_s + t\)

   As stated before \((W_t^2 - t)_{t \geq 0}\) is a martingale.

2. Let us take \(f(x) = e^x = f'(x) = f''(x)\).

   \(e^{W_t} = 1 + \int_0^t e^{W_s} \, dW_s + \frac{1}{2} \int_0^t e^{W_s} \, ds\)

   Here, the stochastic integral is only defined with respect to a Brownian motion.

**Question.** In finance, how can we define the integral with respect to \(dS_t\), where \(S_t\) is the stock price?

\(S_t\) is defined using Brownian motion, but it is not exactly a Brownian motion.

**Answer.** We need to extend stochastic integral and Itô formula.

**Definition 11.** Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space, \((\mathcal{F}_t)_{t \leq T}\) a filtration, and \((W_t)_{t \leq T}\) a one dimensional \((\mathcal{F}_t)_{t \leq T}\)-Brownian motion. A real valued process \((X_t)_{t \leq T}\) is an **Itô process** iff almost surely

\[
X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s
\]

where

- \(X_0\) is \(\mathcal{F}_0\)-measurable
- \((K_t)_{t \leq T}\) and \((H_t)_{t \leq T}\) are progressively measurable processes (for example, right or left continuous and adapted)
- \(\int_0^T |K_s| \, ds < \infty\) \(\mathbb{P}\)-almost surely
- \(\int_0^T |H_s|^2 \, ds < \infty\) \(\mathbb{P}\)-almost surely

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The processes $K$ and $H$ are uniquely determined by $X$ in the following sense:

**Proposition 8.** If $(M_t)_{t \leq T}$ is a continuous martingale such that $M_t = \int_0^t K_s \, ds$, with $\mathbb{P}$-a.s. $\int_0^t |K_s| \, ds < \infty$, then a.s. for all $t \leq T : M_t = 0$.

This implies: If there are two decompositions of an Itô process

$$X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s = X_0' + \int_0^t K'_s \, ds + \int_0^t H'_s \, dW_s$$

then

- $X_0 = X_0'$ $\mathbb{P}$-a.s.
- $H_s(w) = H'_s(w)$ almost everywhere with respect to the measure $ds \otimes d\mathbb{P}$. This means $\{(s, w) | H_s(w) \neq H'_s(w)\}$ has measure zero.
- Similar for $K_s$ and $K'_s$.

Moreover, if a martingale $(X_t)_{t \leq T}$ has the form: $X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s$ then $K_t = 0$ $dt \otimes d\mathbb{P}$ almost everywhere.

**Proposition 9.** Let $(X_t)_{t \leq T}$ be an Itô process: $X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s$ (formally $dX_t = K_t \, dt + H_t \, dW_t$). Let $f$ be a function, continuously, twice differentiable. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s) \, d\langle X, X \rangle_s$$

where

- $\int_0^t f''(X_s) \, dX_s = \int_0^t f'(X_s) K_s \, ds + \int_0^t f'(X_s) H_s \, dW_s$
- $(X, X)_s = \int_0^s H^2_t \, ds$ (it is called “the quadratic variation of $X$’)

$$\Rightarrow \int_0^t f''(X_s) \, d\langle X, X \rangle_s = \int_0^t f''(X_s) H^2_s \, ds$$

Generalization for functions depending also on the time. Let $f$ be a function from $I \times \mathbb{R}$, which is twice differentiable with respect to the second variable (space); once differentiable with respect to the first variable (time); and all the corresponding partial derivatives are continuous on $I \times \mathbb{R}$.

(we also write: $f \in C^{1,2}$).

$$f(t, X_t) = f(0, X_0) + \int_0^t f'(s, X_s) \, ds + \int_0^t f''_2(s, X_s) \, dX_s + \frac{1}{2} \int_0^t f''_2(s, X_s) \, d\langle X, X \rangle_s$$

Another form: (integration by parts formula). Let $X_t, Y_t$ be two Itô processes:

- $X_t = X_0 + \int_0^t K_s \, ds + \frac{1}{2} \int_0^t H_s \, dW_s$
\[ Y_t = Y_0 + \int_0^t K'_s \, ds + \frac{1}{2} \int_0^t H'_s \, dW_s \]

Then, \( XY \) is also an Itô process, more precisely:

\[ X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t \]

where \( \langle X, Y \rangle \) is the quadratic variation of \( X \) and \( Y \), it is defined by:

\[ \langle X, Y \rangle_t = \int_0^t H_s H'_s \, ds \]

(It is application of Itô formula for \( X, Y \) and \( X + Y \), with the function \( f : x \mapsto x^2 \)).

These three forms of Itô formulas are particular cases of the multidimensional Itô formula stated below:

**Proposition 10.** Let \( (W_t)_{t \in I} \) be a \( d \)-dimensional \( \mathcal{F}_t \)-Brownian motion. \( X_t \) is an Itô process with respect to \( (W_t)_{t \in I} \) iff

\[ X_t = X_0 + \int_0^t K_s \, ds + \sum_{i=1}^d \int_0^t H_{s}^{(i)} \, dW_{s}^{(i)} \]

with

- \( (K_t)_{t \in I}, (H_{t}^{(i)})_{t \in I} \) adapted.
- \( T \int_0^T |K_s| \, ds < \infty \).
- \( T \int_0^T (H_{s}^{(i)})^2 \, ds < \infty \).

Then: If \( X_{t}^{(1)}, \ldots, X_{t}^{(n)} \) are \( n \) Itô processes

\[ X_t^{(i)} = X_0^{(i)} + \int_0^t K_s^{(i)} \, ds + \sum_{j=1}^d \int_0^t H_{s}^{(i,j)} \, dW_{s}^{(j)} \]

If \( f \in C^{1,2} \) from \( I \times \mathbb{R}^n \) to \( \mathbb{R} \), then

\[ f(t, X_{t}^{(1)}, \ldots, X_{t}^{(n)}) = f(0, X_0^{(1)}, \ldots, X_0^{(n)}) + \int_0^t \frac{\partial f}{\partial x_i}(s, X_{s}^{(1)}, \ldots, X_{s}^{(n)}) \, ds \]

\[ + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(s, X_{s}^{(1)}, \ldots, X_{s}^{(n)}) \, dX_{s}^{(i)} \]

\[ + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_{s}^{(1)}, \ldots, X_{s}^{(n)}) \, d\langle X^{(i)}, X^{(j)} \rangle_s \]

where formally \( dX_{s}^{(i)} = K_{s}^{(i)} \, ds + \sum_{j=1}^d H_{s}^{(i,j)} \, dW_{s}^{(j)} \)

\[ d\langle X^{(i)}, X^{(j)} \rangle_s = \sum_{m=1}^d H_{s}^{(i,m)} H_{s}^{(j,m)} \, ds \]
3 The Black-Scholes Model

3.1 Description of the Model

As informally seen before, the simplest nontrivial binomial model admits a “kind of limit” when the frequency of the steps tends to infinity.

This limit is called Black (and) Scholes model (described in 1973, with Merton; some ideas go back to Bachelier in 1900).

This model (in its simplest non-trivial form) can be rigorously stated as follows.

The corresponding model contains only two assets (as in the binomial model): a stock (risky) and a bond (risk-less).

- The bond price satisfies the following equation:

\[ dS_0^t = rS_0^t dt \]

i.e.

\[ S_0^t = S_0^0 + r \int_0^t S_s^0 ds \]

where \( S_0^t \) is the price at time \( t \).

- The stock price satisfies the stochastic differential equation:

\[ dS_t = S_t(\mu dt + \sigma dB_t) \]

where \( S_t \) is the price at time \( t \).

This equation means that:

\[ S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dB_s \]

Here \( r, \mu, \sigma > 0 \) are real parameters and \((B_s)_{s \geq 0}\) is a one dimensional Brownian motion (Generally \( \mu > r > \sigma \)).

Intuitively.

- The bond is a risk-less asset, which gives a fixed interest rate \( r \), generally positive.

- The stock is a risky asset, which gives “in average” a gain of \( \mu \) per unit of time, but with fluctuations, driven by the Brownian motion \( B \). (Generally \( \mu > r \), because one expect a better gain in average if one takes some risk).

- The parameter \( \sigma \) represents the amplitude of the fluctuations of the stock price, it is called volatility.
3.1.1 Explicit Description of the Prices

- The equation satisfied by the bond price can be easily solved. One obtains \( S_t^0 = S_0^0 \exp(rt) \).

- By using Itô formula, one can check that for \( S_0 > 0 \) given: \( S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t) \) is a solution of the stochastic differential equation stated before. In fact, this solution is unique (exercise).

Now, since we have a model for stock prices, we have also a model for trading in the financial model described above. We then use this model to replicate options, and hence, to evaluate option prices.

3.1.2 Self-Financing Strategies

A strategy will be defined as a process \( \phi = (\phi_t)_{0 \leq t \leq T} = (H_t^0, H_t)_{0 \leq t \leq T} \) with values in \( \mathbb{R}^2 \), adapted to the natural filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \) of the Brownian motion \( (B_t)_{0 \leq t \leq T} \).

\( H_t^0 \) and \( H_t \) represent the quantities of bonds and stocks hold in the portfolio at time \( t \). The time \( T \in \mathbb{R}_+^* \) is the time when one stops trading (typically the maturity of the option one needs to replicate).

Remarks.
- \( H_t^0 \) and \( H_t \) can change at each instant, i.e. it is possible to trade “continuously” on the market.
- \( H_t^0 \) and \( H_t \) can be negative. For \( H_t^0 \) it corresponds to “borrowing money”, and for \( H_t \), it is “short-selling” of the stock.

One can now define the value of the portfolio \( (H_t^0, H_t)_{0 \leq t \leq T} \) as the process
\[
(V_t(\phi)) = H_t^0 S_t^0 + H_t S_t
\]

Question. How can we define a self-financing portfolio (or strategy)?

Intuitive answer. The value of the portfolio does not change at a time of trading. More concretely, let us consider instants \( t \) and \( t + dt \) (\( dt > 0 \) “small”).

- Between \( t \) and \( t + dt \) let us suppose that there is no trading. One has \( H_t^0 \) bonds and \( H_t \) stocks, which gives a value at \( t + dt \).

\[
\tilde{V}_{t+dt}(\phi) = H_t^0 S_{t+dt}^0 + H_t S_{t+dt}
\]

- Let us trade just after \( t + dt \). The quantity of bonds is \( H_{t+dt}^0 \) and the number of stocks is \( H_{t+dt} \). Value after trading:

\[
\tilde{V}_{t+dt}(\phi) = H_{t+dt}^0 S_{t+dt}^0 + H_{t+dt} S_{t+dt}
\]
• By self-financing property, the value does not change during trading, so

\[ V_{t+dt}(\phi) = \tilde{V}_{t+dt}(\phi) \]
\[ = \tilde{V}_{t+dt}(\phi) \]
\[ = H^0_t S^0_{t+dt} + H_t S_{t+dt} \]
\[ = H^0_t (S^0_t + dS^0_t) + H_t (S_t + dS_t) \]
\[ = H^0_t S^0_t + H_t S_t + H^0_t dS^0_t + H_t dS_t \]
\[ = V_t(\phi) + H^0_t dS^0_t + H_t dS_t \]

So

\[ dV_t(\phi) = H^0_t dS^0_t + H_t dS_t \]

In other words the portfolio \( \phi \) is self-financing (by definition) iff

\[ V_t(\phi) = V_0(\phi) + \int_0^t H^0_s dS_s + \int_0^t H_s dS_s \]

where the stochastic integrals have supposed to be well-defined (\( H^0_s, H_s \) adapted and \( H^0_s, H_s \) “sufficiently bounded”).

3.1.3 Precise Condition of Boundedness

\[ dH^0_s = rH^0_s ds, \quad dH_s = \mu H_s ds + \sigma H_s dB_s \]
\[ \int_0^T |H^0_s| ds < \infty \quad \text{and} \quad \int_0^T |H_s| ds < \infty \quad \text{and} \]
\[ \int_0^T (H_s)^2 ds < \infty \quad \text{almost surely} \]

Let us denote by \( \tilde{S}_t = e^{-\tau t} S_t \) the discounted price of the risky asset. Then the following result holds:

**Proposition 11.** Let \( \phi = (H^0_t, H_t)_{0 \leq t \leq T} \) be an adapted process with values in \( \mathbb{R}^2 \), satisfying:

\[ \int_0^T |H^0_t| dt + \int_0^T (H_t)^2 dt < \infty \quad \text{almost surely} \]

Let \( V_t(\phi) = H^0_t S^0_t + H_t S_t \) and \( \tilde{V}_t(\phi) = e^{-\tau t} V_t(\phi) = (H^0_t S^0_t + H_t \tilde{S}_t) \). Then, \( \phi \) defines a self-financing strategy iff for all \( t \in [0, T] \):

\[ \tilde{V}_t(\phi) = \tilde{V}_0(\phi) = \int_0^t H_s d\tilde{S}_s \quad \text{almost surely} \]
Proof. Suppose that $\phi$ is a self-financing strategy. From the equality:

$$d \tilde{V}_t(\phi) = -r \tilde{V}_t(\phi) + e^{-rt}dV_t(\phi)$$

(No extra term in the Itô formula here, because $e^{-rt}$ has bounded variation). We deduce:

$$d \tilde{V}_t(\phi) = -re^{-rt}(H_0te^{rt} + H_0e^r + H_0S)dt + e^{-rt}H_0d(\text{e}^r+H_0S)dt$$

$$= H_t(-re^{-rt}S + e^{-rt}dS) = H_t\tilde{S}_t$$

In the other direction, the proof is similar. \qed

Intuitively. The variation of the discounted value of the portfolio is the product of the quantity of stocks held by the "discounted gain" per stock.

### 3.2 Change of Probability, Representation of Martingales

Recall (Equivalent Probabilities). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Recall that a probability $Q$ on $(\Omega, \mathcal{A})$ is absolutely continuous with respect to $\mathbb{P}$ iff for all $A \in \mathcal{A}$:

$$Q(A) = \int_A Z(\omega) \, d\mathbb{P}(\omega)$$

Z is called density of $Q$ with respect to $\mathbb{P}$ and is denoted by $\frac{dQ}{d\mathbb{P}}$.

Informally. $x \in \Omega : Q("around x") \approx Z(x)\mathbb{P}("around x")$

Definition 12. Two probabilities $\mathbb{P}$ and $Q$ are equivalent if they are defined on the same measurable space and if they are absolutely continuous with respect to each other.

Criterion. If $Q$ is absolutely continuous with respect to $\mathbb{P}$, with density $Z$, then $\mathbb{P}$ is equivalent to $Q$ iff $\mathbb{P}(Z > 0) = 1$.

If $\mathbb{P}$ is equivalent to $Q$, a property holds $\mathbb{P}$-almost surely iff it holds $Q$-almost surely.

Let us describe a particular case of probability change, which is related to Brownian motion.

Theorem 13 (Girsanov). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ be a filtered probability space, and $(B_t)_{t \leq T}$ a one-dimensional $(\mathcal{F}_t)$-Brownian motion. We suppose that $(\theta_t)_{t \leq T}$ is an adapted process, satisfying

$$\int_0^T \theta^2_s \, ds < \infty$$

almost surely, and such that the process $(\eta_t)_{t \leq T}$ defined by $\eta_t := \exp(-\int_0^t \theta_s \, dB_s - \frac{1}{2} \int_0^t \theta^2_s \, ds)$ is a martingale. (It is the case if $(\theta_t)_{t \leq T}$ satisfies "good integrability conditions").

Then under the probability $\mathbb{P}^{(\theta)}$ with density $\eta_T$ with respect to $\mathbb{P}$ (recall that $\mathbb{E}_\mathbb{P}[^t] = \mathbb{E}_\mathbb{P}[^0]$ by assumption) the process $(W_t)_{t \leq T}$ defined by:

$$W_t := B_t + \int_0^t \theta_s \, ds$$

is an $(\mathcal{F}_t)_{t \leq T}$-Brownian motion.
Remarks.

• If $\mathbb{P}$ is replaced by $\mathbb{P}(\cdot)$, one needs to correct $B$ by a "drift term" $(\theta_s ds)$ in order to obtain again a Brownian motion under the new probability measure.

• By Itô-Formula:

$$\iota_t = \iota_0 + \left[ \text{a stochastic integral with respect to } dB_s \right]$$

Hence $(\iota_t)_{t \leq T}$ is a martingale with respect to $\mathbb{P}$ if it satisfies some integrability conditions. It is, for example, sufficient to have:

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \theta_s^2 \, dt \right) \right] < \infty \quad \text{(called: Novakov’s criterion)}$$

Proof. Admitted (exercise for $\theta$ constant, a simple but important case) \qed

3.2.1 Representation of Martingales

Let $(B_t)_{t \leq T}$ be a one-dimensional Brownian motion built on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t \leq T}$ be its natural filtration. Recall that if $(H_t)_{t \leq T}$ is an adapted process such that:

$$\mathbb{E} \left[ \int_0^T H_t^2 \, dt \right] < \infty$$

then the process $(\int_0^t H_s \, dB_S)_{t \leq T}$ is a square-integrable martingale, equal to zero. The result stated below shows that any "Brownian martingale" can be represented in term of stochastic integral.

**Theorem 14.** Let $(M_t)_{t \leq T}$ be a square-integrable martingale, with respect to the filtration $(\mathcal{F}_t)_{t \leq T}$ (recall that $(\mathcal{F}_t)_{t \leq T}$ is the natural filtration of the Brownian motion $B$). There exists an adapted process $(H_t)_{t \leq T}$ such that

$$\mathbb{E} \left[ \int_0^T H_t^2 \, dt \right] < \infty$$

and for all $t \in [0, T]$:

$$M_t = M_0 + \int_0^t H_s \, dB_s \quad \text{almost surely}$$

Remarks.

• This theorem does not apply for a "general filtration".

• From this theorem, if $U$ is an $(\mathcal{F}_t)_{t \leq T}$-measurable, square-integrable random variable, it can be written as:

$$U = \mathbb{E}[U] + \int_0^T H_s \, dB_s \quad \text{almost surely}$$
where \((H_t)_{t \leq T}\) satisfies the conditions above. (consider: \(M_t = \mathbb{E}[U|\mathcal{F}_t]\)). One can also represent martingales which are not square-integrable. In this case,
\[
\int_0^T H_t^2 \, dt < \infty
\]
but not necessarily the expectation.

### 3.3 Pricing and Hedging Options in the Black-Scholes Model

One begins this section by proving that there exists a probability, equivalent to \(\mathbb{P}\), under which the discounted stock price \((\tilde{S}_t := e^{-rt})_{t \leq T}\) is a martingale.

Indeed, one obtains, from the stochastic differential equation satisfied by \((S_t)_{t \leq T}\):
\[
d\tilde{S}_t = -re^{-rt}S_t \, dt + e^{-r} \, dS_t = \tilde{S}_t((\mu - r) \, dt + \sigma \, dB_t)
\]
(The "drift term" is shifted by \(-r\) in the s.d.e., with respect to \(S_t\)). Let us set:
\[
W_t := B_t + (\mu - r)t \sigma (\sigma > 0)
\]
We check immediately that:
\[
d\tilde{S}_t = \tilde{S}_0 \exp(\sigma W_t - \frac{\sigma^2 t}{2})
\]

We apply Girsanov’s theorem, with \((\theta_t)\) constant, equal to \((\mu - r)/\sigma\), one obtains a probability \(\mathbb{P}^*\) equivalent to \(\mathbb{P}\), under which \((W_t)_{t \leq T}\) is a standard Brownian motion. Moreover, one can prove that the definition of the stochastic integral is invariant by a change of probability which is equivalent. Then, under \(\mathbb{P}^*\), one has the same s.d.e.:
\[
d\tilde{S}_t = \tilde{S}_t dW_t \text{ (here } W \text{ is a Brownian motion)}
\]
which has a unique solution (up to a multiplicative constant):
\[
\tilde{S}_t = \tilde{S}_0 \exp\left(\frac{\sigma W_t - \frac{\sigma^2 t}{2}}{2}\right)
\]
In particular, \((\tilde{S}_t)_{t \leq T}\) is a martingale under \(\mathbb{P}^*\). This "equivalent martingale measure" is very important when we want to compute option prices.

More precisely, let us focus on European contingent claims (more simply said: European option). They are defined as non-negative, \(\mathcal{F}_T\)-measurable random variables, which will generally be denoted by \(h\) in this lecture. Quite often, \(h\) can be written as a function \(f\) of \(S_T\): for example, for a call option with strike price \(K\) and maturity \(T\), \(f(x) = (x - K)_+\), then
\[
h = f(S_T) = (S_T - K)_+
\]
The value of such an option will be defined by a replication argument. For technical reasons, we will limit our study to the following admissible strategies:
Definition 13. A strategy $\phi = (H_0^t, H_t)_{t \leq T}$ is admissible iff it is self-financing and if the discounted value:

$$\tilde{V}_t(\phi) = V_t(\phi)e^{-rt} = H_t^0 + H_t \tilde{S}_t$$

of the corresponding portfolio is, for all $t$, non-negative, and such that: $\sup_{t \in [0,T]} \tilde{V}_t(\phi)$ is square-integrable under $P^*$.

An option is said to be replicable iff its payoff is equal to the terminal value of an admissible strategy. It is clear that, for the option defined by $h$, it can be replicable only if it is non-negative, and square-integrable under $P^*$.

**Example.** A call option satisfies the condition above:

$$\mathbb{E}_{P^*} [ (S_T - K)^2 ] < \infty$$

For a put, $h$ is even bounded.

**Theorem 15.** In Black-Scholes-model, any option defined by a non-negative, $\mathcal{F}_T$-measurable random variable, which is square-integrable under the probability $P^*$, is replicable and the value at time $t$ of any replicating portfolio is given by:

$$V_t := \mathbb{E}_{P^*} \left[ e^{-r(T-t)} h | \mathcal{F}_t \right]$$

Thus, by basic "non-arbitrage argument", the option value at time $t$ can be naturally defined by the expression giving $V_t$.

**Proof.** First, assume that there is an admissible strategy $(H_0^t, H_t)$ replicating the option. The value at time $t$ of the portfolio $(H_0^t, H_t)$ is given by:

$$V_t = H_t^0 S_t^0 + H_t S_t$$

and by assumption, we have $V_T = h$. Let $\tilde{V}_t = V_t e^{-rt}$ be the discounted value:

$$\tilde{V}_t = H_t^0 + H_t \tilde{S}_t$$

Since the strategy is self-financing, we get:

$$\tilde{V}_t = V_0 + \int_0^t H_u \, d\tilde{S}_u = V_0 + \int_0^t H_u \sigma \tilde{S}_u \, dW_u$$

Under $P^*$, $\sup_{t \in [0,T]} \tilde{V}_t$ is square-integrable, by definition of admissible strategies.

Moreover, $(\tilde{V}_t)_{t \leq T}$ is a stochastic integral with respect to $dW_u$. It follows (admitted) that $(\tilde{V}_t)_{t \leq T}$ is a square-integrable martingale under $P^*$.

Hence,

$$\tilde{V}_t = \mathbb{E}^*[\tilde{V}_T | \mathcal{F}_t]$$

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where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$. Equivalently:

$$
\bar{V}_t = \mathbb{E}^*[e^{-r(T-t)}h| \mathcal{F}_t]
$$

We have proved that if a portfolio $(H_0^0, H)$ replicates the option defined by $h$, its value is given by the expression in the theorem.

- The value $V_t$ was found and proved last time.
- To complete the proof of the theorem, it remains to show that the option is replicable, i.e. to find process $(H_t^0)_{t \leq T}$ and $(H_t)_{t \leq T}$ defining an admissible strategy such that:

$$
V_t = H_t^0 S_t^0 + H_t S_t = \mathbb{E}^*[e^{-r(T-t)}h| \mathcal{F}_t]
$$

(In this case $V_T = \mathbb{E}^*[h| \mathcal{F}_T] = h$).

Under the probability $\mathbb{P}^*$, the process defined by

$$
M_t := \mathbb{E}^*[e^{-rT}h| \mathcal{F}_t]
$$

is a martingale, square-integrable because $h$ is square-integrable with respect to $\mathbb{P}^*$. The filtration $(\mathcal{F}_t)_{t \leq T}$, which is the natural filtration of $(B_t)_{t \leq T}$ is also a natural filtration of $(W_t)_{t \leq T}$, and from the martingale representation theorem, there exists an adapted process $(K_t)_{t \leq T}$ such that:

$$
\mathbb{E}^* \left[ \int_0^T K_s^2 \, ds \right] < \infty
$$

and for all $t \in [0, T]$:

$$
M_t = M_0 + \int_0^t K_s \, dW_s
$$

$\mathbb{P}^*$-almost surely, (which is equivalent to $\mathbb{P}$-almost surely, since $\mathbb{P}^*$ is equivalent to $\mathbb{P}$). The strategy

$$
\phi = (H^0, H), \text{ with } H_t^0 := \frac{K_t}{\sigma S_t} \text{ and } H_t := M_t - H_t \bar{S}_t
$$

is a self-financing strategy, with value at $t$:

$$
V_t(\phi) = H_t^0 S_t^0 + H_t S_t
$$

$$
= (M_t - H_t \bar{S}_t) S_t^0 + H_t \cdot S_t
$$

$$
= M_t S_t^0 - H_t \bar{S}_t e^{rt} + H_t S_t
$$

$$
= M_t S_t^0
$$

$$
= \mathbb{E} \left[ e^{-r(T-t)}h| \mathcal{F}_t \right]
$$
Check the self-financing property:
\[
   dV_t = d(M_te^{rt}) \\
   = Md(e^{rt}) + e^{rt}dM_t(l) \\
   = rM_te^{rt}dt + e^{rt}K_tdW_t
\]

(!): No extra term because \( e^{rt} \) has finite variation.

On the other hand:
\[
   H^0_tS^0_t + H_tS_t = H^0_tre^{rt}dt + H_t(e^{rt}d\tilde{S}_t) \\
   = H^0_tre^{rt}dt + H_t(e^{rt}d\tilde{S}_t + re^{rt}\tilde{S}_tdt) \\
   = (M_t - H_t\tilde{S}_t)re^{rt}dt + \frac{K_t}{\sigma\tilde{S}_t}(e^{rt}(\sigma\tilde{S}_tdW_t) + re^{rt}\tilde{S}_tdt) \\
   = rM_te^{rt}dt - H_t\tilde{S}_tre^{rt}dt + K_te^{rt}dW_t + \frac{K_t}{\sigma}re^{rt}dt \\
   = rM_te^{rt}dt - H_t\tilde{S}_tre^{rt}dt + K_te^{rt}dW_t + H_t\tilde{S}_tre^{rt}dt \\
   = rM_te^{rt}dt + e^{rt}K_tdW_t
\]

⇒ the two increments are equal.

With the expression of \( V_t \), we also check that \( V_t(\phi) \geq 0 \) (since \( h \geq 0 \)), with \( \sup V_t(\phi) \) square-integrable (by Doob inequality applied to the martingale \( (M_t)_{t \leq T} \)). We have found an admissible strategy replicating \( h \).

3.3.1 Particular Case where \( h \) is a Function of \( S_T \)

We have \( h = f(S_T) \) (\( f \geq 0 \) such that \( E^*[(f(S_T))^2] < \infty \)). We can express the option value \( V_t \) at time \( t \) as a function of \( t \) and \( S_t \). We have indeed:
\[
   V_t = E^*\left[e^{-r(T-t)}f(S_t)|\mathcal{F}_t\right] \\
   = E^*\left[e^{-r(T-t)}f(S_te^{r(T-t)}e^{r(T-t)}(W_T-W_t)-\frac{\sigma_x^2}{2}(T-t))|\mathcal{F}_t\right]
\]

*: by the explicit expression of \( S_T \): \( S_t = S_0e^{(r-\frac{\sigma^2}{2})t+\sigma Wi} \)

The random variable \( S_t \) is \( \mathcal{F}_t \)-measurable, and under \( P^* \), \( (W_T-W_t) \) is independent of \( \mathcal{F}_t \). One can deduce \( V_t = F(t,S_t) \), where
\[
   F(t,x) = E\left[e^{-r(T-t)}f(xe^{r(T-t)}e^{\sigma\tilde{N}_{T-t}+\frac{\sigma^2}{2}(T-t)})\right]
\]

where \( \tilde{N}_{T-t} \) represents \( W_T-W_t \) and is a normal variable with expectation zero and variance \( T-t \).

In other words:
\[
   F(t,x) = e^{-r(T-t)}\int_{-\infty}^\infty f \left( x \cdot e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma \cdot y \sqrt{T-t}} \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy 
\]

where \( y \sqrt{T-t} \) represents \( \tilde{N}_{T-t} \).

The function \( F \), which depends linearly on \( f \), can be explicitly computed for call \( (f(x) = (x-K)_+) \)
and put \((f(x) = (x - K)_-).\)

**Case** \(f(x) = (x - K)_+:\)

\[
F(t, x) = \mathbb{E} \left[ \left( xe^{\sigma \sqrt{\theta} - \frac{r^2}{2} - Ke^{-r\theta}} \right)_+ \right]
\]

where \(\mathcal{N}\) is the standard Gaussian variable and \(\theta = T - t\).

Let us set:

\[
d_1 := \log \left( \frac{x}{K} \right) + \frac{r + \sigma^2\theta}{\sigma \sqrt{\theta}}
\]

\[
d_2 := d_1 - \sigma \sqrt{\theta} = \log \left( \frac{x}{K} \right) + \frac{r - \sigma^2\theta}{\sigma \sqrt{\theta}}
\]

\[
F(t, x) = \mathbb{E} \left[ \left( xe^{\sigma \sqrt{\theta} - \frac{r^2}{2} - Ke^{-r\theta}} \right) \mathbb{1}_{g + d_2 \geq 0} \right] (\ast)
\]

\[
= \int_{-\infty}^{d_2} \left( xe^{\sigma \sqrt{\theta} - \frac{r^2}{2} - Ke^{-r\theta}} \right) \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy
\]

\[
= \int_{-\infty}^{d_2} \left( xe^{-\sigma \sqrt{\theta} - \frac{r^2}{2} - Ke^{-r\theta}} \right) \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy
\]

\((\ast): g = -d_2 \) corresponds to the change of the expression \(xe^{\sigma \sqrt{\theta} - \frac{r^2}{2} - Ke^{-r\theta}}\)

We can then split the integral into two parts:

\[
F(t, x) = \int_{-\infty}^{d_2} x e^{-\sigma \sqrt{\theta} - \frac{r^2}{2} - Ke^{-r\theta}} dy - Ke^{-r\theta} \int_{-\infty}^{d_2} \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy
\]

where \(\mathcal{N}(d_2) = \int_{-\infty}^{d_2} \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy = \mathbb{P}[\mathcal{N} \leq d_2] \) (repartition function \(\mathcal{N}(d_2)\) of the standard Gaussian variable).

By doing the change of variable \(z = y + \sigma \sqrt{\theta}\) in the first integral we obtain:

\[
F(t, x) = x \mathcal{N}(d_1) - Ke^{-r\theta} \mathcal{N}(d_2)
\]

**Case** \(f(x) = (x - K)_-:\)

we similarly obtain:

\[
F(t, x) = Ke^{-r\theta} \mathcal{N}(-d_2) - x \mathcal{N}(-d_1)
\]

These explicit expressions which give the price of a call and a put option for Black-Scholes-models (the simplest case, studied here) are called **Black-Scholes formulae**.

### 3.3.2 Hedging call and puts

We have used the martingale representation theorem in order to prove the existence of replicating portfolio of a call and a put. It is not satisfactory to have only a theorem of existence. We need
to find explicitly a replicating strategy, to hedge the call and put option. We shall build such a strategy. The strategy can be extended to general functions $f$. The value of a replicating portfolio should be equal to $F(t, S_t)$ where $F$ is described above:

$$F(t, x) = \mathbb{E}^* \left[ e^{-r(T-t)} f(x e^{r(T-t)} e^{\sigma N_{T-t} - \frac{\sigma^2}{2}(T-t)}) \right]$$

The discounted value is:

$$\tilde{V}_t = e^{-rt} F(t, S_t)$$

For a "large class" of functions $f$ (containing the puts and calls), $F$ is $C^\infty$ on $[0, T) \times \mathbb{R}$.

If we set, $\tilde{F}(t, x) = e^{-rt} F(t, xe^{rt})$ ($\tilde{F}$ is also $C^\infty$), we have:

$$\tilde{V}_t = \tilde{F}(t, \tilde{S}_t)$$

and for $t < T$, by the Itô formula:

$$\tilde{F}(t, \tilde{S}_t) = \tilde{F}(0, \tilde{S}_0) + \int_0^t \frac{\partial \tilde{F}}{\partial x}(u, \tilde{S}_u) d\tilde{S}_u + \int_0^t \frac{1}{2} \frac{\partial^2 \tilde{F}}{\partial x^2}(u, \tilde{S}_u) d\langle \tilde{S}, \tilde{S} \rangle_u$$

where $\partial x$ is the derivative with respect to the second variable.

From $d\tilde{S}_t = \sigma \tilde{S}_t dW_t$, we deduce $d\langle \tilde{S}, \tilde{S} \rangle_u = \sigma^2 \tilde{S}_u^2 du$ so that $\tilde{F}(t, \tilde{S}_t)$ can be written:

$$\tilde{F}(t, \tilde{S}_t) = \tilde{F}(0, \tilde{S}_0) + \int_0^t \sigma \frac{\partial \tilde{F}}{\partial x}(u, \tilde{S}_u) \tilde{S}_u dW_u + \int_0^t K_u du$$

for some process $K$ (coming from the last two terms).

The discounted value of the self-financing portfolio is a martingale under $P^*$, the term $\int_0^t K_u du$ vanishes.

$$\Rightarrow \tilde{F}(t, \tilde{S}_t) = \tilde{F}(0, \tilde{S}_0) + \int_0^t \sigma \frac{\partial \tilde{F}}{\partial x}(u, \tilde{S}_u) \tilde{S}_u dW_u$$

It is natural to suppose that the number of stocks for the replicating portfolio is:

$$H_t := \frac{\partial \tilde{F}}{\partial x}(t, \tilde{S}_t) \Leftrightarrow H_t := \frac{\partial F}{\partial x}(t, S_t)$$

Let us define $H^0_t := \tilde{F}(t, \tilde{S}_t) - H_t \tilde{S}_t$, the portfolio $(H^0_t, H_t)$ has discounted value $\tilde{V}_t = \tilde{F}(t, \tilde{S}_t)$.

One can check that it is self-financing.

Remarks.

- In fact it is not necessary the martingale representation theorem to deal with the particular case $h = f(S_t)$.
• In the case of the calls and the puts, by using the notation of the Black-Scholes formulae, we obtain:
  \[
  \frac{\partial F}{\partial x}(t, x) = \mathcal{N}(d_1) \text{ for the call, and} \\
  \frac{\partial F}{\partial x}(t, x) = -\mathcal{N}(-d_1) \text{ for the put.}
  \]

  This derivation is called the delta of the option.

More generally, when the value at time \( t \) of an option is \( \Psi(t, S_t) \), \( \frac{\partial \Psi}{\partial x}(x, S_t) \) measures the sensitivity of the option to the variation of the stock price, is called the delta of the option (also available for portfolio).

The second order derivative \( \frac{\partial^2 \Psi}{\partial x^2}(t, S_t) \) is called gamma.

The derivative \( \frac{\partial \Psi}{\partial t}(t, S_t) \) is the theta.

The derivative with respect to the volatility (\( \sigma \)) is called vega.

Remarks.

• One can easily see that the parameter \( \mu \) does not appear giving option prices.
  
  \[
  \begin{align*}
  - d\tilde{S}_t &= \tilde{S}_t dW_t \\
  - dS_t &= S_t \sigma dW_t + rd\tau \text{ (no } \mu \text{) under } P^* 
  \end{align*}
  \]

  This can be considered as a counterintuitive fact. (One could expect, for example, the call price is higher if the stock price tends to increase than if the stock price tends to decrease).

  However, it is a consequence of the fact that we always work with the probability \( P^* \), under which the parameter \( \mu \) disappears (analogous to the binomial model).

• For call option, it is possible to check that the option price is decreasing in \( t \) (theta is negative), increasing in \( r \), increasing in \( S_t \) (delta is positive) and increasing in \( \sigma \) (important: vega is positive).

3.4 American Options

An American option is an option which can be exercised at any time until maturity \( T \). The decision to exercise or not at time \( t \) is made by the owner of the option, using only the information available at time \( t \) (i.e., the time of exercise has to be a stopping time, chosen by the owner; it has also to be smaller than \( T \)). In our model, an American option can be defined by an adapted non-negative process \( (h_t)_{t \leq T} \). \( h_t \) represents the payoff at time \( t \) if the option is exercised at this time. For the sake of simplicity, we will only consider payoff processes of the form \( h_t = \Psi(S_t) \), where \( \Psi_t \) is a continuous function from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \) satisfying \( \Psi(x) \leq Ax + B \) for some \( A, B \geq 0 \).

An American call option, corresponds to \( \Psi(x) = (x - K)_+ \) and an American put option, corresponds to \( \Psi(x) = (K - x)_+ \).
• A call is the right, but not the obligation to buy the stock at price $K$ at any time $t \leq T$.
• A put is the right, but not the obligation to sell the stock at price $K$ at any time $t \leq T$.

In order to give an information on the "fair price" of an American option, we need to hedge it by trading in the market. Let us state the following

**Definition 14.** A trading strategy with consumption is an adapted process $\phi = (H^0_t, H_t)_{t \leq T}$ with values in $\mathbb{R}^2$, satisfying the following properties:

1. $\int_0^T |H^0_t| \, dt + \int_0^T (H_t)^2 \, dt < \infty$ almost surely

2. $H^0_t S^0_t + H_t S_t := V_t(\phi) = V_0(\phi) + \int_0^t H^0_u \, dS^0_u + \int_0^t H_u \, dS_u - c_t$ almost surely for all $t \in [0,T]$

where $(c_t)_{t \leq T}$ is an adapted, continuous non-decreasing process, null at time $t = 0$. $c_t$ represents the cumulative consumption up to time $t$.

With this definition, let us say that a trading strategy with consumption hedges the American option defined by $h_t = \Psi(S_t)$ if, for all $t \in [0,T]$, $V_t(\phi) \geq \Psi(S_t)$ almost surely. We will denote by $\Phi^\Psi$ the set of all trading strategies with consumption hedging the American option defined by $h_t = \Psi(S_t)$. If the writer (i.e. the seller) of the option follows a strategy $\phi \in \Phi^\Psi$, he possesses at any time $t$ a wealth at least equal to $\Psi(S_t)$, which is precisely the payoff if the option is exercised at time $t$. Hence, he can pay the owner in all possible cases with the wealth corresponding to his portfolio.

The following theorem introduces the minimal value of an hedging strategy for an American option.

with finite time

**Theorem 16.** Let $u$ be the map from $[0,T] \times \mathbb{R}_+$ to $\mathbb{R}$

$$u(t,x) := \sup_{\tau \in \mathcal{C}_{t,T}} \mathbb{E}^\pi \left[ e^{-\tau(u(t,S_t) - \psi_S)} \right]$$

where $\mathcal{C}_{t,T}$ is the set of all the stopping times with values in $[t,T]$. There exists a strategy $\phi \in \Phi^\Psi$, such that $V_t(\phi) = u(t,S_t)$ for all $t \in [0,T]$. Moreover, this strategy is minimal in the following sense:

$$V_t(\phi) \geq u(t,S_t) \text{ for all } t \in [0,T].$$

**Sketch of the proof.** One can show that the process $(e^{-\tau u(t,S_t)})_{t \leq T}$ is the smallest right continuous $\mathbb{P}^\pi$-supermartingale which dominates $(e^{-\tau \Psi(S_t)})_{t \leq T}$. Now, it is provable (and quite intuitive) that the discounted value of a trading strategy consumption is a $\mathbb{P}^\pi$-supermartingale (because of the decreasing term $-c_t$). Hence, the discounted stock price of a hedging strategy $\phi$ (for the payoff $\psi$), which dominates $(e^{-\tau \Psi(S_t)})_{t \leq T}$ by assumption, has to be greater than $(e^{-\tau u(t,S_t)})_{t \leq T}$ which implies $V_t(\phi) \geq u(t,S_t)$ for any strategy $\phi \in \Phi^\Psi$. [Due to the supermartingale property of the discounted value of $\phi$, applied to all the stopping times $\tau \in \mathcal{C}_{t,T}$].

The existence of $\phi$ such that $V_t(\phi) = u(t,S_t)$ can be proved by using the two following ingredients:
• The decomposition of a supermartingale into a martingale and a decreasing process.
• The martingale representation theorem.

It is (quite) natural to consider $u(t, S_t)$ as a price for the American option at time $t$, since it is the minimal value of a strategy hedging the option. The price $u(t, S_t)$ can also be considered in the owners point of view, as follows:

For simplicity, let us assume that $t = 0$. The owner of the option has to optimize the time when the option is exercised by choosing the best possible stopping time. Now, for any stopping time $\tau$ (in $\mathcal{C}_{t,T}$), the discounted value of the payoff $\Psi(S_\tau)$ obtained by exercising the option at time $\tau$ is $e^{-r\tau}\Psi(S_\tau)$, and hence, the value at time zero of an admissible strategy with value $\Psi(S_\tau)$ at time $\tau$ is

$$E^* [e^{-r\tau}\Psi(S_\tau)].$$

Therefore, the price of the American option would be

$$E^* [e^{-r\tau}\Psi(S_\tau)]$$

if the owner were obliged to exercise it at time $\tau$.

By optimizing in $\tau$, one obtains the "fair price" $u(0, S_0)$ of the American option.

The following proposition proves that in fact, there is no difference between European and American call option, if the interest rate $r$ is positive.

**Proposition 17.** The fair price of an American option corresponding to the function $\Psi : x \mapsto (x - K)_+$ (i.e. the American call option with strike price $K$ and maturity $T$) is equal to the price of a European call of strike price $K$ and maturity $T$, if the interest rate is positive.

**Proof.** It is sufficient to show that "the stopping time $T$ is optimal" (i.e. it is never useful to exercise the American call strictly before maturity). It is equivalent to prove (let us assume $t = 0$) that for all stopping times $\tau$:

$$E^* [e^{-r\tau}(S_\tau - K)_+] \leq E^* [e^{-rT}(S_T - K)_+] = E^* [(S_T^* - Ke^{-rT})_+]$$

On the other hand, we have:

$$E^* [(\tilde{S}_T - Ke^{-rT})_+|F_\tau] \geq E^* [(\tilde{S}_T - Ke^{-rT})|F_\tau]$$

Using the martingale property:

$$E^* [(\tilde{S}_T - Ke^{-rT})|F_\tau] = \tilde{S}_\tau - e^{-rT}K$$

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since \((\tilde{S}_t)_{t\leq T}\) is a \(\mathbb{P}^\ast\)-martingale and \(\tau\) is a (bounded) stopping time.

Since \(r \geq 0\), one deduces:

\[
E^\ast \left[ (\tilde{S}_T - K e^{-rT})_+ \big| \mathcal{F}_T \right] \geq \tilde{S}_T - e^{-rT} K
\]

Since the left hand side is non-negative:

\[
E^\ast \left[ (\tilde{S}_T - K e^{-rT})_+ \big| \mathcal{F}_T \right] \geq (\tilde{S}_T - e^{-rT} K)_+
\]

By taking the expectations, we obtain:

\[
E^\ast \left[ (\tilde{S}_T - K e^{-rT})_+ \big| \mathcal{F}_T \right] \geq E^\ast \left[ (\tilde{S}_\tau - K e^{-r\tau})_+ \big| \mathcal{F}_{\tau} \right]
\]

\[
\square
\]

### 3.4.1 Perpetual Put, Critical Price

In the case of the put, the American option price is not equal to the European one and there does not exist a closed form for the function \(u\) giving the price:

**Recall.**

\[ u(t, x) = \sup_{\tau \in \mathcal{C}_{0,T}} E^\ast \left[ (Ke^{-r(\tau - t)} - xe^{\sigma W_t - \tau \frac{\sigma^2}{2}})_+ \big| \mathcal{F}_t \right] \]

By replacing \(T\) by \(T - t\), it is always possible to come down to the case \(t = 0\) which gives:

\[ u(0, x) = \sup_{\tau \in \mathcal{C}_{0,T}} E^\ast \left[ (Ke^{-r\tau} - xe^{\sigma B_t - \tau \frac{\sigma^2}{2}})_+ \big| \mathcal{F}_t \right] \]

Let us now consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and let \((B_t)_{t \geq 0}\) be a standard Brownian motion (with infinite time horizon) defined on this space. The function \(u\) is now equal to the following:

\[ u(0, x) = \sup_{\tau \in \mathcal{C}_{0,\infty}} E \left[ (Ke^{-r\tau} - xe^{\sigma B_{\tau - \tau} \frac{\sigma^2}{2}})_+ \big| \mathcal{F}_{\tau} \right] \]

\[ \leq \sup_{\tau \in \mathcal{C}_{0,\infty}} E \left[ (Ke^{-r\tau} - xe^{\sigma B_{\tau} - \tau \frac{\sigma^2}{2}})_+ \right] 1_{\tau < \infty} \]

where \(\mathcal{C}_{0,\infty}\) denotes the set of all stopping times of the natural filtrations of \((B_t)_{t \geq 0}\) and \(\mathcal{C}_{0,T}\) consists of all the stopping times with values in \([0, T]\). The right hand side of the inequality above can be interpreted naturally as the value of a "perpetual put" (i.e., which can be exercised at any time). (Here, we have no rigorous justification of this interpretation). The following proposition gives an explicit formula for the price at a "perpetual put".

**Proposition 18.** The function \(u^\infty(x) := \sup_{\tau \in \mathcal{C}_{0,\infty}} E \left[ (Ke^{-r\tau} - xe^{\sigma B_{\tau} - \tau \frac{\sigma^2}{2}})_+ \right] 1_{\tau < \infty}\) is given by the formula:

\[ u^\infty(x) = \begin{cases} 
K - x & \text{for } x \leq x^* \\
(K - x^*) \frac{K}{1 + \gamma} & \text{for } x > x^* 
\end{cases} \]

with \(\gamma = \frac{2r}{\sigma^2}\) and \(x^* = \frac{k}{1 + \gamma}\). In particular \(0 < x^* < K\).
Proof. Admitted.

Example. $t = 1 \Rightarrow \sigma^2 = 2r \Rightarrow x^* = K^2$

We see, that the put price is $K - x$ for $x \leq x^*$ for all $T$.

Remark. $T = 0$ gives $(K - x)_+$

3.4.2 Description of the Best Strategy for an American Put

$(T < \infty)$.

One can prove that for any $t \in [0, T)$, there exists a real number $s(t) \in [0, K]$ satisfying:

$$\forall x \leq s(t) : u(t, x) = K - x \quad \text{and} \quad \forall x > s(t) : u(t, x) > (K - x)_+$$

$(T = \infty, s(0) = x^*)$.

The number $s(t)$ is interpreted as the "critical price" at time $t$. If the price of the underlying asset at time $t$ is less then or equal to $s(t)$, the buyer of the option should exercise it immediately, in the opposite case, he should wait.

3.5 Implied Volatility and Local Volatility Models

One of the main features of the Black-Scholes model (and one of the reasons for it success) is the fact that the pricing formula, as well as the hedging formula, depend on only one "non-observable" parameter: the volatility $\sigma$ (recall that the drift parameter $\mu$ does not appear!).

Question. How can we evaluate $\sigma$?

- **Historical Method:** for all $t$, $\log(S_t)$ has variance $\sigma^2 t$ and the variables $\log \left( \frac{S_t}{S_0} \right)$, $\log \left( \frac{S_{t+1}}{S_t} \right)$, $\log \left( \frac{S_{t+1}}{S_{t-1}} \right)$ are independent, identically distributed. Therefore $\sigma$ can be estimated by statistical means using past observation of the asset prices.

  This method is essentially not used by practitioners.

  **Problem.** In order to make a good statistics, you need to quite far in the past (ex. $t = 1$ day, $N = 1000$, you need to consider $\geq 3$ years of data) of taking very short intervals of time ($Nt = 1$ day, $t < 1$ min).

  - **case 1:** Market changes a lot.
case 2: The price is not continuous at very short scale in practice.

- "Implied Volatility" Method: Some options (calls and puts for some particular maturities and strike prices) are directly traded in the market. The “fair price” is observable. In Black-Scholes model this price is an increasing function of the volatility $\sigma$. (Fair price is a certain function of the volatility which is known by the market). (Fair price = function( known unknown))

By inverting Black-Scholes formulae, we can remove $\sigma$. This method is used by practitioners.

**Problem.** The volatility observed in this way depends on the option which is considered. The model for the stock price depends on the option (the parameter $\sigma$ changes). This is due to the fact that Black-Scholes model does not fit in the real market dates. (for example: the tails of distribution of log($S_t$) are larger than Gaussian tails in practice).

Despite of this problem, the Black-Scholes model is used as a reference by practitioners. The implied volatility of an option being “more meaningful than its price”.

One has constructed some models which are more consistent with market data. For example, one can replace the constant volatility $\sigma$ by a stochastic process $(\sigma_t)_{t \geq 0}$, and then obtain:

$$dS_t = S_t(\mu dt + \sigma_t dB_t).$$

If $(\sigma_t)_{t \geq 0}$ is adapted to the natural filtration of $B$, and if $\sigma_t, r/S_t$ are bounded, then the approach of this chapter can be extended. In particular one can construct a martingale measure $\mathbb{P}^*$, replicate European options, and give their prices as an expectation of the discounted payoff with respect to $\mathbb{P}^*$. However, we have no explicit formulae (as Black-Scholes formulae).

A particular case of this model is the local volatility model where $\sigma_t = \sigma(t, S_t)$, $\sigma$ being a deterministic function. Dupire, Derman and Kani have observed that, given market prices of call options, one can construct a local volatility model which provides the same prices as the market.

More precisely, if $C(T, K)$ is the market price of a call option with strike price $K$ and maturity $T$, observed at time $0$, the local volatility is given by the following equation (Dupire’s formula):

$$\frac{\partial C}{\partial T}(T, K) = \frac{\sigma^2(T, K) \cdot K^2}{2} \cdot \frac{\partial^2 C}{\partial K^2}(T, K) - rK \frac{\partial C}{\partial K}(T, K)$$

In particular, this formula involves the derivatives of the option prices $\Rightarrow$ it is difficult to implement. (Not so many options are traded).

**Remark.** The class of stochastic volatility models generally refers to models where $(\sigma_t)$ is driven by another Brownian motion. These markets are not complete, i.e. one cannot, as in the Black-Scholes model, replicate all the European options.

### 3.6 The Black-Scholes Model with Dividends, and Call/Put Symmetry

In the models described in this chapter, there is no distributed dividends. In fact, Black-Scholes methodology can be extended to options on dividend paying stock, when dividends are paid in
continuous time, at a constant rate $\delta$. This means that, in the infinitesimal interval of time $[t, t+dt]$, the holder of one share (i.e. stock) receives $\delta S_t dt$. The interest of this rather unrealistic assumption is that it leads to closed formulae. (Foreign currencies: The model is more realistic, and $\delta$ can be interpreted as an interest rate).

In the context of a dividend paying asset, the self-financing condition takes the following form:

$$dV_t = H_0^0 dS_t^0 + H_t (dS_t + \delta S_t dt)$$

and there is the discounted version:

$$d\tilde{V}_t = H_t (d\tilde{S}_t + \delta \tilde{S}_t dt) = H_t \sigma \tilde{S}_t dW_t$$

where

$$W_t^{(\delta)} = B_t + \frac{(\mu + \delta - r)}{\sigma} t$$

### 3.6.1 Self-Financing Property

$$dV_t = H_0^0 dS_t^0 + H_t (dS_t + \delta S_t dt)$$

$$d\tilde{V}_t = H_t \sigma \tilde{S}_t dW_t^{(\delta)}$$

where

$$W_t^{(\delta)} = B_t + \frac{\mu + \delta - r}{\sigma} t$$

The pricing measure is now the probability $\mathbb{P}^\delta$ under which $(W_t^{(\delta)})_{t \leq T}$ is a Brownian motion. Under $\mathbb{P}^\delta$, one can check that $(e^{(\delta - r)t} S_t)_{t \leq T}$ is a martingale. One can write the option prices as expectations under this probability measure $\mathbb{P}^\delta$. One has a symmetry between call and put prices.

Let us denote by $C_e(t, x, K, r, \delta)$ (resp. $P_e(t, x, K, r, \delta)$) the prices at time $t$ of a European call (resp. put) option with maturity $T$, exercise (strike) price $K$, for a current stock price $x$ ($S_t = x$) when the interest rate is $r$ and the dividend yield is $\delta$. By similar computation as for the option without dividends, one has:

$$C_e(t, x, K, r, \delta) = \mathbb{E}^\delta \left[ e^{-r(T-t)} \left( x e^{-(r-\frac{\delta^2}{2})(T-t)+\sigma(W_t^\delta-W_t^\delta)} - K \right) \mathbb{1}_{C_e(t, x, K, r, \delta)} \right]$$

Let us also define $C_a(t, x, K, r, \delta)$ as the price of the American option with the same parameters (also define $P_a(t, x, K, r, \delta)$).

**Proposition 19.**

- $C_e(t, x, K, r, \delta) = P_e(t, K, x, \delta, r)$
- $C_a(t, x, K, r, \delta) = P_a(t, K, x, \delta, r)$
Proof. We prove the result for American options (the case of European option is similar). We assume that $t = 0$ for simplicity. We have (similarly to the case of options without dividends):

$$C_a(0, x, K, r, \delta) = \sup_{\tau \in C_{0,T}} \mathbb{E}^\delta \left[ e^{-r\tau} \left( xe^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W^\delta_{\tau}} - K \right)_+ \right]$$

For $\tau \in C_{0,T}$, we have, with the notation $\hat{W}^\delta_t := W^\delta_t - \sigma t$, and let us denote by $\hat{P}^\delta$ the probability measure with density given by:

$$\frac{d\hat{P}^\delta}{dP} = \exp(\sigma W^\delta_T - T\frac{\sigma^2}{2})$$

one has:

$$\mathbb{E}^\delta \left[ e^{-r\tau} \left( xe^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W^\delta_{\tau}} - K \right)_+ \right] = \mathbb{E}^\delta \left[ e^{-\delta \tau} e^{\sigma W^\delta_{\tau} - \frac{\sigma^2}{2} \tau} \left( x - K e^{(\delta - r + \frac{\sigma^2}{2})\tau - \sigma W^\delta_{\tau}} \right)_+ \right]$$

The last equality comes from the fact that $\left( e^{\sigma W^\delta_{\tau} - \frac{\sigma^2}{2} \tau} \right)_{t \leq T}$ is a martingale under $P^\delta$. Therefore

$$\mathbb{E}^\delta \left[ e^{-r\tau} \left( xe^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W^\delta_{\tau}} - K \right)_+ \right] = \mathbb{E}^\delta \left[ e^{-\delta \tau} \left( x - K e^{(\delta - r + \frac{\sigma^2}{2})\tau - \sigma W^\delta_{\tau}} \right)_+ \right]$$

under $\hat{P}^\delta$.

Now, under the probability $\hat{P}^\delta$, the process $(\hat{W}^\delta_t)_{t \leq T}$ is a standard Brownian motion (because of the Girsanov theorem), as well as $(-\hat{W}^\delta_t)_{t \leq T}$.

We can deduce, by looking to the last formula that:

$$C_a(t, x, K, r, \delta) = P_a(t, K, x, r, \delta)$$

\[ \square \]

4 Option Pricing and PDE

In the previous chapter we have obtained a formula for European option prices in the case of Black-Scholes model. However, there are not anymore explicit expressions for more general options (e.g. as American option) or for more complex models. In this chapter we study option prices in a more general framework, and we prove that they satisfy some PDE. Despite that, these equations do not give explicit formulas, they can be used in order to do numerical computations.
4.1 European Option Pricing and Diffusions

Recall that in Black-Scholes model the price of an European call or put (vanilla) options is given by:

\[ V_t = E \left[ e^{-r(T-t)} f(S_T) | \mathcal{F}_t \right] \]

with

\[ f(x) = (x - K)_+ \text{ for calls, and} \]
\[ f(x) = (K - x)_+ \text{ for puts, and} \]
\[ S_T = S_0 \exp \left( \left( r - \frac{\sigma^2}{2} \right) T + \sigma W_T \right) \]

(where \( r \) is the interest rate, \( \sigma \) the volatility, \( T \) the maturity, \( K \) the strike price and \( W \) is a standard Brownian motion).

The problem of computing \( V_t \) can be generalized with the following way: Let \((X_t)_{t \leq T}\) be a stochastic process which satisfies the stochastic differential equation:

\[ dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t \]

where \( b \) and \( \sigma \) are functions from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{R} \), satisfying the following technical conditions:

(a) \(|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y| \text{ for some } K > 0 \text{ independent of } t.\]

(b) \(|b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|) \text{ for some } K > 0 \text{ independent of } t.\]

The rigorous meaning of this stochastic differential equation is the following: Consider the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\), and \((W_t)_{t \geq 0}\) an \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion. A solution of the equation above is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted continuous stochastic process \((X_t)_{t \leq T}\) on the interval \([0, T]\) such that \(X_0 = Z\) almost surely.

1. \( \forall t \geq 0 \), the integrals \( \int_0^t b(s, X_s) \, ds, \int_0^t \sigma(s, X_s) \, dW_s \) exist, i.e.

   - \( \int_0^t |b(s, X_s)| \, ds < \infty \)
   - \( \int_0^t |\sigma(s, X_s)|^2 \, dW_s < \infty \)

2. There exists an \( \mathcal{F}_0 \)-measurable random variable \( Z \) such that

\[ X_t = Z + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s \]

If \( b \) and \( \sigma \) satisfy the conditions (a) and (b) above, and if \( Z \) is a square integrable \( \mathcal{F}_0 \)-measurable random variable, then there exists a unique solution (to the stochastic differential equation above) \((X_t)_{t \leq T}\) on the interval \([0, T]\) such that \(X_0 = Z\) almost surely.

(Uniqueness: Two solutions are indistinguishable, i.e. a.s. there are equal at each time).
(For Black-Scholes model under $P^*$: $b = rX_t$ and $\sigma(t, X_t) = X_t \cdot \text{volatility}$).

If $(X_t)_{t \geq 0}$ satisfies the stochastic differential equation above, let us try to compute:

$$V_t = \mathbb{E} \left[ e^{-\int_0^T r(s, X_s) \, ds} f(X_T) \mid F_t \right]$$

where $r : (t, x) \mapsto r(t, x)$ is a bounded, continuous function. In this model, $r$ is the (non-constant) interest rate and $\sigma$ is the (non-constant) volatility, multiplied by the current stock price.

$\Rightarrow$ $V_t$ represents the value at time $t$ of an European option with payoff $f(X_t)$.

In the same way as in the Black-Scholes model, $V_t$ can be written as a function of $t$ and $X_t$:

$$V_t = F(t, X_t)$$

where

$$F(t, X_t) = \mathbb{E} \left[ e^{-\int_0^T r(s, X_s) \, ds} f(X_T) \mid X_t = x \right]$$

and $(X_s^{(t,x)})_{s \in [t,T]}$ is the solution of the stochastic differential equation which starts at the value of $x$ and at time $t$. Intuitively,

$$F(t, x) = \mathbb{E} \left[ e^{-\int_0^T r(s, X_s) \, ds} f(X_T) \mid X_T = x \right]$$

(non rigorous a priori).

As a consequence, the option price is known, if we are able to compute the function $F$. Now, we will find that $F$ satisfies a certain partial differential equation, related with an operator, called infinitesimal generator of the process $(X_t)_{t \geq 0}$. (One says that $X$ is a diffusion, which justifies the title of this section).

### 4.2 Infinitesimal Generator of a Diffusion

Let us now assume that $b$ and $\sigma$ do not depend on the time. Hence $(X_t)_{t \geq 0}$ is the solution of the equation:

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t.$$

We deduce the following result:

**Proposition 20.** Let $f$ be continuously twice differentiable function ($f \in C^2$) with bounded derivatives, and let $A$ be the differential operator that maps a $C^2$-function $f$ to $Af$ such that:

$$(Af)(x) = \frac{[\sigma(x)]^2}{2} f''(x) + b(x) f'(x)$$

Then, the process $(M_t)_{t \geq 0}$, $M_t = f(X_t) - \int_0^t (Af)(X_s) \, ds$ is an $(\mathcal{F}_t)_{t \leq T}$-martingale.
Proof. Itô formula yields: \((f \in C^2)\)

\[
f(X_t) = f(X_0) + \int_0^t f'(X_s) \, ds + \frac{1}{2} \int_0^t f''(X_s) \sigma^2(X_s) \, ds
\]

\(\sigma^2(X_s) \, ds = d\langle X, X \rangle_s\) because of the stochastic differential equation satisfied at \((X_t)_{t \geq 0}\).

\[
\Rightarrow f(X_t) = f(X_0) + \int_0^t f'(X_s) \sigma(X_s) \, dW_s + \int_0^t \left( \frac{1}{2} \sigma^2(X_s) f''(X_s) + b(X_s) f'(X_s) \right) \, ds
\]

\[
\Rightarrow f(X_t) - \int_0^t (Af)(X_s) \, ds = \int_0^t f'(X_s) \sigma(X_s) \, dW_s
\]

To check that the right-hand side is a martingale, it is sufficient to check that the expectation:

\[
E \left[ \int_0^t |f'(X_s)|^2 \cdot |\sigma(X_s)|^2 \, ds \right] < \infty
\]

This is a consequence of the technical assumptions made on \(f', \sigma\) and of some properties of the solutions of stochastic differential equation (unfortunately, we cannot detail the proof here).

Remark. If we denote by \(X_t^{(x)}\) the solution of the stochastic differential equation above, such that \(X_0^{(x)} = x\), we obtain, from the proposition above:

\[
E \left[ f(X_t^{(x)}) \right] = f(x) + E \left[ \int_0^t (Af)(X_s^{(x)}) \, ds \right]
\]

Moreover, since the derivatives of \(f\) are supposed to be bounded by a constant \(K_f\), and since \(|b(x)| + |\sigma(x)| \leq K(1 + |x|)\) we have:

\[
E \left[ \sup_{s \leq T} |Af(X_s^{(x)})| \right] \leq K_f \left( 1 + E \left[ \sup_{s \leq T} |X_s^{(x)}|^2 \right] \right)
\]

where the right-hand side can be proved to be finite (properties of solutions of stochastic differential equation). Therefore, since \(x \mapsto (Af)(x)\) and \(s \mapsto X_s^{(x)}\) are continuous, the dominated convergence theorem is applicable and yields:

\[
\lim_{t \to 0} E \left[ \int_0^t (Af)(X_s^{(x)}) \, ds \right] = \int_0^t (Af)(X_s^{(x)}) \, ds
\]

\[
= (Af)(x)
\]
The differential operator $A$ is called the infinitesimal generator of the diffusion $X$.

The proposition above can be generalized in the time-dependent case, with major difficulties.

**Proposition 21.** If $u(t,x)$ is a $C^{1,2}$-function (i.e. once differentiable with respect to $t$ and twice with respect to $x$ and with continuous partial derivatives) with bounded derivatives with respect to $x$, and if $(X_t)_{t \geq 0}$ is the solution of the stochastic differential equation above:

$$dX_t = b(t,X_t) dt + \sigma(t,X_t) dW_t,$$

(with the technical assumption above)

$$M_t = u(t,X_t) - \int_0^t \left( \frac{\partial u}{\partial t} + A_s u \right) (s,X_s) \, ds$$

is a martingale where $\frac{\partial u}{\partial t}$ denotes the first derivative of $u$ and the operator $A_s$ is defined by:

$$(A_s u)(x) = \frac{\sigma^2(s,x) \partial^2 u}{2} + b(s,x) \frac{\partial u}{\partial x}$$

**Proof.** The proof is similar to the time-independent case. (We apply Itô formula for a function of the time and an Itô process).

In order to deal with discounted values (i.e. multiplied by a factor $\exp \left( - \int_{t_1}^t r(s,X_s) \, ds \right)$), we need to generalize again a little bit.

**Proposition 22.** Under the assumption of the proposition above and for $r : (t,x) \mapsto r(t,x)$ a bounded, continuous function from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$, the process

$$M_t = e^{-\int_{t_1}^t r(s,X_s) \, ds} u(t,X_t) - \int_0^t e^{-\int_{s_0}^s r(v,X_v) \, dv} \left( \frac{\partial u}{\partial t} + A_s u - ru \right) (s,X_s) \, ds$$

is a martingale.

**Sketch of the proof.** We use integration by parts formula to differentiate the product $e^{-\int_{t_1}^t r(s,X_s) \, ds} u(t,X_t)$ and then apply Itô formula to the process $u(t,X_t)$.

**4.3 Conditional Expectations and PDE**

In this section, we emphasize the link between pricing European option and solving a partial differential equation.

**Recall.** We want to compute:

$$V_t = \mathbb{E} \left[ e^{-\int_t^T r(s,X_s) \, ds} f(X_T) | \mathcal{F}_t \right]$$
which is equal to $F(t, X_t)$, where:

$$F(t, x) = \mathbb{E} \left[ -\int_t^T r(s, X_s^{(t,x)}) \, ds \, f(X_T^{(t,x)}) \right]$$

The following result gives a characterization of $F$ as a solution of a partial differential equation.

**Proposition 23.** Let $u$ be a $C^{1,2}$-function, with a bounded derivation with respect to the second variable defined on $[0, T) \times \mathbb{R}$. If $u$ satisfies: $\forall x \in \mathbb{R} : u(T, x) = f(x)$ and $(\frac{\partial u}{\partial t} + A_t u - ru)(t, x) = 0$ for all $(t, x) \in [0, T) \times \mathbb{R}$, where $u(T, x) = \lim_{t \to T} u(t, x)$ (by definition). Then: $\forall (t, x) \in [0, T) \times \mathbb{R} : F(t, x) = u(t, x)$.

**Proof.** Let us prove the equality $F(t, x) = u(t, x)$ for $t = 0$ (for $t > 0$, use time shift with a similar proof). From the previous proposition, we know that the process:

$$M_t = e^{-\int_0^t r(s, X_s^{(0,x)}) \, ds} u(t, X_t^{(0,x)})$$

is a martingale (the non-written term here is equal to zero by the partial differential equation satisfied by $u$). Therefore $\mathbb{E}[M_0] = \mathbb{E}[M_T]$, which yields:

$$u(0, x) = \mathbb{E} \left[ e^{-\int_0^T r(s, X_s^{(0,x)}) \, ds} u(T, X_T^{(0,x)}) \right]$$

$$= \mathbb{E} \left[ e^{-\int_0^T r(s, X_s^{(0,x)}) \, ds} f(X_T^{(0,x)}) \right]$$

since $u(T, x) = f(x)$. \qed

**Remark.** This result suggests the following method to price the options: In order to compute:

$$F(t, x) = \mathbb{E} \left[ -\int_t^T r(s, X_s^{(t,x)}) \, ds \, f(X_T^{(t,x)}) \right]$$

for a given $f$, we first need to find $u$ such that:

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u - ru = 0 & \text{in } [0, T) \times \mathbb{R} \\ u(T, x) = f(x) & \text{for } x \in \mathbb{R} \end{cases}$$

This is a partial differential equation with final condition (at terminal time $T$). If we find a solution $u$ of this equation and if it satisfies the regularity assumptions of the last proposition, of the previous section, we conclude that $F = u$.

### 4.4 Application to Black-Scholes Model

We are working on the probability measure $\mathbb{P}^*$, under which the discounted stock price is a martingale ($\mathbb{P}^*$ is called risk neutral probability measure).
Under $\mathbb{P}^*$, the stock price $S_t$ satisfies $dS_t = S_t(r dt + \sigma dW_t)$, where $(W_t)_{t \leq T}$ is a Brownian motion.

The operator $A_t$ is time-independent and is equal to:

$$A_t = A^{(bs)} = \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + rx \frac{\partial}{\partial x}$$

(bs for Black-Scholes: $A^{(bs)} f(x) = \frac{\sigma^2}{2} x^2 f''(x) + rf'(x)$)

It is straightforward (but a little bit painful) to check that the call price given by

$$F(t, x) = x N(d_1) - Ke^{-r(T-t)} N(d_1 - \sigma \sqrt{T-t})$$

with

$$d_1 = \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$

is a solution of the partial differential equation

$$\begin{cases}
\partial u/\partial t + (\tilde{A}u) = 0 & \text{on } [0, T) \times \mathbb{R}_+ \\
u(T, x) = (x - K)_+ & \text{in } \mathbb{R}_+ \text{ (as limit of } u(t, x) \text{ to } t \rightarrow T)
\end{cases}$$

The same result holds for the put price.

### 4.5 PDE on Bounded Open Sets and Computations of Expectations

In this section we assume that there is only one asset and that $b(x)$, $\sigma(x)$, $r(x)$ are all time-independent. The quantity $r(x)$ can be interpreted as the instantaneous interest rate.

Consider the differential operator $A$, given by:

$$(Af)(x, t) = \frac{1}{2}(\sigma(x))^2 \frac{\partial^2}{\partial x^2} f(x, t) + b(x) \frac{\partial}{\partial x} f(x, t)$$

Let us denote by $\tilde{A}$ the "discount operator", defined by:

$$(\tilde{A} f)(x, t) = Af(x, t) - r(x) f(x, t).$$

The equation giving the option price becomes

$$\begin{cases}
\partial u/\partial t + (\tilde{A}u) = 0 & \text{on } [0, T) \times \mathbb{R} \\
u(T, x) = f(x) & \text{for } x \in \mathbb{R} \text{ (or } x \in \mathbb{R}_+\text{)}
\end{cases}$$

Let us first try to solve the problem on the bounded "space" interval $O = (a, b)$, $a < b$.

We then use some boundary conditions at $a$ and $b$ in order to have an equation with a unique solution. Let us assume that $u$ vanishes at the boundaries of $O$ (called Dirichlet boundary conditions).

We now deal with the following problem: solve:

$$\begin{cases}
\partial u/\partial t + (\tilde{A}u) = 0 & \text{on } [0, T) \times O \\
u(T, x) = f(x) & \forall x \in O \\
u(t, a) = u(t, b) = 0 & \forall t \in (0, T)
\end{cases}$$
It is now more convenient to deal with bonded interval $O = (a, b)$. And we can hope that the solution is ”close” to the solution of the initial problem. In fact the solution of this modified problem can be expressed in terms of the diffusion $X^{(t,x)}$.

**Proposition 24.** Let $u$ be a continuous function of $[0, T] \times [a, b]$. Assume that $u$ is $C^{1,2}$ on $(0, T) \times O$ and that $\frac{\partial u}{\partial x}$ is bounded on $(0, T) \times O$. Then, if $u$ satisfies the conditions given in the ”modified” problem, then for all $(t, x) \in [0, T] \times O$:

$$u(t, x) = \mathbb{E} \left[ \mathbb{I}_{\forall s \in [t, T], X_s^{(t,x)} \in O} \cdot e^{-\int_t^T r(X_s^{(t,x)}) \, ds} | f(X_T^{(t,x)}) \right]$$

**Proof.** We prove the result for $t = 0$ since the argument is similar at the other times. In order to avoid technicalities, we assume that there exists an extension of the function $u$ from $[0, T] \times O$ to $[0, T] \times \mathbb{R}$, that is still of class $C^{1,2}$. We also denote by $u$ such extension. We know that:

$$M_t = \exp \left( - \int_0^t r(X_s^{(0,x)}) \, ds \right) u(t, X_t^{(0,x)}) - \int_0^t e^{-\int_0^s r(X_u^{(0,x)}) \, du} \left( \frac{\partial u}{\partial t} + Au - ru \right)(s, X_s^{(0,x)}) \, ds$$

is a martingale. Now let

$$\tau(x) := \inf\{0 \leq s \leq T | X_s^{(0,x)} \notin O \} \land T \ (\text{with the convention } \inf\emptyset = \infty).$$

Note that $\tau(x)$ is a (bounded) stopping time, because $\tau(x) = T_a^{(x)} \cap T_b^{(x)} \cap T$ where:

$$T_b^{(x)} = \inf\{0 \leq s \leq T | X_s^{(t,x)} = b\},$$

and indeed $\tau(x)$ is a stopping time according to general results on cont. stochastic processes. By applying the optimal stopping theorem with the stopping time $\tau(x)$, we get $\mathbb{E}[M_0] = \mathbb{E}[M_{\tau(x)}]$. Thus, by noticing that for $s \in [0, \tau(x)]$, $(Af)(X_s^{(0,x)}) = 0$, it follows that:

$$u(0, x) = \mathbb{E} \left. \left[ \mathbb{I}_{\forall s \in [t, T], X_s^{(t,x)} \in O} \cdot e^{-\int_t^T r(X_s^{(t,x)}) \, ds} \right. \left| u(\tau(x), X_{\tau(x)}^{(0,x)}) \right] \right]$$

$$= \mathbb{E} \left[ \mathbb{I}_{\exists s \in [t, T], X_s^{(t,x)} \in O} e^{-\int_t^T r(s, X_s^{(0,x)}) \, ds} u(T, X_T^{(0,x)}) \right]$$

Furthermore, $f(x) = u(T, x)$ and $u(\tau(x), X_{\tau(x)}^{(0,x)}) = 0$ on the event $\{\exists s \in [t, T], X_s^{(t,x)} \notin O\}$ (because of the boundary conditions). The second expectation vanishes

$$u(0, x) = \mathbb{E} \left[ \mathbb{I}_{\forall s \in [t, T], X_s^{(t,x)} \in O} e^{-\int_t^T r(s, X_s^{(0,x)}) \, ds} \right. \left| u(T, X_T^{(0,x)}) \right. \left/ f(X_T^{(t,x)}) \right]$$

which completes the proof for $t = 0$. $\square$
\[ u(t, x) = \mathbb{E} [ \text{solution of SDE} ] \rightarrow \text{solution of a partial differential equation}. \]

**Remark.** An option defined by the \( \mathcal{F}_t \)-measurable random variables:

\[
\mathbb{1}_{\forall s \in [t, T], X_s \in O} \exp \left( -\int_t^T r(X_s^{(t,x)}) \, ds \right) \left( X_t^{(t,x)} \right)
\]

is called extinguishable. Indeed, as soon as the asset price exists the open set \( O \), the option becomes worthless (payoff 0).

In the Black-Scholes model, if \( O \) is of the form \((0, l)\) or \((l, \infty)\) we are able to compute explicit formulae for the option price.

### 4.6 American Options

The analysis of American options in continuous time is not straightforward. In the Black-Scholes model, we obtained the following formula for the price of American call \((f(x) = (x - K)_+)\) or of American put \((f(x) = (K - x)_+)\):

\[ V_t = \Phi_t(t, S_t) \]

where

\[ \Phi(t, x) = \sup_{\tau \in \mathcal{C}_{t,T}} \mathbb{E}^{*} \left[ e^{-r(T-t)} f(x, e^{r(T-t) + \sigma \sqrt{(T-t)}}) \right] \]

and under \( \mathbb{P}^* \), \((W_t)_{t \geq 0}\) is a standard Brownian motion, \( \mathcal{C}_{t,T} \) is the set of all stopping times with values in \([t, T]\). We showed that American call price (on a stock offering no dividends) is equal to the European call price. For the American put price, there is no explicit formula and numerical methods are needed. The problem to be solved is a particular case of the following general problem, given a "good" function \( f \) and a diffusion \((X_t)_{t \geq 0}\) in \( \mathbb{R} \), the solution of the s.d.e.:

\[ dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \]

compute the function:

\[ \Phi(t, x) = \sup_{\tau \in \mathcal{C}_{t,T}} \mathbb{E} \left[ e^{-\frac{1}{2} \int_0^\tau r(s, X_s) \, ds} \Phi(\tau, X_\tau) \right] \]

By considering the stopping time \( \tau = t \), we get \( \Phi(t, x) \geq f(x) \). Also note that, for \( t = T \), we clearly have \( \Phi(T, x) = f(x) \).

**Remark.** It can be proved that the process \( e^{-\frac{1}{2} \int_0^\tau r(s, X_s) \, ds} \Phi(\tau, X_\tau) \) is the smallest supermartingale which dominates the process \( f(X_t) \) at all times.

For European options, we have constructed some corresponding PDEs. For American options, we will obtain some partial differential inequalities. The following properties tries to explain that.
Proposition 25. Assume that \( u \) is a "regular" solution of the following system of partial differential inequalities (we don’t give here the meaning of "regular"):

\[
\begin{align*}
\frac{\partial u}{\partial t} + A_t u - ru &\leq 0, \quad u \geq f \quad \text{in } [0,T) \times \mathbb{R} \\
\frac{\partial u}{\partial t} + A_t u - ru &\geq 0 \quad \text{or } u = f \quad \text{in } [0,T) \times \mathbb{R} \\
u(T, x) &= f(x)
\end{align*}
\]

Then, \( u(t, x) = \Phi(t, x) = \sup_{\tau \in \mathcal{C}_{t,T}} \mathbb{E} \left[ e^{-\int_0^{\tau} r(s, X^{(t,x)}_s) ds} f(X^{(t,x)}_{\tau}) \right] \)

Sketch of the proof. We suppose \( t = 0 \), and we denote \( X^{(t,x)}_t \) the solution of the s.d.e. satisfied by \( X \), which starts at \( x \). The process

\[
M_t = \exp \left( -\int_0^t r(s, X^{(t,x)}_s) ds \right) u(t, X^{(t,x)}_t) - \int_0^t \exp \left( -\int_0^s r(v, X^{(t,x)}_v) dv \right) \left( \frac{\partial u}{\partial t} + A_t u - ru \right)(s, X^{(t,x)}_s) ds
\]

is a martingale. By applying the optimal stopping theorem to this martingale, we get \( \mathbb{E}[M_0] = \mathbb{E}[M_{\tau}] \), and since \( \frac{\partial u}{\partial t} + A_t u - ru \leq 0 \):

\[
u(0, x) \geq \mathbb{E} \left[ e^{-\int_0^{\tau} r(s, X^{(t,x)}_s) ds} u(\tau, X^{(t,x)}_{\tau}) \right]
\]

Recall that \( u(t, x) \geq f(x) \), thus:

\[
u(0, x) \geq \mathbb{E} \left[ e^{-\int_0^{\tau} r(s, X^{(t,x)}_s) ds} f(X^{(t,x)}_{\tau}) \right]
\]

This proves that

\[
u(0, x) \geq \sup_{\tau \in \mathcal{C}_{0,T}} \mathbb{E} \left[ e^{-\int_0^{\tau} r(s, X^{(t,x)}_s) ds} f(X^{(t,x)}_{\tau}) \right] = \Phi(0, x)
\]

(one side of the "double" inequality).

We need to prove the reverse inequality.

Let \( \tau_{\text{opt}} = \inf\{0 \leq s \leq T | u(s, X^{(t,x)}_s) = f(X^{(t,x)}_s) \} \).

It can be proved that \( \tau_{\text{opt}} \) is a stopping time. Also, for \( s \) between 0 and \( \tau_{\text{opt}} \), we have:

\[
\left( \frac{\partial u}{\partial t} + A_t u - ru \right)(s, X^{(t,x)}_s) = 0
\]

The optimal stopping theorem yields:

\[
u(0, x) = \mathbb{E} \left[ e^{-\int_0^{\tau_{\text{opt}}} r(s, X^{(t,x)}_s) ds} u(\tau_{\text{opt}}, X^{(t,x)}_{\tau_{\text{opt}}}) \right]
\]

but at time \( \tau_{\text{opt}} \): \( u(\tau_{\text{opt}}, X^{(t,x)}_{\tau_{\text{opt}}}) = f(X^{(t,x)}_{\tau_{\text{opt}}}) \), so that

\[
u(0, x) = \mathbb{E} \left[ e^{-\int_0^{\tau_{\text{opt}}} r(s, X^{(t,x)}_s) ds} f(X^{(t,x)}_{\tau_{\text{opt}}}) \right]
\]

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This proves
\[ u(0, x) \leq \sup_{\tau \in \mathcal{C}_0, T} \mathbb{E} \left[ e^{-\int_0^\tau r(s, X_s^{(x)}) \, ds} f(X_{\tau}^{(x)}) \right] = \Phi(0, x) \]
\[ \Rightarrow u(0, x) = \Phi(0, x). \] We have also proved that \( \tau_{\text{opt}} \) is an optimal stopping time for the owner of the option.

Remark. The precise meaning of differential inequalities given in the proposition is awkward because even for a "regular" function \( f \), the solution is generally not in \( C^2 \). (And hence the second derivative is not well-defined a priori). This is a problem to define the infinitesimal generator of \( X \) on \( a \) and then one cannot apply directly Itô formula. (The sketch of the proof above is not rigorous). However, it is possible to give a meaning to everything here, and to make rigorous statements and proofs.

4.6.1 Example of Black-Scholes Model

We work under the measure \( \mathbb{P}^* \Rightarrow dS_t = S_t(r \, dt + \sigma dW_t) \). Let us introduce \( X_t := \log(S_t) = \log(S_0) + (r - \sigma^2/2)t + \sigma W_t \). Its infinitesimal generator \( A \) is actually time-independent, and
\[ A_{\text{BS-log}} := \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + (r - \frac{\sigma^2}{2}) \frac{\partial}{\partial x} \]
Let us define:
\[ \tilde{A}_{\text{BS-log}} := A_{\text{BS-log}} - r \]
(multiplication of functions by \( r \)).

If we consider \( \Phi(x) := (K - e^x)_+ \), the partial differential inequality corresponding to the price of the stock price of the American put is:
\[
\begin{cases}
\frac{\partial v}{\partial t}(t, x) + \tilde{A}_{\text{BS-log}} v(t, y) \leq 0 & \text{a.e. in } [0, T) \times \mathbb{R} \\
v(t, x) \geq \Phi(x) \\
v(t, x) = \Phi(x) \text{ or } \frac{\partial v}{\partial t}(t, x) + \tilde{A}_{\text{BS-log}} v(t, x) = 0 & \text{a.e. in } [0, T) \times \mathbb{R} \\
v(T, x) = \Phi(x)
\end{cases}
\]

The following proposition states existence and uniqueness results for a solution of this system and establishes the connection with American put price.

Proposition 26. The system of inequalities above has a unique continuous solution \( v(t, x) \) such that its partial derivatives in the distribution sense (extension of the notion of derivative to more general framework; not precised here):
\[
\frac{\partial v}{\partial x}, \quad \frac{\partial^2 v}{\partial x^2}
\]
are locally bounded. Moreover, this solution satisfies:

$$v(t, \log x) = \Phi(t, x) = \sup_{\tau \in \mathbb{C}_{t,T}} \mathbb{E}^* \left[ e^{-r(t-\tau)} f(xe^{(r-\frac{\sigma^2}{2})(\tau-t)+\sigma(w_{\tau}-w_t)}) \right]$$

(formula for the price of American option: put with strike price $K$)

### 4.7 Solving the PDE Numerically

(case of European options).

We saw under which conditions the option price coincided with the solutions of the PDE:

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + \tilde{A}u(t, x) = 0 & \text{on } [0, T) \times \mathbb{R} \\ u(T, x) = f(x) & \forall x \in \mathbb{R} \end{cases}$$

(for European options)

We have seen that it is more convenient to deal with a bounded interval instead of $\mathbb{R}$. It is possible by using the expression of the solution as an expectation of a function $f(X_s(x))$ to give some bound for the difference between the solution with $\mathbb{R}$ and the solution with $(-l,l)$. We have to solve the following system:

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + \tilde{A}u(t, x) = 0 & \text{on } [0, T) \times (-l,l) \\ u(t, l) = u(t, -l) = 0 & \text{if } t \in [0, T) \\ u(T, x) = f(x) & \text{if } x \in (-l,l) \end{cases}$$

The finite difference method: basically, this is a discretization of time and space of this system. We shall start by discretizing the differential operator $\tilde{A}$ on $(-l,l)$. In order to do this, we need to associate to a function $(f(x))_{x \in (-l,l)}$, which is an element of an infinite-dimensional vector space, a vector $(f_i)_{1 \leq i \leq N}$ in $\mathbb{R}^N$. We proceed as follows: for $i = 0, 1, \ldots, N$, let $x_i = -l + \frac{2i}{N+1}$. The number $f_0$ is supposed to be an approximation of $f(x_i)$. The boundary conditions can be stated as $f_0 = 0, f_{N+1} = 0$. Let $h := \frac{2l}{(N+1)}$ be the mesh of the subdivision of $(-l,l)$ coming from the $x_i$'s. The discretization of the operator $\tilde{A}$ is the operator $\tilde{A}_h$ on $\mathbb{R}^N$, defined as follows:

Think of a vector $u_n = (u^{(i)}_n)_{1 \leq i \leq N}$ in $\mathbb{R}^N$ as the discrete approximation of a function $u$ (i.e. $u^{(i)}_n = u(x_i)$) and replace the first derivative $\frac{\partial u}{\partial x}(x_i)$ with

$$\partial_n u^i = \frac{u^{(i+1)}_n - u^{(i-1)}_n}{2h}$$

"discrete derivative".

Similarly, replace the second derivative $\frac{\partial^2 u}{\partial x^2}(x_i)$ by

$$\partial^2_n u^i = \frac{u^{(i+1)}_n - u^{(i-1)}_n}{2h} = \frac{u^{(i+1)}_n - 2u^i_n + u^{(i-1)}_n}{h^2}$$

The discretized operator $A_n u_n$ is defined by:

$$(\tilde{A}_n u_n)_i = \frac{\sigma^2(x_i)}{2} \partial^2_n u^i + b(x_i) \partial_n u^i_n - ru^i_n, \quad i = 1 : N$$
Black-Scholes case: (after logarithmic change of variable).

\[
\tilde{A}^{\text{BS-log}}_{t}u(x) = \sigma^2 \frac{\partial^2 u}{\partial x^2}(x) + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial u}{\partial x}(x) - ru(x)
\]

\[
\left( \tilde{A}^{\text{BS-log}}_{n}u_{n} \right)_{i} = \frac{\sigma^2}{2h^2} \left( u_{n+1}^{i} - 2u_{n}^{i} + u_{n-1}^{i} \right) + \left( r - \frac{\sigma^2}{2} \right) \frac{1}{2h} \left( u_{n+1}^{i} - u_{n-1}^{i} \right) - ru_{n}^{i}
\]

\( \tilde{A}^{\text{BS-log}}_{n} \) can be represented by the following matrix:

\[
\left( (\tilde{A}^{\text{BS-log}}_{n})_{ij} \right)_{i,j=1}^{N} = \begin{bmatrix}
\beta & \gamma & 0 & \cdots & 0 \\
\alpha & \beta & \gamma & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \alpha & \beta & \gamma \\
0 & \cdots & 0 & \alpha & \beta \\
\end{bmatrix}
\]

where

\[
\alpha = \frac{\sigma^2}{2h^2} - \frac{1}{2h} \left( r - \frac{\sigma^2}{2} \right)
\]

\[
\beta = -\frac{\sigma^2}{h^2} - r
\]

\[
\gamma = \frac{\sigma^2}{2h^2} + \frac{1}{2h} \left( r - \frac{\sigma^2}{2} \right)
\]

By doing the discretization above, we replace the initial PDE by an ODE:

\[
\begin{cases}
\frac{du_n(t)}{dt} + \tilde{A}^{\text{BS-log}}_{n}u_n(t) = 0 & \text{if } 0 \leq t \leq T \\
u_n(T) = f_n
\end{cases}
\]

where \( f_n = (f_n^i)_{1 \leq i \leq N} \) is the vector \( f^i = f(x_i) \).

The time discretization can be obtained, using the so called "\( \Theta \)-Schemes". Consider \( \Theta \in [0,1] \) and let \( k \) be a time step such that \( T = Mk \) (\( M \) is a large positive integer). We approximate the solution \( u_n \) of the previous equation at the time \( nk \) by \( u_{n,h,k} \) where the sequence \( (u_{n,h,k})_{n=0}^M \) solves the following recursive equation:

\[
\begin{cases}
u_{n,h,k}^M = f_n \\
u_{n,h,k}^{n+1} - u_{n,h,k}^{n} = \Theta \tilde{A}^{\text{BS-log}}_{n}u_{n,h,k} + (1 + \Theta)\tilde{A}^{\text{BS-log}}_{n}u_{n,h,k}^{n+1} = 0 & 0 \leq n \leq M - 1
\end{cases}
\]

We can solve the system by going backwards from \( n = M \) to \( n = 0 \), since the vector \( u_{n,h,k}^M \) is known.

**Particular values of \( \Theta \):**

- \( \Theta = 0 \) ("explicit scheme"), \( u_{n,h,k}^n \) is computed directly from \( u_{n,h,k}^{n+1} \) (without solving a system of equations).
- \( \Theta = 1 \) ("completely implicit scheme"), we need to solve systems of equations.
- \( \Theta = \frac{1}{2} \), more symmetric, we also need to compute systems. (More precise).