JARNÍK-TYPE INEQUALITIES

STEFFEN WEIL

ABSTRACT. It is well known due to Jarník [19] that the set $\text{Bad}_1^R$ of badly approximable numbers is of Hausdorff-dimension one. If $\text{Bad}_1^R(c)$ denotes the subset of $x \in \text{Bad}_1^R$ for which the approximation constant $c(x) \geq c$, then Jarník was in fact more precise and gave nontrivial lower and upper bounds of the Hausdorff-dimension of $\text{Bad}_1^R(c)$ in terms of the parameter $c > 0$. Our aim is to determine simple conditions on a framework which allow to extend 'Jarník’s inequality’ to further examples; among the applications, we discuss the set $\text{Bad}_{\vec{r}}^R$ of badly approximable vectors in $\mathbb{R}^n$ with weights $\vec{r}$ and the set of geodesics in the hyperbolic space $H^n$ which avoid a suitable collection of convex sets.

1. INTRODUCTION AND MAIN RESULTS

An irrational number $x \in \mathbb{R}$ is called \textit{badly approximable} if there exists a positive constant $c = c(x) > 0$, called \textit{approximation constant} such that

$$|x - \frac{p}{q}| \geq \frac{c}{q^2}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The set $\text{Bad}_1^R$ of badly approximable numbers is a Lebesgue null-set, yet it is well known due to Jarník [19] that $\text{Bad}_1^R$ is of Hausdorff-dimension one. Note that a positive irrational number $x \in \mathbb{R}$ is badly approximable if and only if the entries $a_n \in \mathbb{N}$ of the continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$ of $x$ are bounded by some integer $N \in \mathbb{N}$. Moreover, a small bound $N$ on the entries corresponds to a larger approximation constant $c(x)$ in (1.1). In fact, if $p_n/q_n$ are the approximates given by the continued fraction expansion of $x$, then

$$\frac{1}{(a_{n+1} + 2)q_n^2} < |x - \frac{p_n}{q_n}| < \frac{1}{a_{n+1}q_n^2}.$$ 

Moreover, if $|x - p/q| < 1/(2q^2)$, then $p/q = p_n/q_n$ for a suitable $n$. Using this correspondence, Jarník was more precise and gave nontrivial lower and upper estimates on the Hausdorff-dimension of the set of badly approximable numbers with an approximation constant bounded below.

\textbf{Theorem 1.1 ([19], Satz 4).} If $M_N$ denotes the set of irrational numbers for which the entries of the continued fraction expansion are bounded by $N$, where $N > 8$, then

$$1 - \frac{4}{N \log(2)} \leq \dim(M_N) \leq 1 - \frac{1}{8N \log(N)}.$$ 

(1.2)

Here and in the following, ‘$\dim$’ stands for the Hausdorff-dimension.

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\textsuperscript{1} In fact, the approximation constant should be given in terms of the supremum of all the constants $c$ satisfying (1.1). However, this plays no role in the following.
In particular, inequality (1.2), which we call Jarník’s inequality, implies Jarník’s theorem on full Hausdorff-dimension of $\text{Bad}^{1}_{\mathbb{R}}$.

There is a further correspondence between Diophantine approximation and hyperbolic geometry. Let $\mathbb{H}^2/\text{SL}(2, \mathbb{Z})$ be the modular surface, which is a hyperbolic orbifold with a cusp; for details, we refer to Section 3. Let $H_t$ be the maximal standard cusp neighborhood and denote by $H_t \subset H_0$ the standard cusp neighborhood at height $t$ with $d(H_t, H_0) = t$. The set of complete ‘cuspidal’ geodesics $\gamma$ with $\gamma(0) \in \partial H_0$, $\gamma(-t) \in H_t$ (hence starting from the cusp) can be identified with the set $[0, 1]$ via the endpoint $\tilde{\gamma}(\infty) \in [0, 1)$ of a suitable lift $\tilde{\gamma}$ of $\gamma$, starting from $\infty$. We say that $\gamma$ is bounded with height $t = t(\gamma)$ if $\gamma|_{\mathbb{R}_+}$ does not enter $H_t$. Again, $\gamma$ is bounded if and only if $x = \tilde{\gamma}(\infty) \in [0, 1) \setminus \mathbb{Q}$ is a badly approximable number and a small height $t(\gamma)$ corresponds to a large approximation constant $c$. Hence, Jarník’s inequality (1.2) also shows that the Hausdorff-dimension of the cuspidal geodesics in the modular surface with a sufficiently large given upper bound on the height can be bounded below and above nontrivially.

While Kristensen, Thorn and Velani [24] extended Jarník’s Theorem on full Hausdorff-dimension to a more general setting, our intention is to determine simple conditions on a framework which enable to extend Jarník’s inequality to further examples. We remark that implicitly in the proof of [24], a lower bound on the Hausdorff-dimension of a given set of badly approximable points with a lower bound on the approximation constant can be determined. However, the bound is neither stated explicitly, nor is it effective. In particular, they only use the trivial upper bound, which is the dimension of the space.

1.1. Main results. Among the applications in Section 3 we now present two of the main results in their simplest settings. For $n \geq 1$, let $\vec{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ be the weight vector with $r_1, \ldots, r_n \geq 0$ such that $\sum_i r_i = 1$. Let $\text{Bad}^{1}_{\mathbb{R}^n}$ be the set of points $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ for which there exists a positive constant $c(\vec{x}) > 0$ such that

$$\max_{i=1,\ldots,n} |q x_i - p_i|^{1/r_i} \geq \frac{c(\vec{x})}{q},$$

for every $q \in \mathbb{N}$ and $\vec{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$. The set $\text{Bad}^{1}_{\mathbb{R}^n}$ is the classical set of badly approximable numbers and the set $\text{Bad}^{n}_{\mathbb{R}^n} \equiv \text{Bad}^{1}_{\mathbb{R}^n}$, where $\vec{r} = (\frac{1}{n}, \ldots, \frac{1}{n})$, agrees with the set of badly approximable vectors. For $c > 0$, let moreover $\text{Bad}^{n}_{\mathbb{R}^n}(c)$ be the set of badly approximable vectors $\vec{x}$ with approximation constant $c(\vec{x}) \geq c$.

**Theorem 1.2.** There exist positive constants $k_l, k_u, \tilde{k}_l, \tilde{k}_u > 0$ and $c_0 > 0$ such that for all $c > c_0$ we have

$$n - k_l \frac{[\log(1 - \tilde{k}_l e^{-c/2})]}{c} \leq \dim(\text{Bad}^{n}_{\mathbb{R}^n}(e^{-c})) \leq n - k_u \frac{[\log(1 - \tilde{k}_u e^{-(n+1)c})]}{c}.$$ 

In particular, $\dim(\text{Bad}^{n}_{\mathbb{R}^n}) = n$. Using the Taylor expansion, we recover for $n = 1$ and large $c = \log(N)$ an inequality which is similar to Jarník’s inequality (1.2).

We will in fact prove a similar result for the set $\text{Bad}^{n}_{\mathbb{R}^n}$ and for intersections with suitable ‘diffuse’ sets which are, more precisely, supports of absolutely decaying measures. It is worth pointing out that a positive lower bound on the Hausdorff-dimension, is a lower bound for the Hurwitz-constant of the spectrum of approximation constants. Very little seems to be known about the Hurwitz-constant (of $\text{Bad}^{n}_{\mathbb{R}^n}$).

Now let $M = \mathbb{H}^{n+1}/\Gamma$ be a $(n + 1)$-dimensional finite volume hyperbolic manifold with exactly one cusp. As above, let $H_0$ be a standard cusp neighborhood and let $H_t \subset H_0$ be the standard cusp neighborhood at height $t$. Fix a base point $o \in M - H_0$ in the compact part of $M$ and let $SM_o$ be the $n$-dimensional unit tangent space of $M$ at $o$. Identify a vector...
\( v \in SM_o \) with the unique geodesic ray \( \gamma_v \) starting at \( o \) such that \( \dot{\gamma}_v(0) = v \). For a constant \( t_0 > 0 \), we define for \( t > t_0 \) the set of rays \( \gamma_v \) which avoid \( H_t \), i.e. stay in the compact part \( H^C_t \), by

\[
\text{Bad}_{M,H_0,o}(t) \equiv \{ v \in SM_o : \gamma_v(s) \notin H_t \text{ for all } s \geq 0 \}.
\]

**Theorem 1.3.** There exist positive geometric constants \( k_l, k_u, \bar{k}_l, \bar{k}_u > 0 \), depending on \( \Gamma \), and a height \( t_0 \) such that for all \( t > t_0 \) we have

\[
n - k_l \frac{\log(1 - \bar{k}_l e^{-nt/2})}{t} \leq \dim(\text{Bad}_{M,H_0,o}(t)) \leq n - k_u \frac{\log(1 - \bar{k}_u e^{-2nt})}{t}.
\]

In particular, the set of ‘bounded’ rays is of full Hausdorff-dimension.

We will prove a similar result even in a geometrically finite setting, which yields information about the distribution of the horoballs (lifts of \( H_0 \) in \( \mathbb{H}^{n+1} \)) as well as of the orbit of the parabolic fixed points (base points of the lifts of \( H_0 \)) in the limit set of \( \Gamma \).

Note that the Hurwitz-constant is given in terms of the infimum of the heights of closed geodesics in \( M \) (see [16]).

**Outline of the paper.** In Section 2 we introduce the framework and conditions which lead to an abstract formalism for the lower and upper bound on the Hausdorff dimension of a set of badly approximable points with respect to a given lower bound on the approximation constant (see Subsections 2.2 and 2.3 respectively). We can distinguish between ‘separation conditions’ and ‘measure conditions’, which both concern the parameter space as well as the structure and distribution of the resonant sets.

In Section 3 we apply the deduced bounds to the set of badly approximable vectors with weights (Subsection 3.1), to the set of words in the Bernoulli shift which avoid a periodic word (Subsection 3.2), to the set of geodesics in a geometrically finite hyperbolic manifold which are bounded with respect to a suitable collection convex sets (Subsection 3.3) and to the set of orbits of toral endomorphisms which avoid separated sets of \( \mathbb{R}^n \) (Subsection 3.4).

1.2. **Further remarks.** The property of full Hausdorff-dimension of a set of badly approximable points (with respect to a suitable setting of Diophantine approximation) has been established for various examples specifically and, as mentioned above, by [24] in an abstract fashion. With respect to the examples we consider in Section 3 we point out Patterson [31], for the case of Diophantine approximation in Fuchsian groups, and Pollington, Velani [32], for the set \( \text{Bad}^{(r_1,r_2)}_{\mathbb{H}^2} \). Again, a lower bound on the Hausdorff-dimension of the set of badly approximable points with a lower bound on the approximation constants can be determined from the proofs for these specific examples.

Moreover, Schmidt [34] showed that \( \text{Bad}^1_{\mathbb{R}} \) is actually a Schmidt-winning set. Schmidt’s game also applies to further examples from number theory and dynamical systems (see for instance [10]) when there is a suitable set of badly approximable points. Since winning sets of Schmidt’s game (and modifications of it) enjoy a remarkable rigidity, only recently several modifications, adopted to the specific setting of the considered examples, have been introduced. In particular, the property of full Hausdorff-dimension is, at least in a reasonably nice setting, a ‘byproduct’ of a winning set. We want to refer, for instance, to

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2 By ‘geometric’ we mean that the constants depend on geometric quantities such as the diameter of \( M - H_0 \) as well as further universal constants depending on the group \( \Gamma \) and on the hyperbolic space.

3 Further details can be found in the probably incomplete list [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37].
the work of Kleinbock, Weiss \[23\] in which a technique is given that determines a lower bound for the Hausdorff-dimension of the set \(S_{\alpha, \beta}\) in terms of \(\alpha, \beta\), where \(\alpha, \beta\) are parameters of the Schmidt game. However, we remark that, although we will use this technique for our purpose, the set \(S_{\alpha, \beta}\) in general contains badly approximable elements with arbitrarily small approximation constants.

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2. The Geometry of Parameter Spaces and the Abstract Formalism

The idea of the formalism and the required conditions are simple, yet hidden below technicalities. We therefor want to explain it for the basic example \(\text{Bad}^1_{\mathbb{R}}\), the set of badly approximable numbers (see Subsection 3.1). For \(r > 0\), let \(R(r) \equiv \{p/q \in \mathbb{Q} : 0 < r \leq q \leq r\}\). Fix a sufficiently large parameter \(c > 0\). For the lower bound, we start with any closed metric ball \(B_1 = B(x, 1)\). Now, given a closed metric ball \(B = B_{1i_1...i_k}\) of radius \(r = e^{-2kc}\) at the \(k\)th step, we consider the ‘relevant set’ \(A^1_k = \bigcup_{p/q \in R(r \cdot l_u)} B(p/q, e^{-2c_2r})\). The constant \(l_u = 3\) guarantees that at most one of the balls \(B(p/q, e^{-2c_2r})\) with \(p/q \in R(r \cdot l_u)\) can intersect \(B\). Hence, with respect to the Lebesgue measure \(\mu\), the following condition is satisfied

\[
\mu(B \cap \bigcup_{p/q \in R(r \cdot l_u)} B(p/q, e^{-2c_2r})) \leq e^{-2c_2c} \mu(B) \equiv \tau_c \cdot \mu(B). \tag{2.1}
\]

Up to further separation constants, we can find disjoint balls \(B_{1i_2...i_k}\) of radius \(e^{-2c_2r}\) contained in \(B\) and in the complement of \(A^1_k\). The number of these balls can be estimated from below in terms of \(\tau_c\). Thus, step by step, we construct a treelike collection of ‘sub-cover’ of the set \(\text{Bad}^1_{\mathbb{R}}(e^{-2c_2})\) with \(c_2\) related to \(c\). This will yield a lower bound on the Hausdorff-dimension of \(\text{Bad}^1_{\mathbb{R}}(e^{-2c_2})\) in terms of \(\tau_c\).

For the upper bound, given again a closed metric ball \(B = B_{1i_1...i_k}\) of radius \(r_k = u_k e^{-4kc}\) at the \(k\)th step, we consider the ‘relevant set’ \(A^u_k = \bigcup_{p/q \in R(r_k \cdot u^c)} B(p/q, e^{-2c_2r})\). The parameter \(u^c = u_k e^{-4c_2c}\) guarantees that either \(B\) is contained in a set \(B(p/q, e^{-2c_2r})\) with \(p/q \in R(r_k \cdot u^c)\) or that there exists a point \(p/q \in R(r_k \cdot u^c)\) with \(B(p/q, e^{-4c_2r_k}) \subset B\). Hence, the following condition is satisfied

\[
\mu(B \cap \bigcup_{p/q \in R(r_k \cdot u^c)} B(p/q, e^{-2c_2r})) \geq e^{-4c_2c} \mu(B) \equiv \tau^c \cdot \mu(B). \tag{2.2}
\]

Again, up to further separation constants, we can find closed balls \(B_{1i_1...i_k}\) of radius \(u_k e^{-4c_2r_k}\) covering the complement of \(A^u_k\) in \(B\), for which the number can be estimated from above in terms of \(\tau^c\). Thus, step by step, we construct a treelike collection of covers of the set \(\text{Bad}^1_{\mathbb{R}}(e^{-2c_2}) \cap B_1\). This will yield an upper bound on the Hausdorff-dimension of \(\text{Bad}^1_{\mathbb{R}}(e^{-2c_2})\) in terms of \(\tau^c\).

For our abstract formalism, we will in fact assume the conditions (2.1) and (2.2) as well as separation conditions and construct treelike collections of sub-covers and covers respectively as above.

Remark. Our setting and formalism is similar to the local ubiquity setup of Beresnevich, Dickinson and Velani \[4\]. In particular, our main conditions (2.1) and (2.2) (as well as
(2.13) and (2.23) respectively) are similar to their intersection conditions. However, their formalism served the purpose of determining the Hausdorff-dimension of the complementary set, that is the set of well-approximable points and of 'limsup sets' in general.

2.1. The general framework. We first introduce the setting of this section that bases on the notion of [23] and was adopted in the author’s earlier work [40]. However, some of the following terminology differs from these works.

Let \((X, d)\) be a proper metric space. Fix \(t_s \in \mathbb{R} \cup \{-\infty\}\) and define the parameter space \(\bar{\Omega} \equiv X \times (t_s, \infty)\), the set of formal balls in \(X\). Let \(C(X)\) be the set of nonempty compact subsets of \(X\). Assume that there exists a function \(\bar{\psi} : \bar{\Omega} \to C(\bar{X})\) which is monotonic, that is, for all \((x, t) \in \bar{\Omega}\) and \(s \geq 0\) we have

\[
\bar{\psi}(x, t + s) \subset \bar{\psi}(x, t).
\]

(2.3)

For a subset \(Y \subset \bar{X}\) and \(t > t_s\), we call \((Y, t) \equiv \{(y, t) : y \in Y\}\) formal neighborhood, and define \(\mathcal{P} = \mathcal{P}(\bar{X}) \times (t_s, \infty)\) to be the set of formal neighborhoods. Define the \(\bar{\psi}\)-neighborhood of \((Y, t) \in \mathcal{P}\) by

\[
\bar{\psi}(Y, t) \equiv \bigcup_{y \in Y} \bar{\psi}(y, t).
\]

Note that by monotonicity (2.3), \(\bar{\psi}(Y, t + s) \subset \bar{\psi}(Y, t)\) for all \(s \geq 0\).

For instance, since \(\bar{X}\) is proper, set \(t_s = -\infty\) and for \(x \in \bar{X}\), \(r > 0\), let \(B(x, r) \equiv \{y \in \bar{X} : d(x, y) \leq r\}\) to be the formal balls. For \(\sigma > 0\), the standard function \(\bar{\psi}_\sigma \equiv B_\sigma\) is given by the monotonic function

\[
\bar{\psi}_\sigma(x, t) \equiv B(x, e^{-\sigma t}).
\]

(2.4)

In many applications, we are interested in badly approximable points of a closed subset \(X\) of \(\bar{X}\) which is, with the induced metric, a complete metric space. However, we do not require the resonant sets to be contained in \(X\) but in \(\bar{X}\). Therefore, let also \(\Omega = X \times (t_s, \infty) \subset \bar{\Omega}\). The monotonic function \(\bar{\psi}\) induces the monotonic function \(\psi : \Omega \to C(X)\), defined by

\[
\psi(\omega) \equiv \bar{\psi}(\omega) \cap X, \quad \omega \in \Omega.
\]

2.1.1. The family of resonant sets. Now, let \(\Lambda\) be a countable index set and \(\{R_\lambda \subset \bar{X} : \lambda \in \Lambda\}\) be a family of resonant sets in \(\bar{X}\), where we assign a size \(s_\lambda \geq s_s\) to every \(R_\lambda\) with \(t_s < s_s \in \mathbb{R}\). We consider the contractions of the \((\bar{\psi}, s_\lambda)\)-neighborhoods of \(R_\lambda\),

\[
f_\lambda(s) \equiv \bar{\psi}(R_\lambda, s_\lambda + s) \subset \bar{\psi}(R_\lambda, s_\lambda), \quad s \geq 0.
\]

Denote this family by

\[
\mathcal{F} = (\Lambda, R_\lambda, s_\lambda).
\]

Assume that the family \(\mathcal{F}\) satisfies the following conditions.

(N) The resonant sets \(\{R_\lambda\}\) are nested with respect to their sizes, that is, for \(\lambda, \beta \in \Lambda\) we have

\[
s_\lambda \leq s_\beta \implies R_\lambda \subset R_\beta.
\]

(D) The sizes \(\{s_\lambda\}\) are discrete, that is, for all \(t > t_s\) we have

\[
|\{\lambda \in \Lambda : s_\lambda \leq t\}| < \infty.
\]
We then define the set of badly approximable points with respect to \( F \) by
\[
\text{Bad}_X(\mathcal{F}) = \{ x \in X : \exists c = c(x) < \infty \text{ such that } x \notin \bigcup_{\lambda \in \Lambda} f_\lambda(c) \},
\]
or simply by \( \text{Bad}(\mathcal{F}) \) if there is no confusion about the parameter spaces under consideration. The constant \( c(x) \equiv \sup \{ c \in \mathbb{R} : x \in \bigcup_{\lambda \in \Lambda} f_\lambda(c) \} \) is called the approximation constant of \( x \in \text{Bad}(\mathcal{F}) \). In the following, we are interested in the subset
\[
\text{Bad}(\mathcal{F}, c) \equiv \{ x \in \text{Bad}(\mathcal{F}) : c(x) \leq c \}.
\]

Note moreover that the resonant sets can be ordered with respect their sizes by (N). We will therefore assume in the following that \( \Lambda = \mathbb{N} \), \( s_n \leq s_m \) for \( n \leq m \). For a parameter \( t \geq s_1 \), we define the relevant resonant set with respect to the parameter \( t \) by
\[
R(t) \equiv \bigcup_{s_n \leq t} R_n = R_{nt},
\]
where \( n_t \in \mathbb{N} \) is the largest integer such \( s_n \leq t \) (see (N) and (D)), and we call \( s_{nt} \) the relevant size.

2.1.2. Rigidity assumptions. The following requirements will be standing assumptions in Section 2. For \( c > 0 \) assume there exist constants \( d_c, d^c \geq 0 \) such that, for \( (y, t) \in \Omega \),
\[
\begin{align*}
&x \in \tilde{\psi}(y, t + d_c) \implies \tilde{\psi}(x, t + c) \subset \tilde{\psi}(y, t) \quad (2.5) \\
&x \in \tilde{\psi}(y, t) \implies \tilde{\psi}(x, t + c) \subset \tilde{\psi}(y, t - d^c).
\end{align*}
\]
Moreover, require that \( (\Omega, \tilde{\psi}) \) is \( d_s \)-separating with respect to \( \mathcal{F} \), that is, there exists a constant \( d_s \geq 0 \) such that for all resonant sets \( Y = R_n \subset \bar{X} \), or points \( Y = y \in X, t > t_s \) and for all \( x \in X \), \( \tilde{\psi} \) satisfies
\[
\begin{align*}
x &\notin \tilde{\psi}(Y, t) \implies \psi(x, t + d_s) \cap \tilde{\psi}(Y, t + d_s) = \emptyset. \quad (2.6)
\end{align*}
\]
Assume in addition that for every Borel set \( Y \subset \bar{X} \) as above also the \( \tilde{\psi} \)-neighborhood \( \tilde{\psi}(Y, t) \) is a Borel set. Let \( \mu \) be a locally finite Borel measure on \( \bar{X} \) which is positive on \( \psi \)-balls, that is, for all \( \omega \in \Omega \) we have
\[
\mu(\psi(\omega)) > 0. \quad (2.7)
\]
Finally, we require that for all \( \omega = (x, t) \in \Omega \), the diameter of \( \psi(\omega) \) is bounded by
\[
\text{diam}(\psi(x, t)) \leq c_{\sigma} e^{-\sigma t}, \quad (2.8)
\]
where \( c_{\sigma}, \sigma > 0 \).

2.1.3. Further considerations. Denote by \( O(x, r) \equiv \{ y \in \bar{X} : d(x, y) < r \} \) the open metric ball around \( x \in \bar{X} \). Let \( \mu \) be a locally finite Borel measure on \( \bar{X} \). The lower pointwise dimension of \( \mu \) at \( x \in \text{supp}(\mu) \) is defined by
\[
d_\mu(x) \equiv \liminf_{r \to 0} \frac{\log(\mu(O(x, r)))}{\log r}.
\]
If \( \mu \) satisfies a power law, that is, there exist constants \( \delta, c_1, c_2 \) and \( R > 0 \) such that for every \( 0 < r < R \) and \( x \in \text{supp}(\mu) \) we have
\[
c_1 r^\delta \leq \mu(O(x, r)) \leq c_2 r^\delta,
\]
then we have \( d_\mu(x) = \delta \).
We say that \((\Omega, \psi, \mu)\) satisfies a power law with respect to the parameters \((\tau, c_1, c_2)\), where \(\tau > 0\), \(c_2 \geq c_1 > 0\), if \(\sup(\mu) = X\) and
\[
c_1 e^{-\tau t} \leq \mu(\psi(x, t)) \leq c_2 e^{-\tau t}
\]
(2.9) for all formal balls \((x, t) \in \Omega\).

Note that, depending on the considered function \(\psi\), the exponent \(\tau\) from (2.9) may differ from \(\delta\). For \(x \in X\), define
\[
\Delta_{\mu, \psi}(x) \equiv \limsup_{t \to \infty} \frac{1}{t} \log \left( \frac{\mu(O(x, 2c_\sigma e^{-\sigma t}))}{\mu(\psi(x, t))} \right),
\]
if the limit exists. We remark that by (2.8) we have \(\psi(x, t) \subset O(x, 2c_\sigma e^{-\sigma t})\) for all \(t > t_*\); hence \(\Delta_{\mu, \psi}(x) \geq 0\). The following lemma is readily checked.

**Lemma 2.1.** Let \((\Omega, \psi, \mu)\) satisfy a power law. If the limit exists, we have
\[
\tau = \sigma d_\mu(x) + \Delta_{\mu, \psi}(x).
\]

Finally, let \((\Omega, \psi, \mu)\) satisfy a power law with respect to the parameters \((\tau, c_1, c_2)\). For later purpose, we state that the following inequalities are satisfied. For \(c > 0\), a constant \(u^c \geq 0\) and \(\omega = (x, t - d^c) \in \Omega\), we have for all \(y \in \psi(\omega)\)
\[
\mu(\psi(y, t + c)) \geq \frac{c_2}{c_1} e^{-\tau c} \mu(\psi(x, t)) \equiv \tilde{k}_c \mu(x, t),
\]
(2.10)
\[
\mu(\psi(x, t + d_c)) \geq \frac{c_2}{c_1} e^{\tau(c - d_c - d_\sigma)} \mu(\psi(y, t + c - d_\sigma)) \equiv \tilde{k}_c^{-1} \mu(\psi(y, t + c - d_\sigma)),
\]
\[
\mu(\psi(y, t + (c + u^c) + d_\sigma)) \geq \frac{c_2}{c_1} e^{-\tau(c + u^c + d_\sigma)} \mu(\psi(x, t - d^c)) \equiv k^c \mu(\psi(x, t - d^c)).
\]
Moreover, if \(\tilde{\psi}\) is given by the standard function \(B_\sigma(x, t) \equiv B(x, e^{-\sigma t})\), then for \(c > 0\) we have
\[
d_\sigma \leq \log(2)/\sigma, \quad d_c \leq -\log(1 - e^{-\sigma c})/\sigma, \quad d^c \leq \log(1 - e^{-\sigma c})/\sigma.
\]

### 2.2. The lower bound

We fix a constant \(c > 0\) and let \(l_c \geq 0\). For \(k \geq 1\), define \(t_k \equiv s_1 + kc + l_c\) and
\[
L_k(c) = L_k^\psi(c) \equiv \bigcap_{i=1}^k \tilde{\psi}(R(t_i - l_c, t_i + c)^C).
\]
(2.11)

Assume that there exist positive constants \(\tilde{k}_c, k_c > 0\) such that \((\Omega, \psi, \mu)\) satisfies, for all formal balls \(\omega = (x, t_k) \in \Omega\) with \(x \in L_{k-1}(c - d_\sigma)\) and \(y \in \psi(\omega) \cap L_k(c - d_\sigma)\),
\[
k_c \mu(\psi(x, t_k)) \leq \mu(\psi(y, t_{k+1})) \leq \mu(\psi(y, t_{k+1} - d_\sigma)) \leq \tilde{k}_c \mu(\psi(x, t_k + d_\sigma)).
\]
(2.12)

The concept of (absolutely) decaying measures was introduced in [20] and we adopted it to our setting in [40]. \((\Omega, \psi, \mu)\) is called \(\tau_c\)-decaying with respect to \(\mathcal{F}\) and the parameters \((c, l_c)\) if all formal balls \(\omega = (x, t_k + d_c) \in \Omega\) with \(x \in L_{k-1}(c - d_\sigma)\) we have
\[
\mu(\psi(\omega) \cap \tilde{\psi}(R(t_k - l_c, t_k + c - d_\sigma))) \leq \tau_c \cdot \mu(\psi(\omega)),
\]
(2.13)
where \(\tau_c < 1\)[4]

**Remark.** For \(c \geq d_\sigma\), the condition that \(x \in L_{k-1}(c - d_\sigma)\) implies that \(\psi(x, t_k)\) is disjoint to \(\tilde{\psi}(R(t_k - l_c, t_k + c - d_\sigma)) \subset \tilde{\psi}(R(t_k - l_c, t_k + c - d_\sigma))\) by (2.6). Hence it would actually suffice to consider the set \(R(t_k - l_c, c) \equiv R(t_k - l_c) - R(t_{k-1} - l_c)\) in (2.13). Note that also the proof of Lemma 2.4 will work if we only consider the sets \(R(t_k - l_c, c)\).

---

[4] In fact, we should call this condition ‘absolutely \(\tau_c\)-decaying’ rather than \(\tau_c\)-decaying according to [20]. For the sake of simplicity we omit the term ‘absolutely’.
In order to determine the lower bound of $\dim(Bad(F, 2c + l_c))$, we first construct a strongly treelike family of sets such that its limit set, $A_{\infty}$, is a subset of $Bad(F, 2c + l_c)$. Using the method of \cite{23,22} (which is a generalization of the ones of \cite{25,39}), based on the 'Mass Distribution Principle', we derive a lower bound of $\dim(A_{\infty})$.

Let $\omega_1 = (x_1, s_1 + l_c) \in \Omega$ be a formal ball and note that $L_0(c) = X$.

**Lemma 2.2.** Given a formal ball $\omega_{i_1,..,i_k} = (x_{i_1,..,i_k}, t_k) \in \Omega$ with $x_{i_1,..,i_k} \in L_{k-1}(c - d_s)$ there exist formal balls $\omega_{i_1,..,i_k,i_{k+1}} = (x_{i_1,..,i_k,i_{k+1}}, t_{k+1}) \in \Omega$ satisfying

$$\psi(\omega_{i_1,..,i_k,i_{k+1}}) \subset \psi(\omega_{i_1,..,i_k}) - \psi(R(t_k - l_c), t_k + c)$$

(2.14)

where $x_{i_1,..,i_k,i_{k+1}} \in L_k(c - d_s)$, such that $\psi(\omega_{i_1,..,i_k,i_{k+1}})$ are disjoint, and moreover,

$$\mu(\bigcup_{i_{k+1}} \psi(\omega_{i_1,..,i_k,i_{k+1}})) \geq \frac{(1 - \tau_c)k_c}{2k_c} \mu(\psi(\omega_{i_1,..,i_k}))$$

(2.15)

**Proof.** Given the formal ball $\omega_{i_1,..,i_k} = (x_{i_1,..,i_k}, t_k) \in \Omega$ where $x_{i_1,..,i_k} \in L_{k-1}(c - d_s)$, assume that we have $m \geq 0$ formal balls $\omega_{i_1,..,i_k,1} = (x_{i_1,..,i_k,1}, t_{1+1}) \in \Omega$, for which (3.6) is satisfied and such that $\psi(\omega_{i_1,..,i_k,1})$ are disjoint for $i_{k+1} = 1, \ldots, m$.

We apply (2.13) on the formal ball $\omega_0 \equiv (x_{i_1,..,i_k}, t_k + d_s) \in \Omega$ and use (2.12) so that we obtain

$$\mu(\psi(\omega_0) - \psi(R(t_k - l_c), t_k + c - d_s)) - \bigcup_{i_{k+1}} \psi(x_{i_1,..,i_k,i_{k+1}}, t_{k+1} + d_s))$$

$$= \mu(\psi(\omega_0)) - \mu(\psi(\omega_0) \cap \psi(R(t_k - l_c), t_k + c - d_s) \cup \bigcup_{i_{k+1}} \psi(x_{i_1,..,i_k,i_{k+1}}, t_{k+1} + d_s))$$

$$\geq (1 - \tau_c - m \cdot \bar{k}_c) \mu(\psi(\omega_0)).$$

As long as $m < (1 - \tau_c)\bar{k}_c^{-1}$, by (2.7) there exists a point

$$x' \in \psi(\omega_0) - \psi(R(t_k - l_c), t_k + c - d_s) - \bigcup_{i_{k+1}} \psi(x_{i_1,..,i_k,i_{k+1}}, t_{k+1} + d_s).$$

Define $\omega_{i_1,..,i_k,1} \equiv (x', t_k + c) \in \Omega$. By (2.6) we know that $\psi(\omega_{i_1,..,i_k,1})$ is disjoint from both, $\bigcup_{i_{k+1}=1}^m \psi(\omega_{i_1,..,i_k,i_{k+1}})$ as well as $\psi(R(t_k - l_c), t_k + c)$. Moreover, by (2.5) we have that $\psi(\omega_{i_1,..,i_k,1}) \subset \psi(\omega_{i_1,..,i_k})$. In particular, by construction, $x' \in L_k(c - d_s) \cap L_{k-1}(c)$. Iterating this argument until

$$m + 1 \geq \frac{1 - \tau_c}{k_c}$$

we see by (2.12) that

$$\mu\left(\bigcup_{i_{k+1}=1}^m \psi(\omega_{i_1,..,i_k,i_{k+1}})\right) \geq m \cdot k_c \cdot \mu(\psi(\omega_{i_1,..,i_k}))$$

(2.16)

$$\geq \frac{m + 1}{2} \cdot k_c \cdot \mu(\psi(\omega_{i_1,..,i_k}))$$

$$\geq \frac{(1 - \tau_c)k_c}{2k_c} \cdot \mu(\psi(\omega_{i_1,..,i_k})),$$

which shows the claim. \qed
We now construct a strongly treelike family $\mathcal{A}$ of subsets of $X \cap \psi(\omega_1)$ relative to $\mu$ as follows. Let $\mathcal{A}_1 = \{ \psi(\omega_1) \}$. Given the subfamily $\mathcal{A}_k$ at the $k$.th step and a set $\psi(\omega_{i_1,..,i_k}) \in \mathcal{A}_k$, Lemma 2.2 implies the existence of sets $\psi(\omega_{i_1,..,i_k,i_{k+1}})$, which are disjoint subsets of $\psi(\omega_{i_1,..,i_k})$, disjoint to $\psi(R(t_k-l_c), t_k + c)$ and satisfy (2.15). We therefore define

$$\mathcal{A}_{k+1} = \bigcup_{i_1,..,i_k} \{ \psi(\omega_{i_1,..,i_k,i_{k+1}}) \}.$$  

If $\mathcal{A}$ (a countable family of compact subsets of $X$) denotes the union of the subcollections $\mathcal{A}_k, k \in \mathbb{N}$, the following properties are satisfied with respect to $\mu$:

(TL0) $\mu(A) > 0$ for all $A \in \mathcal{A}$,

(TL1) for all $k \in \mathbb{N}$, for all $A, B \in \mathcal{A}_n$, either $A = B$ or $\mu(A \cap B) = 0$,

(TL2) for all $k \in \mathbb{N}_{\geq 2}$, for all $B \in \mathcal{A}_k$, there exists $A \in \mathcal{A}_{k-1}$ such that $B \subset A$,

(TL3) for all $k \in \mathbb{N}$, for all $A \in \mathcal{A}_k$, there exists $B \in \mathcal{A}_{k+1}$ such that $B \subset A$.

We can therefore define $\cup \mathcal{A}_k = \bigcup_{A \in \mathcal{A}_k} A$ and obtain a decreasing sequence of nonempty compact subsets $X \supset \cup \mathcal{A}_1 \supset \cup \mathcal{A}_2 \supset \cup \mathcal{A}_3 \supset \ldots$. Since $X$ is complete, the limit set

$$A_\infty \equiv \bigcap_{k \in \mathbb{N}} \cup \mathcal{A}_k$$

is nonempty. Define moreover the $k$.th stage diameter $d_k(A) \equiv \max_{A \in \mathcal{A}_k} \text{diam}(A)$, which by (2.8) satisfies $d_k(A) \leq c_{d,e} e^{-\sigma k}$, and hence

(STL) $\lim_{k \to \infty} d_k(A) = 0$.

Finally, by (2.15), we obtain a lower bound for the $k$.th stage 'density of children'

$$\Delta_k(A) \equiv \min_{B \in \mathcal{A}_k} \frac{\mu(\cup \mathcal{A}_{k+1} \cap B)}{\mu(B)} \geq \frac{(1 - \tau_c)k_c}{2k_c}$$

(2.17)

of $A$. This gives a lower bound on the Hausdorff-dimension of $A_\infty$.

**Lemma 2.3.** If $\mathcal{A}$ as above satisfies (TL0-3) and (STL), then

$$\dim(A_\infty) \geq \inf_{x_0 \in A_\infty} d_\mu(x_0) - \lim_{k \to \infty} \frac{\sum_{i=1}^k \log(\Delta_i(A))}{\log(d_k(A))}. $$

**Proof.** In [22], Lemma 2.5 (which is stated for $\widetilde{X} = \mathbb{R}^n$ but also true for general complete metric spaces, see [23]) a measure $\nu$ is constructed for which its support equals $A_\infty$. Moreover, $\nu$ satisfies for every $x \in A_\infty$ that

$$d_\nu(x) \geq d_\mu(x) - \lim_{k \to \infty} \frac{\sum_{i=1}^k \log(\Delta_i(A))}{\log(d_k(A))}. $$

For every open set $U \subset \widetilde{X}$ with $\nu(U) > 0$, let

$$d_\nu(U) \equiv \inf_{x \in U \cap \text{supp}(\mu)} d_\nu(x), $$

which is known to be a lower bound for the Hausdorff-dimension of $\text{supp}(\nu) \cap U = A_\infty \cap U$ (see [13], Proposition 4.9 (a)). Setting $U = \widetilde{X}$ shows the claim. □
Using Lemma 2.3 (2.8) and (2.17), we obtain

\[
\dim(A_\infty) \geq \inf_{x_0 \in A_\infty} d_\mu(x_0) - \limsup_{k \to \infty} \frac{k \log((1 - \tau_c) k_c (2k_c)^{-1})}{\log(c \sigma e^{-\sigma k_c})} \geq \inf_{x_0 \in A_\infty} d_\mu(x_0) - \limsup_{k \to \infty} \frac{k(\log(1 - \tau_c) - \log(2k_c k_c^{-1}))}{-\sigma k_c} = \inf_{x_0 \in A_\infty} d_\mu(x_0) - \frac{\log(2k_c k_c^{-1}) + \|\log(1 - \tau_c)\|_{\sigma c}}{\sigma c}.
\]

We establish our lower bound by showing the following Lemma.

**Lemma 2.4.** $A_\infty \subset \psi(\omega_1) \cap \text{Bad}(\mathcal{F}, 2c + l_c)$; hence, $\dim(\text{Bad}(\mathcal{F}, 2c + l_c)) \geq \dim(A_\infty)$.

**Proof.** Let $x_0 \in A_\infty$. Let $\{i_1 \ldots i_k\}_{k \in \mathbb{N}}$ be a sequence such that $x_0 \in \cap_{k \in \mathbb{N}} \psi(\omega_{i_1 \ldots i_k})$. For the above case, since for every $k \in \mathbb{N}$ the sets $\psi(\omega_{i_1 \ldots i_k})$ of the construction of $\mathcal{A}$ are in particular disjoint, the sequence $\{i_1 \ldots i_k\}_{k \in \mathbb{N}}$ is in fact unique - this might not be true for the standard case in the following.

Assume that $x_0 \in \psi(R_m, s_m)$ for some $m \in \mathbb{N}$ (if no such $m$ exists, then the claim already follows). Let $k \in \mathbb{N}$ such that $s_m + l_c \in [t_k, t_{k+1})$. By construction, $x_0 \in \psi(\omega_{i_1 \ldots i_{k+2}})$ which is disjoint to $\psi(R(t_{k+1} - l_c), t_{k+1} + c)$ by (3.6). Since $t_k - l_c \leq s_m < t_{k+1} - l_c$ we have $R_m \subset R(t_{k+1} - l_c)$ and

\[
x_0 \notin \bar{\psi}(R(t_{k+1} - l_c), t_{k+1} + c) = \psi(R(t_{k+1} - l_c), t_k - l_c + 2c + l_c) \supset \psi(R_m, s_m + (2c + l_c)),
\]

by monotonicity of $\psi$. This shows that $x_0 \notin \text{Bad}(\mathcal{F}, 2c + l_c)$. □

### 2.2.1. The Standard Case $X = \mathbb{R}^n$.

Let $X = \mathbb{T} = \mathbb{R}^n$ and $\mu$ be the Lebesgue measure. For $\sigma > 0$, let $\psi(x, t) = B_\sigma(x, t) \equiv B(x, e^{-\sigma t})$. Then $\mu$ satisfies a power law. However, even in this case, our given bound (2.18) might not be sharp, because the constants $k_c$ and $\bar{k}_c$ respectively depend sensitively on the separation constants $d_\epsilon$ and $d_\epsilon$, respectively.

For this standard case we now want to sharpen the lower bound. We only need to modify the above arguments by shifting the separation constants into $\tau_c$. Consider therefore the monotonic function on $\Omega = \mathbb{R}^n \times \mathbb{R}^+$ given by

\[
Q_\sigma(x, t) \equiv B(x_1, e^{-\sigma t}) \times \cdots \times B(x_n, e^{-\sigma t}),
\]

which denotes the $n$-dimensional cube of edge length $2e^{-\sigma t}$ with center $x = (x_1, \ldots, x_n)$. Note that for all formal balls $(x, t) \in \Omega$ we have

\[
Q_\sigma(x, t + \sqrt{n}/\sigma) \subset B_\sigma(x, t) \subset Q_\sigma(x, t).
\]

Thus, $\text{Bad}_{\mathbb{R}^n}^Q(\mathcal{F}, c) \subset \text{Bad}_{\mathbb{R}^n}^B(\mathcal{F}, c) \subset \text{Bad}_{\mathbb{R}^n}^Q(\mathcal{F}, c + \sqrt{n}/\sigma)$. Moreover, the Lebesgue measure satisfies $\mu(Q_\sigma(x, t)) = 2^n e^{-n \sigma t}$ so that $(\Omega, Q_\sigma, \mu)$ satisfies a power law. Also, if $(\Omega, B_\sigma)$ is $d_\epsilon$-separating with respect to $\mathcal{F}$, let $d_\epsilon \equiv d_\epsilon + \sqrt{n}/\sigma$, and we see that $(\Omega, Q_\sigma)$ is at least $d_\epsilon$-separating with respect to $\mathcal{F}$.

The crucial point is that, given any cube $Q_\sigma(x, t) \subset \mathbb{R}^n$ and $c = \log(m)/\sigma$ for some $m \in \mathbb{N}$, we can find a partition into $m^n$ cubes $Q_i = Q_i(x, t) \equiv Q_\sigma(x, t + c)$ satisfying

\[
\mu(Q_i \cap Q_j) = 0 \quad \text{for } i \neq j, \quad \text{and} \quad \bigcup_i Q_i = Q_\sigma(x, t).
\]
Now let \( c = \log(m)/\sigma \geq \tilde{d}_* + \log(2)/\sigma \) for some integer \( m \in \mathbb{N}, \tilde{l}_c \geq 0 \) and modify (2.13) such that for all formal balls \( \omega = (x, t_k) \in \Omega \) with \( x \in L^Q_\sigma(c) \) we have
\[
\mu(Q_\sigma(\omega) \cap Q_\sigma(R(t_k - \tilde{l}_c), t_k + c - \tilde{d}_* - \log(2)/\sigma)) \leq \bar{\tau}_c \cdot \mu(Q_\sigma(\omega)),
\]
(2.20)
where \( \bar{\tau}_c < 1 \). Note that for all \( (x, t) \in \Omega, y \in \mathbb{R}^n \) and \( s \geq 0 \) we already have
\[
\mu(Q_\sigma(y, t + s)) = e^{-n\sigma} \mu(Q_\sigma(x, t))
\]
and in particular (2.12).

We modify the arguments from Lemma 2.2 where we replace the choices \( \psi(\omega_{i_1 \ldots i_k}) \) by cubes \( Q_{i_1 \ldots i_k} \) in order to construct a strongly treelike family \( A \) with a limit set contained in \( \text{Bad}^Q_\sigma(\mathcal{F}, 2c + \tilde{l}_c) \cap Q_1 \).

In fact, if \( Q = Q_{i_1 \ldots i_k} = Q_\sigma(x_{i_1 \ldots i_k}, t_k) \) is a given cube, let \( Q_{i_1 \ldots i_k,i_{k+1}} = Q_\sigma(x_{i_1 \ldots i_k,i_{k+1}}, t_k + c) \) be precisely the cubes of the partition of \( Q \) as above, which intersect
\[
Q \cap Q_\sigma(R(t_k - \tilde{l}_c), t_k + c - \tilde{d}_* - \log(2)/\sigma)^C,
\]
and hence cover \( Q \cap Q_\sigma(R(t_k - \tilde{l}_c), t_k + c - \tilde{d}_* - \log(2)/\sigma)^C \). From (2.20), we obtain
\[
\mu( \bigcup_{i_{k+1}} Q_{i_1 \ldots i_k,i_{k+1}} ) \geq (1 - \bar{\tau}_c) \mu(Q_{i_1 \ldots i_k}),
\]
which improves (2.15). Moreover, let \( \tilde{Q} = Q_{i_1 \ldots i_k,i_{k+1}} \) be such a cube intersecting \( Q \cap Q_\sigma(R(t_k - \tilde{l}_c), t_k + c - \tilde{d}_* - \log(2)/\sigma)^C \) in a point \( y \). Then \( \tilde{Q} \subset Q_\sigma(y, t_{k+1} - \log(2)/\sigma) \), and, since \( y \notin Q_\sigma(R(t_k - \tilde{l}_c), t_k + c - \tilde{d}_* - \log(2)/\sigma) \), the superset is disjoint to
\[
Q_\sigma(R(t_k - \tilde{l}_c), t_k + c - \log(2)/\sigma) \supset Q_\sigma(R(t_k - \tilde{l}_c), t_k + c).
\]

Hence, every cube chosen as above is contained in \( L^Q_\sigma(c) \) which shows (3.6) for the setting of cubes. Using the results of the strongly treelike construction as well as (2.18) and Lemma 2.4, we obtain our lower bound in the standard case
\[
\dim(\text{Bad}^Q_\sigma(\mathcal{F}, 2c + \tilde{l}_c) \cap Q_1) \geq n - \frac{\log(1 - \bar{\tau}_c)}{\sigma c}.
\]
(2.21)

Remark. The improvement relies on the partition (2.19) of cubes. This is no longer possible in general, not even for subsets of the Euclidean space. Note also that the restriction to \( c = \log(m)/\sigma \) will not be a severe restriction in the applications, since, for sufficiently large \( c > 0 \) we can choose a \( \bar{c} = \log(m)/\sigma \) with \( \bar{c} \leq c \) and obtain a lower bound with respect to \( \bar{c} \). The defect can again be shifted to a multiplicative constant in \( \bar{\tau}_c \).

In special cases, when (2.13) is satisfied with respect to \( B_\sigma \) the parameters \( (c, l_c) \) and even for all \( x \in L^Q_\sigma(c) \), we can in fact already estimate \( \bar{\tau}_c \) using a sufficiently small \( \tau_c \) - however, a more precise bound can be determined in the particular examples.

Lemma 2.5. If \( \mu \) is \( \tau_c \)-decaying with respect to \( \mathcal{F}, B_\sigma \) and the parameters \( (c, l_c) \), let \( \tilde{l}_c = l_c + a \) where \( a = 2\sqrt{n}/\sigma + \log(2)/\sigma + d_c \). Then for all \( x \in L^B_{\sigma-1}(c - d_*) \) and \( t_k \) we have (2.20) with
\[
\bar{\tau}_c \leq e^{\sigma(a - d_*)} \tau_c.
\]

Proof. Recall that for all formal balls \( \omega = (x, t_k + d_c) \in \Omega \) with \( x \in L^B_{\sigma-1}(c) \) we have
\[
\mu(B_\sigma(\omega) \cap B_\sigma(R(t_k - l_c), t_k + c - d_*)) \leq \tau_c \cdot \mu(B_\sigma(\omega)),
\]
where
where $\tau_c < 1$. Define $\tilde{l}_c = l_c + a$ as well as $t_k = s_1 + kc + \tilde{l}_c$ and $\tilde{t}_k = s_1 + kc + \tilde{l}_c$. Then note that $R(\tilde{t}_k - \tilde{l}_c) = R(s_1 + kc) = R(t_k - l_c)$ and
\[
\tilde{t}_k + c - \tilde{d}_s - \log(2)/\sigma = (t_k + a) + c - d_s - \sqrt{\mu}/\sigma - \log(2)/\sigma \\
\geq t_k + c - d_s + \sqrt{\mu}/\sigma.
\]
as well as $\tilde{t}_k \geq t_k + d_c + \sqrt{\mu}/\sigma$. Hence, $Q_\sigma(x, \tilde{t}_k) \subset B_\sigma(x, t_k + d_c)$ and
\[
Q_\sigma(R(\tilde{t}_k - \tilde{l}_c), \tilde{t}_k + c - \tilde{d}_s - \log(2)/\sigma) \subset B_\sigma(R(t_k - l_c), t_k + c - d_s).
\]
For $x \in L^B_{k-1} (c - d_s)$ and $\omega = (x, \tilde{t}_k)$ this shows
\[
\mu(Q_\sigma(\omega) \cap Q_\sigma(R(\tilde{t}_k - \tilde{l}_c), \tilde{t}_k + c - \tilde{d}_s - \log(2)/\sigma)) \leq \tau_c \cdot \mu(B_\sigma(x, t_k + d_c)) \\
\leq \tau_c \cdot \mu(Q_\sigma(x, t_k + d_c)) \\
= e^{n_\sigma(a-d_c)} \tau_c \cdot \mu(Q_\sigma(\omega)),
\]
proving the claim. 

2.3. The upper bound. For a given parameter $c > 0$ we let $u^c \geq 0$ and define $t_k = s_1 + (k - 1)(c + u^c)$ as well as $\tilde{t}_k = t_k - u^c$ for $k \geq 1$. Moreover, we consider the sequence of the sets
\[
U_k(c) = U_k^\psi(c) \equiv \bigcap_{s_n \leq t_k} \bar{\psi}(R_n, s_n + c)^C \\
\supset \bigcap_{n \in \mathbb{N}} \bar{\psi}(R_n, s_n + c)^C \\
= (\bigcup_{n \in \mathbb{N}} \bar{\psi}(R_n, s_n + c))^C = \text{Bad}(\mathcal{F}, c).
\]

At this step, we require that $(\Omega, \psi, \mu)$ satisfies for every $k \in \mathbb{N}$ and for all formal balls $(x, t_k - d^c) \in \Omega$ with $x \in U_{k-1}(c)$ and $y \in \psi(x, t_k) \cap U_k(c)$, that
\[
\mu(\psi(y, \tilde{t}_{k+1} + d_s)) \geq k^c \mu(\psi(x, \tilde{t}_k - d^c)), \tag{2.22}
\]
where $k^c$ is a positive constant. In addition, consider a further condition on $\mu$, which is the counterpart of the notion of decaying measures. $(\Omega, \psi, \mu)$ is called $\tau^c$-Dirichlet with respect to the family $\mathcal{F}$ and the parameters $(c, u^c)$ if for any formal ball $\omega = (x, t_k - d^c) \in \Omega$ with $x \in U_{k-1}(c)$ we have
\[
\mu(\psi(\omega) \cap \bigcup_{s_n \leq t_k} \bar{\psi}(R_n, s_n + c + d_s)) \geq \tau^c \cdot \mu(\psi(\omega)), \tag{2.23}
\]
where $\tau^c \geq 0$. Note that we called this condition 'Dirichlet' since (2.23) will follow from Dirichlet-type results in the applications.

Remark. It would actually suffice to require (2.22) and (2.23) for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$ and modify the arguments below. For clarity of the proof, we however let $k_0 = 1$.

Assume moreover that $X - \bar{\psi}(R_1, s_1 + c)$ can be covered by countably many $\psi$-balls $X_{i_1} \equiv \psi(x_{i_1}, s_1 - u_c)$ where $x_{i_1} \notin \bar{\psi}(R_1, s_1 + c)$. Using the arguments given below, this is for instance the case if $X$ has finite $\mu$-measure and $(\Omega, \psi, \mu)$ satisfies a power law with respect to $(\Omega, \psi)$. Note that, by the countable stability of the Hausdorff-dimension - that is,
\[
\dim(\text{Bad}(\mathcal{F}, c)) \leq \dim(\bigcup_i \text{Bad}(\mathcal{F}, c) \cap X_{i_1}) \leq \sup_i \dim(\text{Bad}(\mathcal{F}, c) \cap X_{i_1}) - \]
it suffices to estimate the dimension of each $\text{Bad}(F, c) \cap X_{i_1}$. In order to determine the upper bound of $\text{dim}(\text{Bad}(F, c) \cap X_{i_1})$, we construct a suitable covering of $\text{Bad}(F, c) \cap X_{i_1}$ with uniform bounds on the diameters converging to zero.

We start with $X_{i_1}$. Suppose that the we are already given $\omega_{i_1 \ldots i_k} = (x_{i_1 \ldots i_k}, \bar{t}_k) \in \Omega$ with $x_{i_1 \ldots i_k} \in U_{k-1}(c)$ and let $U_{i_1 \ldots i_k} \equiv U_{k}(c) \cap \psi(\omega_{i_1 \ldots i_k})$. If possible, let $x_{i_1 \ldots i_k}, \ldots, x_{i_1 \ldots i_k m} \in U_{i_1 \ldots i_k}$ be chosen such that $\psi(x_{i_1 \ldots i_k k+1}, \bar{t}_{k+1} + d_s)$ are disjoint. Note that if there exists $x' \in U_{i_1 \ldots i_k}$ such that $x' \notin \bigcup_{i' \in \{k+1, \ldots, m\}} \psi(x_{i_1 \ldots i_k k+1}, \bar{t}_{k+1} + d_s)$, then $\psi(x', \bar{t}_{k+1} + d_s)$ is disjoint to $\bigcup_{i' \in \{k+1, \ldots, m\}} \psi(x_{i_1 \ldots i_k k+1}, \bar{t}_{k+1} + d_s)$ by (2.6) and we set $\omega_{i_1 \ldots i_k (m+1)} \equiv (x', \bar{t}_{k+1}) \in \Omega$. Therefore, we obtain a covering of $U_{i_1 \ldots i_k}$ by the $\psi$-balls $\psi(\omega_{i_1 \ldots i_k k+1})$ which is finite (see (2.24)), bounded by a number $N_k$.

In fact, by (2.6) and since

$$\bar{t}_{k+1} + d_s = t_{k+1} - u^c + d_s = t_k + c + d_s \geq s_n + c + d_s,$$

for all $s_n \leq t_k$, the $\psi$-balls $\psi(x_{i_1 \ldots i_k k+1}, \bar{t}_{k+1} + d_s)$ are disjoint to $\bar{\psi}(R_n, s_n + c + d_s)$, $s_n \leq t_k$. Moreover, they are contained in $\psi(x_{i_1 \ldots i_k k+1}, \bar{t}_{k+1} - d^c)$ by (2.5). Hence, (2.22) and (2.23) applied to the formal ball $\omega_0 = (x_{i_1 \ldots i_k}, \bar{t}_k - d^c)$ imply

$$\mu(\psi(\omega_0)) \geq \mu(\psi(\omega_0) \cap \bigcup_{s_n \leq t_k} \bar{\psi}(R_n, s_n + c + d_s)) + \sum_{i_{k+1} = 1}^{N_k} \mu(\psi(x_{i_1 \ldots i_k k+1}, \bar{t}_{k+1} + d_s)) \geq (\tau^c + N_k \cdot k^c) \mu(\psi(\omega_0)).$$

Using (2.7), this shows that the above collection of $\psi$-balls $\psi(\omega_{i_1 \ldots i_k k+1})$, $i_{k+1} = 1, \ldots, N_k$, must be finite where $N_k$ is bounded by

$$N_k \leq \frac{(1 - \tau^c)}{k^c}.$$

For every $k \in \mathbb{N}$ we thus constructed a finite cover of

$$\text{Bad}(F, c) \cap X_{i_1} \subset U_k(c) \cap X_{i_1} \subset \bigcup_{i_1 \ldots i_k} U_{i_1 \ldots i_k} \subset \bigcup_{i_1 \ldots i_k k+1} \psi(\omega_{i_1 \ldots i_k k+1}),$$

where the indices run over all $i_2 \ldots i_{k+1}$ from the above construction. The sets of the covering are of diameter at most $c_\sigma e^{-\sigma t_{k+1}}$ (by (2.8)) and, using (2.24), the number of this covering is bounded by

$$\tilde{N}_k \equiv \prod_{i=1}^{k} N_i \leq \frac{(1 - \tau^c)^k}{(k^c)^k}.$$

Finally, it is readily checked (or seen from [13], Proposition 4.1) that

$$\text{dim}(\text{Bad}(F, c) \cap X_i) \leq \liminf_{k \to \infty} \frac{\log(N_k)}{\log(c_\sigma e^{-\sigma t_{k+1}})} \leq \liminf_{k \to \infty} \frac{k \left( \log(1 - \tau^c) - \log(k^c) \right)}{\sigma t_{k+1}} \leq \frac{\log(1 - \tau^c) - \log(k^c)}{\sigma(c + u^c)}.$$

This gives our first formula for the upper bound.

In the case that $\bar{\psi}$ is not the standard function $B_\sigma$, the constructed covering might not be the best suitable for an optimal upper bound and we want to consider further conditions. Let $\mu$ satisfy a power law with respect to both parameter spaces $(\Omega, \psi)$ and $(\Omega, B_1)$ and the parameters $(\tau, c_1, c_2)$ and $(\delta, c_1, \delta, c_2, \delta)$ respectively. Assume moreover that there exists
Replacing the same arguments as above we can assume that
\[ \sigma \geq \tilde{\sigma} \] and \( d^* \geq 0 \) such that \( B_{\tilde{\sigma}}(y, t) = B(y, e^{-\tilde{\sigma} t}) \cap X \subset \psi(x, t - d^*) \), whenever \( y \in \psi(x, t) \).

Then, given one of the \( N_{k-1} \) formal balls \( \omega_{i_1...i_k} \) constructed above, we can cover \( \psi(\omega_{i_1...i_k}) \) by \( Z(\omega_{i_1...i_k}) \) metric balls \( B_j(\omega_{i_1...i_k}) = B(z_j(\omega_{i_1...i_k}), e^{-\sigma_j t_j}) \). Moreover, using the same arguments as above we can assume that \( z_j(\omega_{i_1...i_k}) \in \psi(\omega_{i_1...i_k}) \) for \( j = 1, \ldots, Z(\omega_{i_1...i_k}) \) and that the balls \( B(z_j(\omega_{i_1...i_k}), e^{-\sigma_j t_j}) \) are disjoint and contained in \( \psi(x_{i_1...i_k}, \tilde{t}_k - d^*) \). Hence, we obtain
\[
c_2 e^{-\tau(\tilde{t}_k - d^*)} \geq \mu(\psi(x_{i_1...i_k}, \tilde{t}_k - d^*)) \geq \mu(\bigcup_{j=1}^{Z(\omega_{i_1...i_k})} B(z_j(\omega_{i_1...i_k}), e^{-\sigma_j t_j})) \geq Z(\omega_{i_1...i_k}) \cdot c_1 \delta \frac{1}{\tilde{\sigma}} e^{-\delta \tilde{\sigma} t_k},
\]
and the number \( Z(\omega_{i_1...i_k}) \) can be bounded by
\[
Z_k \equiv \frac{2^d c_2 e^{\tau d^*}}{c_1 \delta} e^{(\delta \tilde{\sigma} - \tau) \tilde{t}_k}.
\]
Thus, this gives a covering of \( \text{Bad}(\mathcal{F}, c) \cap X_i \) by \( N_k \cdot Z_k \) metric balls of diameter \( 2e^{-\tilde{\sigma} t_k} \).

Replacing \( N_k \) by \( N_k \cdot Z_k \) in (2.24), we obtain a new upper bound
\[
\dim(\text{Bad}(\mathcal{F}, c) \cap X_i) \leq \delta - \frac{\tau}{\tilde{\sigma}} + \frac{\log(1 - \tau c) - \log(k \tilde{c})}{\tilde{\sigma} (c + \tilde{c})} \leq \delta - \frac{[\log(1 - \tau c)] + \log(c_1 \tilde{c}) - \tau (d_1 + d^*)}{\tilde{\sigma} (c + \tilde{c})},
\]
where we used (2.10) in the last inequality.

2.3.1. The Standard Case \( X = \mathbb{R}^n \). Let again \( X = \mathbb{R} = \mathbb{R}^n \), \( \psi = B_{\sigma} \) and \( \mu \) be the Lebesgue measure. Note that, even in this case, our given bounds (2.25) and (2.26) respectively might not be sharp, that is, the bounds might in fact be bigger than the actual dimension of \( \mathbb{R}^n \) because of the strictly positive separation constant \( d_1 \) and \( d^* \).

For this standard case we now want to sharpen the upper bound. We again only need to modify the above arguments by shifting the separation constant into \( \tau \).

For \( c > 0 \), let \( \bar{u}^c \geq 0 \) such that
\[
c + \bar{u}^c = \log(m)/\sigma
\]
for some \( m \in \mathbb{N} \), and modify (2.23) such that for any formal ball \( \omega = (x, \tilde{t}_k) \in \Omega \) with \( x \in U_{k-1}^{Q_{\sigma}}(c + d_1) \equiv \bigcap_{s_n \leq \tilde{t}_k - d_1 - \log(2)/\sigma} Q_{\sigma}(R_n, s_n + c + d_1)^C \) we have
\[
\mu(Q_{\sigma}(\omega) \cap \bigcup_{s_n \leq \tilde{t}_k - d_1 - \log(2)/\sigma} Q_{\sigma}(R_n, s_n + c + d_1)) \geq \bar{\tau}^c \cdot \mu(Q_{\sigma}(\omega)),
\]
where \( \bar{\tau}^c \) is a positive constant; here, \( t_k \) and \( \tilde{t}_k \) are with respect to \( \bar{u}_c \). Note that we already have \( \mu(Q_{\sigma}(x, \tilde{t}_k + 1)) = e^{-\sigma \bar{u}^c} \cdot \mu(Q_{\sigma}(x, \tilde{t}_k)) \) and hence (2.22).

We modify our construction (as before (2.24)) where we replace the choices \( \psi(\omega_{i_1...i_k}) \) by cubes \( Q_{i_1...i_k} \) in order to obtain a covering of \( \text{Bad}_{\mathbb{R}^n}^d(\mathcal{F}, c) \cap Q_{i_1} \) by the cubes \( Q_{i_1...i_k} \). In fact, if \( Q = Q_{i_1...i_k} = Q_{\sigma}(x_{i_1...i_k}, \tilde{t}_k) \) is a given cube, let \( Q_{i_1...i_k, i_{k+1}} = Q_{\sigma}(x_{i_1...i_k, i_{k+1}}, \tilde{t}_k + (c + \bar{u}_c)) \) be precisely the cubes of the partition of \( Q \) as in (2.19), which intersect
\[
U_{i_1...i_k}(c) \equiv Q \cap \bigcap_{s_n \leq \tilde{t}_k - d_1 - \log(2)/\sigma} Q_{\sigma}(R_n, s_n + c)^C,
\]
and hence cover \( U^{Q^\sigma}_{t_1 \ldots t_k}(c) \). Let \( \bar{Q} = Q_{t_1 \ldots t_k} \) be such a cube intersecting \( U^{Q^\sigma}_{t_1 \ldots t_k}(c) \) in a point \( y \). Then \( \bar{Q} \subset Q^\sigma(y, \bar{t}_{k+1} - \log(2)/\sigma) \). Moreover, for every \( n \) with \( s_n \leq t_k - \bar{d}_s - \log(2)/\sigma \), we have

\[
\bar{t}_k + (c + \bar{u}_c) - \log(2)/\sigma = t_k + c - \log(2)/\sigma \geq s_n + c + \bar{d}_s
\]

so that \( Q^\sigma(y, \bar{t}_{k+1} - \log(2)/\sigma) \subset Q^\sigma(y, s_n + c + \bar{d}_s) \) where the supset is disjoint to \( Q^\sigma(R_n, s_n + c + \bar{d}_s) \). Hence, every cube chosen as above is contained in \( U^{Q^\sigma}_{t_1 \ldots t_k}(c + \bar{d}_s) \).

Using the above arguments with (2.27) shows

\[
N_k \leq (1 - \tau^c)e^{n\sigma(c+\bar{u}_c)},
\]

which improves (2.24). As in (2.25), we obtain our upper bound in the standard case

\[
\dim(\text{Bad}_{\mathbb{E}^n}^Q(F, c) \cap Q_{t_1}) \leq \frac{\log(1 - \tau^c) - \log(e^{-n\sigma(c+\bar{u}_c)})}{\sigma(c + \bar{u}_c)} \tag{2.28}
\]

\[
= n - \frac{\log(1 - \tau^c)}{\sigma(c + \bar{u}_c)}.
\]

Remark. Again, it suffices to require (2.27) for all \( k \geq k_0 \) for some \( k_0 \in \mathbb{N} \) and start with a covering of \( X = \mathbb{R}^n \) by cubes \( Q^\sigma(x_{t_1}, t_{k_0}) \).

In special situations, when (2.13) is satisfied with respect to \( B_\sigma \) and the parameters \((\bar{c}, u^c)\) (as below) and is even satisfied for all times \( t \), we can in fact already estimate \( \tau^c \) using \( \tau^c \). More precisely, we have the following, where we remark that (2.29) corresponds to (2.23) with \( t = \bar{t}_k - \bar{c}^\sigma \) and the parameters \((c, \bar{u}^c)\) (where \( \bar{t}_k \) is with respect to the parameters \((c, \bar{u}^c)\) in the following).

**Lemma 2.6.** For \( c > 0 \), let \( \bar{c} = c + a \) and \( \bar{u}^c \geq u^c + a \), where \( a = d^c + \bar{a}_s + \log(2)/\sigma + \sqrt{n}/\sigma \). Then, \( U^{Q^\sigma}_{k-1}(c + \bar{d}_s) \subset \bigcap_{s_n \leq \bar{t}_{k-1} - \bar{c}^\sigma + d^c} B_\sigma(R_n, s_n + \bar{c} + \bar{d}_s)^C \). Moreover, assume that for all \( \omega = (x, t) \in \Omega \) with \( x \in \bigcap_{s_n \leq \bar{t}_k - \bar{c} + d^c} B_\sigma(R_n, s_n + \bar{c})^C \) we have

\[
\mu(B_\sigma(\omega) \cap \bigcup_{s_n \leq \bar{t}_k - \bar{c} + d^c} B_\sigma(R_n, s_n + \bar{c} + \bar{d}_s)) \geq \tau^c \cdot \mu(B_\sigma(\omega)) \tag{2.29}
\]

where \( \tau^c \) is a positive constant. Then, (2.27) is satisfied for \( \omega \) with

\[
\tau^c \geq e^{-n\sqrt{n}} \tau^c.
\]

**Proof.** Note that \( \bar{t}_k - \bar{c} + d^c = t_{k-1} - a + d^c \leq t_{k-1} - \bar{d}_s - \log(2)/\sigma \), since \( a \geq d^c + \bar{a}_s + \log(2)/\sigma \) (as \( d^c \leq d^c \)). Using that \( Q^\sigma(y, s + \sqrt{n}/\sigma) \subset B_\sigma(y, s) \) and \( a \geq \bar{d}_s + \sqrt{n}/\sigma \), we see that

\[
U^{Q^\sigma}_{k-1}(c + \bar{d}_s) = \bigcap_{s_n \leq \bar{t}_{k-1} - \bar{d}_s - \log(2)/\sigma} Q^\sigma(R_n, s_n + c + \bar{d}_s)^C \subset \bigcap_{s_n \leq \bar{t}_k - \bar{c} + d^c} B_\sigma(R_n, s_n + \bar{c})^C,
\]

which shows the first claim.

Similarly, since \( t + u^c + d^c \leq t + \bar{u}^c - \bar{d}_s - \log(2)/\sigma \) and \( \bar{c} + \bar{d}_s \geq c + \bar{d}_s \), we have

\[
B_\sigma(x, t) \cap \bigcup_{s_n \leq t + u^c + d^c} B_\sigma(R_n, s_n + \bar{c} + \bar{d}_s) \subset Q^\sigma(x, t) \cap \bigcup_{s_n \leq t + \bar{u}^c - \bar{d}_s - \log(2)/\sigma} Q^\sigma(R_n, s_n + c + \bar{d}_s).
\]
Hence, by (2.29), we obtain
\[
\mu(Q_\sigma(x, t) \cap \bigcup_{s_n \leq t + \tilde{a}e^{-d_* - \log(2)/\sigma}} Q_\sigma(R_n, s_n + c + \tilde{d}_*) \geq \tau^\varepsilon \cdot \mu(B_\sigma(x, t))
\]
\[
\geq \tau^\varepsilon \cdot \mu(Q_\sigma(x, t + \sqrt{n}/\sigma))
\]
\[
= \tau^\varepsilon e^{-n\sqrt{n}} \cdot \mu(Q_\sigma(x, t)),
\]
proving the lemma if we set \( t = \tilde{t}_k \).
}\)

2.4. Dirichlet and decaying measures. Let \( S \equiv \{ S \subset \overline{X} \} \) be a given collection of nonempty Borel sets. For instance, consider \( S \) to be the collection of metric spheres \( S(x, t) \equiv \{ y \in X : d(x, y) = e^{-t} \} \) in \( \overline{X} \), or the set of hyperplanes in the Euclidean space \( \mathbb{R}^n \). Assume moreover, that \( \psi(S, t) \) is a Borel-set for all \( t > t_s \) and \( S \in S \).

For the lower bound, given a locally finite Borel measure \( \mu \) on \( X \), \( (\Omega, \psi, \mu) \) is said to be absolutely \( (c_\delta, \delta) \)-decaying with respect to \( S \) if for all \( (x, t) \in \Omega \) and for all \( S \in S \) and \( s \geq 0 \) we have
\[
\mu(\psi(x, t) \cap \psi(S, t + s)) \leq c_\delta e^{-\delta s} \mu(\psi(x, t)).
\]

Moreover, we say that a nested discrete family \( F \) is locally contained in \( S \) (with respect to \( (\Omega, \psi) \)) if there exists \( l_s \geq 0 \) and a number \( n_s \in \mathbb{N} \) such that for all \( (x, t) \in \Omega \) we have
\[
\psi(x, t + l_s) \cap R(t) \subset \bigcup_{i=1}^{n_s} S_i
\]
is contained in at most \( n_s \) sets \( S_i \) of \( S \).

We say that \( (\Omega, \psi) \) is \( d_\sigma \)-separating if for all formal balls \( (x, t) \in \Omega \) and for any set \( M \) disjoint to \( \psi(x, t) \), we have
\[
\psi(x, t + d_\sigma) \cap \overline{\psi}(M, t + d_\sigma) = \emptyset.
\]

Clearly, the standard function \( B_\sigma \) is \( \log(3)/\sigma \)-separating in a proper metric space \( \overline{X} \).

Proposition 2.7. Let \( (\Omega, \psi, \mu) \) be \( d_\sigma \)-separating and let \( F \) be locally contained in \( S \). Then, if \( (\Omega, \psi) \) is absolutely \( (c_\delta, \delta) \)-decaying with respect to \( S \), it is \( \tau_c \)-decaying with respect to \( F \) and the parameters \( (c, l_s + d_\sigma) \), where
\[
\tau_c = n_s c_\delta e^{-\delta(c - 2d_\sigma)},
\]
for all \( c \geq 2d_\sigma \) such that \( \tau_c < 1 \).

Proof. Fix \( c \geq 2d_\sigma \). Given \( \omega = (x, t + l_s + d_\sigma + d_\sigma) \in \Omega \) and \( l_s, n_s \in \mathbb{N} \) as well as \( S_1, \ldots, S_{n_s} \) from the definition of (2.31), we claim that
\[
\psi(\omega) \cap \psi(R(t), t + l_s + d_c + c - d_\sigma) \subset \psi(\omega) \cap \bigcup_{i=1}^{n_s} \psi(S_i, t + l_s + c - d_\sigma).
\]
In fact, let \( M \) be the set \( R(t) \cup S_i \) which is disjoint to \( \psi(x, t + l_s) \) by (2.31). By monotonocity of \( \psi \), we have
\[
\psi(x, t + l_s + d_c + d_\sigma) \subset \psi(x, t + l_s + d_\sigma)
\]
which, by (2.32), is disjoint to
\[
\psi(M, t + l_s + d_\sigma) \supset \psi(M, t + l_s + c - d_\sigma),
\]
for \( c \geq 2d_\sigma \) again by monotonocity of \( \psi \). This shows the above claim.
Set \( l_c = l_s + d_s \) so that \( \omega = (x, t + l_c + d_c) \in \Omega \). Finally, (2.30) implies
\[
\mu(\psi(\omega) \cap \bigcup_{i=1}^{n_s} \psi(S_i, t + l_s + d_c + c - d_s)) = \mu(\psi(\omega) \cap \bigcup_{i=1}^{n_s} \psi(S_i, t + l_c + d_c + (c - 2d_s))) \\
\leq n_s c_3 e^{-\delta(c - 2d_s)} \mu(\psi(\omega)),
\]
which shows that \( \mu \) is \( \tau_c \)-decaying with respect to \( F \) and the parameters \((c, l_s + d_s)\). \( \square \)

As a special case, let \( \tilde{\psi} = \tilde{B}_\sigma \) be the standard function and \( \bar{X} \) be a proper metric space. Recall that \( d_s \leq \log(3)/\sigma \), and assume that for all distinct points \( x, y \in \bar{R}_n \), we have
\[
d(x, y) > \bar{c} \cdot e^{-\sigma n_s}, \tag{2.33}
\]
for some constant \( \bar{c} > 0 \).

**Lemma 2.8.** Let \((\Omega, \psi, \mu)\) satisfy a power law with respect to the parameters \((\tau, c_1, c_2)\). If (2.33) is satisfied, then \( \mu \) is \( \tau_c \)-decaying with respect to \( F \), where \( \tau_c = c_1^2 e^{\tau(c - 2d_s)} \), for all \( c \geq 2d_s \) and \( l_c = -\log(c)/\sigma + d_s + \log(2) \).

**Proof.** Let \( l_s = -\log(c)/\sigma + \log(2) \). Given a formal ball \((x, t + l_s) \in \Omega\), at most one point \( y \in R(t) \) can lie in \( \tilde{B}(x, e^{-\sigma(t + l_s)}) \). Indeed, for distinct \( y, y' \in \bar{R}_n \), where \( n_t \in \mathbb{N} \) was the largest integer such that \( s_t \leq t \), (2.33) implies
\[
d(y, y') > e^{-\sigma(s_n + \log(c))/\sigma} \geq 2e^{-\sigma(t + l_s)}.
\]
Hence, \( F \) is locally contained in the set \( S = \{ y \in \bar{R}_n : n \in \mathbb{N} \} \) with \( n_s = 1 \). Since \( \mu \) satisfies the power law, it is \((c_1^2, \tau)\)-decaying with respect to \( S \) and \( \tilde{B}_\sigma \). The proof follows from Proposition 2.7. \( \square \)

Analogously, for the upper bound and a possibly different collection of Borel sets \( S \), for a locally finite Borel measure \( \mu \) on \( X \), \((\Omega, \psi, \mu)\) is called \((c_\delta, \delta)\)-Dirichlet with respect to \( S \) if for all \( \omega = (x, t) \in \Omega \), for all \( S \in \mathcal{S} \) such that \( S \cap \tilde{\psi}(\omega) \neq \emptyset \) and \( s \geq 0 \) we have
\[
\mu(\psi(\omega) \cap \tilde{\psi}(S, t + s)) \geq c_\delta e^{-\delta s} \mu(\psi(\omega)). \tag{2.34}
\]

We say that the family \( F \) locally contains \( S \) (with respect to \((\Omega, \psi)\)) if there exists \( u_s \geq 0 \) such that for all formal balls \( \omega = (x, t - u_s) \in \Omega \) there exists \( S \in \mathcal{S} \) with
\[
\tilde{\psi}(\omega) \cap R(t) \supset \tilde{\psi}(\omega) \cap S. \tag{2.35}
\]

**Proposition 2.9.** If \( F \) locally contains \( S \) and \((\Omega, \psi, \mu)\) is \((c_\delta, \delta)\)-Dirichlet with respect to \( S \), then \((\Omega, \psi, \mu)\) satisfies a power law with respect to \( F \) and the parameters \((c, u_s)\), where \( \tau_c \geq c_\delta e^{-\delta(c + d_s)} \).

In the special case when \( F \) locally contains \( S \), where \( S \) consists of subsets of \( X \), and \((\Omega, \psi, \mu)\) satisfies a power law with respect to \((\tau, c_1, c_2)\), we have that \((\Omega, \psi, \mu)\) is \( \tau_c \)-decaying with respect to \( F \) and the parameters \((c, u_s)\), where
\[
\tau_c \geq c_1^2 e^{-\tau(c + d_s + u_s + d_s)}.
\]

**Proof.** The first statement is readily checked. For the second one, let \( \omega = (x, t - u_s) \in \Omega \) and \( S \in \mathcal{S} \) such that \( S \cap \tilde{\psi}(\omega) \subseteq \bar{R}_n \cap \tilde{\psi}(\omega) \). Let \( y \in S \cap \tilde{\psi}(\omega) \). By monotonicity of \( \tilde{\psi} \) and (2.5), \( \tilde{\psi}(y, t + c + d_s) \subset \tilde{\psi}(y, t - u_s + c) \subset \tilde{\psi}(x, t - u_s) \). Hence, for \( \omega_0 = (x, t - u_s - d_s) \in \Omega \) we see that
\[
\mu(\psi(\omega_0) \cap \bigcup_{s_n \leq t} \tilde{\psi}(R_n, s_n + c + d_s)) \geq \mu(\psi(y, t + c + d_s)) \geq \mu(\psi(y, t + c + d_s)) \geq c_1^2 e^{-\tau(c + d_s + u_s + d_s)} \mu(\psi(\omega_0)),
\]
which shows the second claim.  

3. Applications

We want to determine the upper and lower bounds on the Hausdorff-dimension of \( \text{Bad}(\mathcal{F}, c) \) of several examples by checking the conditions of the abstract formalism.

3.1. \( \text{Bad}^p_{\mathbb{R}^n} \). For \( n \geq 1 \), let \( \bar{r} \in \mathbb{R}^n \) with \( r^1, \ldots, r^n \geq 0 \) such that \( \sum r^i = 1 \). Recall that \( \text{Bad}^p_{\mathbb{R}^n} \) is the set of points \( \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \) for which there exists a positive constant \( c(\bar{x}) > 0 \) such that

\[
\max_{i=1,\ldots,n} |q x_i - p_i|^{1/r_i} \geq c(\bar{x})/q,
\]

for every \( q \in \mathbb{N} \) and \( p = (p_1, \ldots, p_n) \in \mathbb{Z}^n \).

As in [23, 40], we let \( \bar{\Omega} = \mathbb{R}^n \times \mathbb{R} \) and define \( \bar{\psi}_r : \bar{\Omega} \to C(\mathbb{R}^n) \) to be monotonic function given by the rectangle determined by \( \bar{r} \), that is, the product of metric balls

\[
\bar{\psi}_r(\bar{x}, t) = B(x_1, e^{-(1+r^1)t}) \times \cdots \times B(x_n, e^{-(1+r^n)t}).
\]

Denote by \( r^+ = \max\{|r^i|\} \) and by \( r^- = \min\{|r^i|\} \) which we assume to be non-zero. Clearly, we have \( \text{diam}(\bar{\psi}_r(x, t)) \leq 2 e^{-(1+r^-)t} \), hence \( \sigma = 1 + r^- \). Moreover, for \( c > 0 \) it is readily checked that

\[
d^c = \frac{\log(1+e^{-(1+r^-)c})}{1+r^-}, \quad d_c = -\frac{\log(1-e^{-(1+r^+)c})}{1+r^+}.
\]

In the following, let \( S \) be the set of affine hyperplanes in \( \mathbb{R}^n \). Note that, if \( \mu \) denotes the Lebesgue-measure on \( \mathbb{R}^n \), it follows from [20, Lemma 9.1], that \( (\Omega, \psi_r, \mu) \) is absolutely \((\delta, c_\delta)\)-decaying with respect to \( S \) for \( \delta = 1 + \min\{r^1, \ldots, r^n\} = 1 + r^- \) and some \( c_\delta > 0 \). Moreover, \((\Omega, \psi_r, \mu) \) satisfies a power law with respect the exponent \( n + 1 \); in fact, for all \( (x, t) \in \Omega \) we have \( \mu(\bar{\psi}_r(x, t)) = 2^n e^{-(n+1)t} \).

For \( c > 0 \), define

\[
l_* = \frac{\log(n!) + n+1}{n+1} \log(2) + \frac{\log(3)}{1+r^-}, \quad u_* = \frac{1}{r^-} \left( c + (1 + r^+) \log(2) \right) + \log(n+1) + 2 \log(2) \equiv \frac{c}{r^-} + u_*,
\]

\[
d_* = \frac{\log(3)}{1+r^+}.
\]

Theorem 3.1. Let \( X \subset \mathbb{R}^n \) be the support of a locally finite Borel measure \( \mu \) on \( \mathbb{R}^n \) such that \((\Omega, \psi_r, \mu) \) satisfies a power law with respect to the parameters \( (\tau, c_\delta, c_\tau) \).

For the lower bound, assume that \((\Omega, \psi_r, \mu) \) is absolutely \((\delta, c_\delta)\)-decaying with respect to \( S \). Then, for \( c > \log(2) + 2d_* \), we have

\[
\text{dim}(\text{Bad}^p_{\mathbb{R}^n}(\frac{1}{2} e^{-(1+r^+)\left(2c+L_\tau\right)} \cap X) \geq d_\mu(X) \geq \frac{\log(n!)}{n+1} + \frac{n+1}{n+1} \log(2) + \frac{\log(3)}{1+r^-} + \log(n+1) + 2 \log(2) \equiv \frac{c}{r^-} + u_* - \frac{\log(2) + 2 \log\left(\frac{\tau}{1+r^+}\right) + \log\left(\frac{d_\mu(X)}{e^{1+r^-}}\right)}{1+r^-}.
\]

For the upper bound, assume that \((\Omega, B_1, \mu) \) satisfies also a power law with respect to the exponent \( \delta \) and with \( X = \mathbb{R}^n \). Then, for \( c > 0 \), we have

\[
\text{dim}(\text{Bad}^p_{\mathbb{R}^n}(e^{-(1+r^+)c})) \leq \delta - \frac{\log\left(\frac{\tau}{e^{1+r^-}}\right) + \log\left(\frac{d_\mu(X)}{e^{1+r^-}}\right)}{1+r^-}.
\]

The theorem will be sharpened for the standard case when \( \mu \) is the Lebesgue measure.\footnote{See also Lemma ??? in [40].}
Proof. For $k \in \Lambda \equiv \mathbb{N}$ we define the set of rational vectors

$$R_k \equiv \{ \tilde{p}/q : \tilde{p} \in \mathbb{Z}^n, 0 < q \leq k \}$$

as resonant set and define its size by $s_k \equiv \log(k+1)$. The family $\mathcal{F} = (\mathbb{N}, R_k, s_k)$ is nested and discrete. Moreover, since $R(t)$ is a discrete set for all $t \geq s_1$ and $\psi_r$ is a product of metric balls, it is readily checked that $(\Omega, \psi_r)$ is $\frac{\log(2)}{1+r^*}$-separating with respect to $\mathcal{F}$, and that $(\Omega, \psi_r)$ is $d_*$-separating.

For the lower bound, choose any $\bar{l}_* > \log(n!)/(n+1) + n/(n+1) \log(2)$. Note that for a formal ball $\omega = (\bar{x}, t + \bar{l}_*)$ such that $t \leq s_k$ the sidelights $\rho_i$ of the box $\psi_r(\omega)$ satisfy

$$\rho_1 \cdot \rho_n = 2e^{-(1+r^*)(t+\bar{l}_*)} \cdots 2e^{-(1+r^*)(t+\bar{l}_*)} < 2^ne^{-(1+n)s_k+\log(n!)} \log(2) \leq \frac{1}{n!(k+1)^{n+1}}.$$

We now use the following version of the 'Simplex Lemma' due to Davenport and Schmidt where the version of this lemma can be found in [24], Lemma 4.

**Lemma 3.2.** Let $D \subset \mathbb{R}^n$ be a box of side lengths $\rho_1, \ldots, \rho_n$ such that $\rho_1 \ldots \rho_n < 1/(n!(k+1)^{n+1})$. Then there exists an affine hyperplane $L$ such that $R_k \cap D \subset L$.

This shows that $\mathcal{F}$ is locally contained in the collection of affine hyperplanes $S$ with $n_* = 1$. Since $(\Omega, \psi_r, \mu)$ is absolutely $(\delta, c_\delta)$-decaying with respect to $S$, it follows from Proposition 2.7 that $(\Omega, \psi_r, \mu)$ is $\tau_c$-decaying with respect to $\mathcal{F}$ for all $c > 2d_*$ where $l_c \equiv l_* + d_* + \tau_c = c_\delta e^{-\delta d_* + \delta}$. Note that $c > 0$, such that $\tau_c < 1$, is given when $c > \log(c_\delta)/\delta + 2d_*$. Finally, if $\bar{x} \in \text{Bad}_X^{\psi_r}(\mathcal{F}, c)$, then for every $\bar{p}/q$, where $\bar{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, and $q \in \mathbb{N}$, $\bar{x} \notin \psi_r(R_q, s_q + c) \supset \psi_r(\bar{p}/q, s_q + c)$. Hence, for some $i \in \{1, \ldots, n\}$, we have

$$|x_i - p_i/q| \geq e^{-(1+r^*)(s_q+c)} \geq \frac{e^{-(1+r^*)c}}{(q+1)^{1+r^*}} \geq \frac{e^{-(1+r^*)c}}{2q},$$

which shows that $\text{Bad}_X^{\psi_r}(\mathcal{F}; 2c + l_* \subset \text{Bad}_{2c}(\frac{1}{2} e^{-(1+r^*)(2c + l_*)}) \cap X$. Applying (2.18), the formula for the lower bound, together with (2.10) gives that $\dim(\text{Bad}_{2c}(\frac{1}{2} e^{-(1+r^*)(2c + l_*)}) \cap X)$ is bounded below by

$$d_\mu(X) - \frac{\log(2c_\delta e^{-\tau(c-d_* - d_*)}) - \log(1 - c_\delta e^{-\delta(c-2d_*)}) - \log(c_\delta e^{-\tau c})}{\sigma c} \leq d_\mu(X) - \frac{\log(2) + 2 \log(c_\delta) + \tau(d_* + d_*) + \log(1 - c_\delta e^{2d_* \delta} e^{-\delta \tau c})}{1 + r^*}.$$

For the upper bound, note that using the pigeon-hole lemma as for the classical Dirichlet Theorem, the following Lemma can be shown.

**Lemma 3.3.** Let $x \in \mathbb{R}^n$. For every $N \in \mathbb{N}$ there exists a vector $(p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $1 \leq q \leq (n + 1)N$ such that, for $i = 1, \ldots, n$, we have

$$|x_i - p_i/q| \leq \frac{1}{qN^r}.$$
1)) + \log(2). Let \( \overline{p}, q \) as in the above lemma and note that \( s_q \leq \log(q) + \log(2) \leq t_k \). In the case when \( \log(q) \leq t_k - (c + u^c) \), we have

\[
|x_i - \frac{p_i}{q}| \leq \frac{1}{qN^{1/s}}
\]
\[
\leq \frac{2^{1/4}}{q(N + 1)^{1/2}}
\]
\[
\leq e^{-\log(q) - r^t(t_k - \log(n + 1) - 2 \log(2))}
\]
\[
\leq e^{-(1 + r^t)(\log(q) - r^t(c + u^c) - \log(n + 1) - 2 \log(2))}
\]
\[
\leq e^{-(1 + r^t)(\log(q) + 1) + c} = e^{-(1 + r^t)(s_q + c)},
\]

for every \( i = 1, \ldots, n \), since

\[
r^t(u^c - \log(n + 1) - 2 \log(2)) = \frac{2}{r^t}(c + (1 + r^t)\log(2))
\]
\[
\geq c + (1 + r^t)(\log(q) + 1 - \log(q)).
\]

This shows

\[\bar{x} \in \tilde{\psi}_p(\bar{p}/q, s_q + c) \subset \bigcup_{s_q \leq t_k - 1} \tilde{\psi}_p(R_q, s_q + c).\]

Hence, we may assume \( \log(q) > t_k - (c + u^c) \) and obtain that for every \( i = 1, \ldots, n \),

\[
|x_i - \frac{p_i}{q}| \leq e^{-\log(q) - r^t t_k}
\]
\[
\leq e^{-(1 + r^t)(t_k + (c + u^c))} \leq e^{-(1 + r^t)(t_k - u^c)},
\]

since \( r^t u^c \geq c \). This yields \( \tilde{p}/q \in \tilde{\psi}_p(\bar{x}, t_k - u^c) \). Since by assumption \( X = \mathbb{R}^n \), replacing \( u^c \) by \( u^c \) in the proof of Proposition 2.9 shows that \( (\Omega, \psi, \mu) \) is \( \tau^c \)-Dirichlet with respect to \( \mathcal{F} \) and the parameters \( (c, u^c) \), where \( \tau^c = \frac{d_m}{c^2} e^{-\tau(d_m + d^c)} \cdot e^{-\tau(c + u^c)} \).

Finally, let \( \bar{x} \in \text{Bad}_{\mathbb{R}^n}(e^{-(1 + \frac{1}{r^t})c}) \), where \( c > 0 \). Thus, for every \( \tilde{p}/q \) with \( \tilde{p} \in \mathbb{Z}^n \) and \( q \in \mathbb{N} \) there exists \( i \in \{1, \ldots, n\} \) such that

\[
|qx_i - p_i|^{1/r^t} > e^{-(1 + \frac{1}{r^t})c}/q.
\]

Equivalently, we have

\[
|x_i - p_i/q| > (e^{-(1 + \frac{1}{r^t})c})^{r^t} q^{-1/(1 + r^t)} \geq e^{-(1 + r^t)(s_q + c)},
\]

which shows that \( \text{Bad}_{\mathbb{R}^n}(e^{-(1 + \frac{1}{r^t})c}) \cap X \subset \text{Bad}_{\mathbb{F}^n}(\mathcal{F}; c). \) Applying (2.26) with \( \sigma = 1 + r^t \) and \( d^* = 2 \log(2) \), yields the upper bound

\[
\dim(\text{Bad}_{\mathbb{R}^n}(e^{-(1 + \frac{1}{r^t})c}) \cap X) \leq \delta - \frac{[\log(1 - \tau^c)] + [\log(\frac{c^t}{c^2}) - \tau(d^* - d^c)]}{(1 + r^t)(c + u^c)}
\]
\[
\leq \delta - \frac{[\log(1 - \frac{c^t}{c^2} e^{-\tau(d_m + d^c)} \cdot e^{-\tau(c + u^c)})] + [\log(\frac{c^t}{c^2}) - \tau(d_m + d^c)]}{(1 + r^t)(c + u^c)}.
\]

This finishes the proof. \( \Box \)

**The Standard Case.** Let \( X = \mathbb{R}^n \) and \( \sigma = 1 + 1/n \). For the lower bound, we let \( c = \log(m)/\sigma > d^* + \log(2)/\sigma \) be sufficiently large for some \( m \in \mathbb{N} \) (such that \( \tilde{c} < 1 \) below). Note that we nowhere used the condition that \( x \in L_b^{B_1+1/n}(c) \) which hence becomes obsolete in this setting. Thus, Lemma 2.5 shows that (2.20) is satisfied for the
parameters \( \tilde{l}_c = \tilde{l}_e + d_e + a \), where \( a \equiv 2\sqrt{n} + \log(2) + d_e \), and \( \tilde{\tau}_e \leq e^{n\sigma(a-d_e)} \tau_e \equiv k_i e^{-(1+1/n)c} \). Thus, (2.21) yields the lower bound

\[
\dim(\text{Bad}_n^{Q_1/n}(e^{-2(1+1/n)c})) \geq \dim(\text{Bad}_n^{Q_1/n}(\mathcal{F}, 2c + \tilde{l}_c)) \geq n - \frac{|\log(1 - k_i e^{-(1+1/n)c})|}{(1 + 1/n)c}.
\]

Up to modifying \( \tilde{k}_i \) to a suitable constant depending on \( c > 0 \) sufficiently large, the lower bound also follows for general \( c > c_0 \).

For the upper bound, for \( c > 0 \) we let \( \tilde{c} = c + a \), where \( a = d_e + \tilde{d}_e + \log(2)/\sigma + \sqrt{n}/\sigma \) and \( \tilde{u}^c \geq u^c + a \) such that \( c + \tilde{u}^c = \log(m)/\sigma \) (with \( m \) minimal). Recall that from Lemma 2.6 \( U^Q_{\tilde{l}_c}(c + \tilde{d}_e) \subseteq \bigcap_{1+\tilde{e} \leq t \leq t_k} B_{\tilde{\tau}_e}(R_n, s_n + \tilde{e})^C \). Moreover, we remark that in the above arguments for determining \( \tilde{\tau}_e^c \), it was in fact nowhere necessary to require \( t = t_k \) and we showed that, if \( (x, \tilde{t}_k) \in \Omega \) with \( x \in U^Q_{\tilde{l}_c}(c + \tilde{d}_e) \) then (2.29) is satisfied. Hence, Lemma 2.6 implies (2.27) with respect to \( \tilde{\tau}_e^c \geq e^{-n\sqrt{n}/\sigma} \equiv k_u e^{-(n+1)c} \). Finally, since \( m \) above was chosen minimal there exists a constant \( k_u \geq 0 \) (independent on \( c \)) such that \( \tilde{c} + \tilde{u}^c \leq (n + 1)c + k_u \), and (2.28) shows

\[
\dim(\text{Bad}_n^{Q_1/n}(e^{-(n+1)c})) \leq \dim(\text{Bad}_n^{Q_1/n}(\mathcal{F}, c)) \leq n - \frac{|\log(1 - k_u e^{-(n+1)c})|}{(1 + 1/n)(n + 1)(c + k_u)}.
\]

This proves Theorem 1.2.

Remark. Let \( n = 2 \) and \( X = \mathbb{R}^2 \). If \( \frac{1+r_1}{1+r_2} \in \mathbb{Q} \), then there exist parameters \( c > 0 \) such that \( c = \frac{\log(m_1)}{1+r_1} = \frac{\log(m_2)}{1+r_2} \) with \( m_1, m_2 \in \mathbb{N} \). For these parameters a partition of the rectangles \( \psi(x, t) \) as in (2.19) is possible, which in turn, following the arguments of the formalism, allows more precise bounds as in the standard case.

3.2. The Bernoulli shift \( \Sigma^+ \). For \( n \geq 1 \), let \( \Sigma^+ = \{1, \ldots, n\}^\mathbb{N} \) be the set of one-sided sequences in symbols from \( \{1, \ldots, n\} \). Let \( T \) denote the shift and let \( d^+ \) be the metric given by \( d^+(w, \tilde{w}) \equiv e^{-\min\{\geq 1; w(i) \neq \tilde{w}(i)\}} \) for \( w \neq \tilde{w} \) and \( d(w, w) \equiv 0 \).

Fix a periodic word \( \tilde{w} \in \Sigma^+ \) of period \( p \in \mathbb{N} \). For \( c \in \mathbb{N} \), consider the set

\[
S_{\tilde{w}}(c) = \{w \in \Sigma^+ : T^k w \notin B(\tilde{w}, e^{-(c+1)}) \text{ for all } k \in \mathbb{N}\}.
\]

Theorem 3.4. For every \( c \in \mathbb{N} \) we have

\[
\dim(S_{\tilde{w}}(c)) \leq \log(n) - \frac{|\log(1 - n^{-c})|}{c},
\]

as well as

\[
\dim(S_{\tilde{w}}(2c + p + 1)) \geq \log(n) - \frac{|\log(1 - n^{-c})|}{c}.
\]

Remark. Note that the Morse-Thue sequence \( w \) in \( \{0, 1\}^\mathbb{N} \) is a particular example of a word in \( S_{\tilde{w}}(2p) \) for any periodic word \( \tilde{w} \) or period \( p \). In fact, \( w \) does not contain any subword of the form \( WWa \) where \( a \) is the first letter of the subword \( W \); for details and more general words in \( S_{\tilde{w}} \), we refer to an earlier work of Schroeder and the author [35].

Proof. For \( k \in \mathbb{N} \) and \( w_k \in \{1, \ldots, n\}^k \), let \( \tilde{w}_k \in \Sigma_+ \) denote the word \( \tilde{w}_k = w_k \tilde{w} \). Let \( \Lambda \equiv \mathbb{N}_0 \) and consider the resonant sets

\[
R_0 = \{\tilde{w}\}, \quad R_k = \{\tilde{w}_l \in \Sigma^+ : w_l \in \{1, \ldots, n\}^l, l \leq k\} \cup R_0, \text{ for } k \in \mathbb{N}.
\]
which we give the size $s_k = k + 1$. Then, the family $\mathcal{F} = (\mathbb{N}_0, R_k, s_k)$ is nested and discrete.

Note that we let $\Omega = \Sigma^+ \times \mathbb{N}$ and consider the standard function $\psi_1$. Hence, we have $d_1(c) = d_\varnothing(c) = d_\varnothing = 0$. Moreover, we have $\text{Bad}(\mathcal{F}, c) = S_w(c)$. In fact, $d^+(T^{k-1}w, \bar{w}) = e^c e^{-c}$. Moreover, if $d^+(w(k) \ldots w(k + c) = \bar{w}(1) \ldots \bar{w}(c))$. Thus, for $w_k = \ldots w(k)$ and $\bar{w}_k = w_k \bar{w}$ we have $d^+(w, \bar{w}_k) \leq e^{-(k+c+1)}$ if and only if $w \in B(\bar{w}_k, e^{-s_k+1}) \subset \psi_1(R_k, s_k + c)$.

For the lower bound, let $\bar{w}_m$ and $\bar{w}_m \in R_m$ be distinct. By definition of $\bar{w}_m$ and $\bar{w}_m$ there exists $i \in \{1, \ldots, m + p\}$ such that $\bar{w}_m(i) \neq \bar{w}_m(i)$; hence

$$d^+(\bar{w}_m, \bar{w}_m) \geq e^{-e^{-e^{-e^{-e^{-(p+m+1)}}}}} = e^{-e^{-e^{-e^{-e^{-(p+m+1)}}}}},$$

and we are given the special case (2.33) with $\bar{c} = e^{-p}$. Moreover, for the probability measure $\mu = \{1/n, \ldots, 1/n\}^{\mathbb{N}}$, $\Omega, B_1, \mu$ satisfies

$$\mu(B(w, e^{-t+1})) = n^{-t} = n e^{-\log(n)(t+1)},$$

and hence a $(\log(n), n)$-power law. From Lemma 2.8 we see that $(\Omega, B_1, \mu)$ is $(\log(n), 1)$-decaying with respect to $\mathcal{F}$ and $l_c = p + 1$. Applying (2.18), we obtain

$$\dim(S_w(2c + p + 1)) \geq \log(n) - \frac{\log(2) + |\log(1 - n^{-c})|}{c}.$$

Finally, note that, checking the arguments in (2.16) and (2.15) respectively, we can see that the constant 'log(2)' can be omitted. (In fact, we even have a partition as in (2.19)).

For the upper bound, let $(w, s_k) = (w, k + 1) \in \Omega$. If $w_k \equiv w(1) \ldots w(k)$, let $\bar{w}_k \equiv w_k \bar{w} \in R_k$ which lies in $B(w, e^{-s_k})$; hence, $R_k \cap \psi_1(w, s_k) \neq \emptyset$. Thus, Lemma 2.9 shows that $(\Omega, B_1, \mu)$ is $(\log(n), 1)$-Dirichlet with respect to $\mathcal{F}$ for $u_\varnothing = 0$. Hence, (2.25) yields

$$\dim(S_w(c)) \leq \log(n) - \frac{|\log(1 - n^{-c})|}{c},$$

finishing the proof. \hfill \Box

3.3. The geodesic flow in $\mathbb{H}^{n+1}$. Although the following setting is even suitable for proper geodesic CAT(0)-metric spaces, we restrict to the real hyperbolic space $\mathbb{H}^{n+1}$. The reason is that, given a non-elementary and geometrically finite Kleinian group $\Gamma$, there exists a nice measure satisfying the Global Measure Formula (see Theorem 3.9). We start by introducing the setting and a model of Diophantine approximation developed by Hersensky, Paulin and Parkkonen in [16, 17, 30], which allows a dynamical interpretation of badly approximable elements.

In the following, $\mathbb{H}^{n+1}$ denotes the $(n+1)$-dimensional real hyperbolic ball-model. For $o \in \mathbb{H}^{n+1}$, we define the visual metric $d_o : S^n \times S^n \rightarrow [0, \infty]$ at $o$ by $d_o(\xi, \xi) \equiv 0$ and $d_o(\xi, \eta) \equiv e^{-(\xi, \eta)_o}$, for $\xi \neq \eta$, where $(\cdot, \cdot)_o$ denotes the Gromov-product at $o$. Note that if $o = 0$ is the center of the ball $\mathbb{H}^{n+1}$ then the visual distance $d_0$ is bi-Lipschitz equivalent to the angle metric on the unit sphere $S^n$. The boundary $S^n = \partial_\infty \mathbb{H}^{n+1}$ is a compact metric space with respect to $d_0$ and we will consider all metric balls to be with respect to $d_0$ in the following.

Let $\Gamma$ be a discrete subgroup of the isometry group of $\mathbb{H}^{n+1}$. The limit set $\Lambda\Gamma$ of $\Gamma$ is given by the set $\bar{\Gamma}_0 \cap S^n$, which is the set of all accumulation points of subsequences $\Gamma.o$. Recall that a subgroup $\Gamma_0 \subset \Gamma$ is called convex cocompact if $\Lambda\Gamma_0$ contains at least two points and the action of $\Gamma_0$ on the convex hull $\mathcal{C}\Gamma_0$ has compact quotient. We call $\Gamma_0$ bounded parabolic if $\Gamma_0$ is the maximal subgroup of $\Gamma$ stabilizing a
parabolic fixed point \( \xi_0 \in \Lambda \Gamma \) and \( \Gamma_0 \) acts cocompactly on \( \Lambda \Gamma - \{ \xi_0 \} \). Moreover, we call \( \Gamma_0 \) *almost malnormal* if \( \varphi \Delta \Gamma \Lambda \Gamma_0 = \emptyset \) for every \( \varphi \in \Gamma - \Gamma_0 \).

Let \( \Gamma \) be a non-elementary geometrically finite group where we refer to [33] for the following. Recall that for the convex hull \( \Lambda \Gamma \) of \( \Lambda \Gamma \), the subset \( \Lambda \Gamma \cap \mathbb{H}^{n+1} \) is closed, convex and \( \Gamma \)-invariant. The convex core \( \Lambda \mathbb{M} \subset \mathbb{M} = \mathbb{H}^{n+1}/\Gamma \) is the convex closed connected set \( \Lambda \mathbb{M} \equiv (\Lambda \Gamma \cap \mathbb{H}^{n+1})/\Gamma = K \cup \bigcup_i V_i \), which can be decomposed into a compact set \( K \), and, unless \( \Gamma \) is convex cocompact, finitely many open disjoint sets \( V_i \) corresponding to the conjugacy classes of maximal parabolic subgroups of \( \Gamma \) which are bounded parabolic and almost malnormal. Moreover, if \( \pi \) denotes the projection to \( \mathbb{M} = \mathbb{H}^n/\Gamma \) we may assume that each \( V_i = \pi(C_i) \cap \Lambda \mathbb{M} \) is the projection of a horoball \( C_i \), where the collection \( \varphi(C_i), \varphi \in \Gamma - \text{Stab}_\Gamma(C_i) \), is disjoint.

### 3.3.1. The setting

Let \( \Gamma \) be a non-elementary geometrically finite group without elliptic elements as above and \( \Gamma_i \subset \Gamma, i = 1, 2 \), be an almost malnormal subgroup in \( \Gamma \) of infinite index. We treat the following two 'disjoint' cases simultaneously.

1. There is precisely one conjugacy class of a maximal parabolic subgroup \( \Gamma_i \) of \( \Gamma \). Let \( m \) be the rank of \( \Gamma_1 \) and let \( C_1 \) be a horoball based at the parabolic fixed point \( \xi_0 \) of \( \Gamma_1 \) as above.
2. Let \( \Gamma \) be convex-cocompact such that \( \Lambda \Gamma \subset S^n \) is not contained in a finite union of spheres of \( S^n \) of codimension at least 1. Let \( \Gamma_2 \) be a convex-cocompact subgroup and \( C_2 = \Lambda \Gamma \) be the convex hull of \( \Gamma_2 \) which is a hyperbolic subspace (that is, \( C_2 \) is totally geodesic and isometric to the hyperbolic space \( \mathbb{H}^m \)).

**Remark.** The requirements that there is only one parabolic subgroup in *Case 1.* or that \( \Gamma \) itself is convex-cocompact in *Case 2.* will be necessary in the Global Measure Formula. In fact, we need to control the 'depth of geodesic rays in the cuspidal end’ which would not be possible in *Case 2.* if \( \Lambda \mathbb{M} \) was not compact.

Note that, since \( \Gamma_i \) is almost malnormal, we have \( \Gamma_i = \text{Stab}_\Gamma(C_i) \) so that \( \Gamma_i \) is determined by \( C_i \). In addition, \( C_i \) is \( (\varepsilon, T) \)-embedded, that is, for every \( \varepsilon > 0 \) there exists \( T = T(\varepsilon) \geq \varepsilon \) such that for all \( \varphi \in \Gamma - \Gamma_i \), we have that \( \text{diam}(\mathcal{N}(C_i) \cap \varphi(\mathcal{N}(C_i))) \leq T \); see [30]. In the first case, we therefore assume, after shrinking \( C_1 \), that the images \( \varphi(C_1) \), \( [\varphi] \in \Gamma/\Gamma_1 \), form a disjoint collection of horoballs. For the second case, we let \( \varepsilon = \delta_0 \) and \( T_0 = T(2\delta_0) \) where \( \delta_0 \) is the constant such that \( \mathbb{H}^{n+1} \) is a tripod-\( \delta_0 \)-hyperbolic space.

**Example.** Clearly, if \( \mathbb{M} = \mathbb{H}^{n+1}/\Gamma \) is a finite volume hyperbolic manifold with exactly one cusp, then *Case 1.* is satisfied with \( m = n \). If \( \Gamma \) is even cocompact, then every closed geodesic \( \alpha \) in \( \mathbb{M} \) determines a subgroup \( \Gamma_2 \) as in *Case 2.* and \( \mathcal{C}_2 \) (a lift of \( \alpha \)) is one-dimensional. Moreover, \( T \) can be estimated in terms of the length of \( \alpha \) and the length of a systole of \( \mathbb{M} \).

### 3.3.2. A model of Diophantine approximation and the main result

Given \( \Gamma, \Gamma_i, i = 1, 2 \), as above, we fix a base point \( o \in \mathbb{H}^{n+1} \) such that \( \pi(o) \in K \). For technical reasons, we

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6 Note that an isometry \( \varphi \) of \( \mathbb{H}^{n+1} \) extends to a homeomorphism of \( S^n \). We denote the image of a set \( S \subset S^n \) under \( \varphi \) by \( \varphi.S \).

7 With respect to the induced metric on \( C_2 \).
also fix a sufficiently large constant \( t_0 \geq 0 \). For the respective cases, \( i = 1, 2 \), denote the quadruple of data by

\[
\mathcal{D}_i = (\Gamma, C_i, o, t_0).
\]

For \( r = [\varphi] \in \Gamma / \Gamma_i \) we define

\[
D_i(r) = d(o, \varphi C_i)
\]

which does not depend on the choice of the representative \( \varphi \) of \( r \). Note that the set \( \{D_i(r) : r \in \Gamma / \Gamma_i\} \) is discrete and unbounded (see [30, 40]); that is, for every \( D \geq 0 \) there are only finitely many elements \( r \in \Gamma / \Gamma_i \) such that \( D_i(r) \leq D \) and there exists an \( r \in \Gamma / \Gamma_i \) with \( D_i(r) > D \).

Now, for \( i = 1, 2 \) and for \( \xi \in \Lambda \Gamma - \Gamma . \Lambda \Gamma_i \) define the approximation constant

\[
c_i(\xi) = \inf_{r=|[\varphi]|\in\Gamma / \Gamma_i : D(r) > t_0} e^{D_i(r)} d_o(\xi, \varphi . \Lambda \Gamma_i),
\]

If \( c_i(\xi) = 0 \) then \( \xi \) is called well approximable, otherwise it is called badly approximable (with respect to \( \mathcal{D}_i \)). Define the set of badly approximable limit points by

\[
\text{Bad}(\mathcal{D}_i) = \{ \xi \in \Lambda \Gamma - \Gamma . \Lambda \Gamma_i : c_i(\xi) > 0 \} \subset \Lambda \Gamma,
\]

and \( \text{Bad}(\mathcal{D}_1, \epsilon^{-\tau}) \) the subset of elements for which \( c_i(\xi) \geq \epsilon^{-c} \).

**Theorem 3.5.** Let \( \delta \) be the Hausdorff-dimension of \( \Lambda \Gamma \) and \( \tau > 0 \) be the exponent of Theorem 3.10 below. There exists \( c_0 > 0 \) and geometric constants \( k_l, k_l, k_u, k_u, k_u > 0 \), determined in the following, such that for all \( c > c_0 \) we have

\[
\delta - \frac{k_l + \log(1 - \tilde{k}_l \epsilon^{-c(2\delta-m)c/2})}{c/2 - (\delta_0 + \log(2))} \leq \text{dim}(\text{Bad}(\mathcal{D}_1, \epsilon^{-\tau})) \leq (2\delta - m) - \frac{|\log(1 - \tilde{k}_u \epsilon^{-c})| - \tilde{k}_u}{2c + k_u},
\]

as well as

\[
\delta - \frac{k_l + \log(1 - \tilde{k}_l \epsilon^{-c(2\delta-m)c/2})}{c/2 - (T_0 + \delta_0 + 2 \log(3))} \leq \text{dim}(\text{Bad}(\mathcal{D}_2, \epsilon^{-\tau})) \leq \delta - \frac{|\log(1 - \tilde{k}_u \epsilon^{-c})| - \tilde{k}_u}{c + k_u}.
\]

**Remark.** It is well known (see [28]) that \( 2\delta \geq m \). In fact, it follows from the lower and upper bound that \( \delta \geq m \) in our case. Therefore, the upper bound is only suitable for \( c > 0 \) such that the right hand side is smaller than the trivial bound \( \delta \). For the second case, note that if \( C_2 \) is an axis, we can choose \( \tau = \delta \). We moreover expect that \( \tau \) is dependent on the dimension of \( C_2 \) (and of course on \( \delta \)).

In the special case, when \( \Gamma \) is of the first kind, that is \( \Lambda \Gamma = S^n \) (for instance if \( \Gamma \) is a lattice), we can improve the above theorem to the following.

**Theorem 3.6.** Let again \( \tau > 0 \) be the exponent of Theorem 3.10 below. If in addition \( \Lambda \Gamma = S^n \), then there exists \( c_0 > 0 \) and geometric constants \( k_l, k_l, k_u, k_u, k_u > 0 \), such that for all \( c > c_0 \) we have

\[
n - \frac{|\log(1 - \tilde{k}_l \epsilon^{-nc/2})|}{c/2 - k_l} \leq \text{dim}(\text{Bad}(\mathcal{D}_1, \epsilon^{-\tau})) \leq n - \frac{|\log(1 - \tilde{k}_u \epsilon^{-2nc})|}{2c + k_u},
\]

as well as

\[
n - \frac{|\log(1 - \tilde{k}_l \epsilon^{-nc/2})|}{c/2 - k_l} \leq \text{dim}(\text{Bad}(\mathcal{D}_2, \epsilon^{-\tau})) \leq n - \frac{|\log(1 - \tilde{k}_u \epsilon^{-nc})|}{c + k_u}.
\]

---

8 The constants may differ from the ones in the proof.
The above theorem and the following dynamical interpretation of the set $\text{Bad}(\mathcal{D}_t, e^{-c})$ yield the proof of Theorem 1.3. The proof can be found in [40], Lemma 3.16, for the context of CAT(-1)-spaces.

**Lemma 3.7.** There exist positive constants $c_0$, $\kappa_0 > 0$ (we may assume $\kappa_0 \geq 1$) and $t_0 \geq 0$ such that, if $C_1$ is a horoball based at $\partial_{\infty}C = \eta \in \partial_{\infty}\mathbb{H}^n$ or $C_2$ is a hyperbolic subspace with $d(o, C_i) \geq t_0$, then for all $\xi \in \Lambda \Gamma$ and $c > c_0$ we have

1. $\gamma_{o,\xi}([t, t + c]) \subset C_1$,
2. $\gamma_{o,\xi}([t, t + c]) \subset \mathcal{N}_{\delta_0}(C_2),$

for some $t \geq d(o, C_i)$, if and only if

1. $d_o(\xi, \eta) \leq \kappa_0 e^{-c/2} \cdot e^{-d(o, C_1)}$,
2. $d_o(\xi, \partial_{\infty}C_2) \leq \kappa_0 e^{-c} \cdot e^{-d(o, C_2)}$.

**3.3.3. A measure on $\Lambda \Gamma$.** Let $o = 0$ be the center so that the visual distance $d_o$ is bi-Lipschitz equivalent to the angle metric on the unit sphere $S^n$. Hence, if $\Gamma$ is of the first kind, then the Lebesgue measure on $S^n$ satisfies a power law with respect to the visual metric $d_o$ and the exponent $n$. More generally, recall that the critical exponent of a discrete group $\Gamma \subset I(\mathbb{H}^{n+1})$ is given by

$$\delta(\Gamma) \equiv \inf \left\{ s > 0 : \sum_{\varphi \in \Gamma} e^{-sd(x,\varphi(x))} < \infty \right\},$$

for any $x \in \mathbb{H}^{n+1}$. If $\Gamma$ is non-elementary and discrete then the Hausdorff-dimension of the conical limit set of $\Lambda \Gamma$ equals $\delta(\Gamma)$ and if $\Gamma$ is moreover geometrically finite, then $\dim(\Lambda \Gamma) = \delta(\Gamma)$ (see [25]).

Moreover, associated to $\Gamma$, there is a canonical measure, the Patterson-Sullivan measure $\mu_{\Gamma}$, which is a $\delta(\Gamma)$-conformal probability measure supported on $\Lambda \Gamma$. For a precise definition we refer to [23]. There are various results concerning the Patterson-Sullivan measure. Here, we will make use of the following.

Let $\Gamma$ be a geometrically finite Kleinian group as in Cases 1. and 2. above. Let moreover $D_0$ be the diameter of the compact part $K$ of the convex core $CM$ of $M$.

For a limit point $\xi \in \Lambda \Gamma$, we let $\gamma_{o,\xi}$ be the unique geodesic ray starting in $o$ and asymptotic to $\xi$. In Case 1. define the depth $D_t(\xi)$ of the point $\gamma_{o,\xi}(t)$ in the collection of horoballs $\{ \varphi(C_1) \}_{\varphi \in \Gamma}$, where $D_t(\xi) \equiv 0$ if $\gamma_{o,\xi}(t)$ does not belong to $\cup_{\varphi \in \Gamma} \varphi(C_1)$, and $D_t(\xi) \equiv d(\gamma_{o,\xi}(t), \partial \varphi(C_1))$ otherwise; in Case 2. we simply set $D_t(\xi) = 0$ for all $t > 0$.

We need the following Lemma.

**Lemma 3.8.** We have

$$D_t(\xi) \leq d(\gamma_{o,\xi}(t), \Gamma.o) \leq D_t(\xi) + 4 \log(1 + \sqrt{2}) + D_0.$$

**Proof.** By the arguments given below, the proof is obvious if $\Gamma$ is convex-cocompact (and hence the set $V$ is empty) and we may assume that we are given Case 1. Recall that the convex core $CM = (\Gamma T \cap \mathbb{H}^{n+1})/\Gamma$ consists of (the disjoint union of) the compact set $K$ and the set $V$ which we may assume to be the projection of $C_1 \cap \Gamma{T}$. Since $\Gamma{T}$ is convex and $o \in \Gamma{T}$, for every limit point $\xi \in \Gamma{T}$ the ray $\gamma_{o,\xi}(\mathbb{R}^+) \subset \Gamma{T}$ is contained in $\Gamma{T}$ and hence covered by lifts of $K$ and of $V$. Since $\pi(o) \in K$, if $\gamma_{o,\xi}(t) \in \Gamma{T} - \cup \varphi(C_1)$ for some $t > 0$, then $d(\gamma_{o,\xi}(t), \Gamma.o) \leq D_0$.

Hence, fix $t > 0$ such that $\gamma_{o,\xi}(t) \in \varphi(C_1) \equiv C$ for some $\varphi \in \Gamma$, where we let $\eta \equiv \varphi(\xi_0)$. If we let $t_0$ be the entering time of $\gamma_{o,\xi}$ in $C$, that is, $\gamma_{o,\xi}(t_0) \in \partial C$, then clearly
by the above remark and since \( \gamma_{o,\xi}(t_0) \) belongs to some lift of \( K \), we have

\[
d(\gamma_{o,\xi}(t), \Gamma, o) \leq d(\gamma_{o,\xi}(t), \gamma_{o,\xi}(t_0)) + D_0 \equiv \bar{d} + D_0.
\]

Moreover, let \( \hat{C} \) be the horball based at \( \eta \) (and contained in \( C \)) such that \( \gamma_{o,\xi}(t) \in \partial \hat{C} \) and note that \( \gamma_{o,\eta}(d(o, C) + \bar{d}(\xi)) \in \partial \hat{C} \). It then follows from [29], Lemma 2.9, that both \( d(\gamma_{o,\xi}(t_0), \gamma_{o,\eta}(d(o, C))) \) and \( d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(d(o, C) + \bar{d}(\xi))) \) are bounded above by the constant \( 2 \log(1 + \sqrt{2}) \). This shows

\[
\bar{d} = d(\gamma_{o,\xi}(t), \gamma_{o,\xi}(t_0)) \\
\leq d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(d(o, C) + \bar{d}(\xi))) + d(\gamma_{o,\eta}(d(o, C) + \bar{d}(\xi)), \gamma_{o,\xi}(t_0)) \\
\leq 2 \log(1 + \sqrt{2}) + (\bar{d}(\xi) + d(\gamma_{o,\eta}(d(o, C)), \gamma_{o,\xi}(t_0))) \\
\leq \bar{d}(\xi) + 4 \log(1 + \sqrt{2}).
\]

Finally, since \( o \notin C \) (used in the first inequality) we have

\[
D_1(\xi) \leq d(\gamma_{o,\xi}(t), \Gamma, o) \\
\leq \bar{d} + D_0 \\
\leq D_1(\xi) + 4 \log(1 + \sqrt{2}) + D_0,
\]

proving the claim.

In the following, let \( \mu = \mu_o \) be the Patterson-Sullivan measure given at the base point \( o \). By the above lemma, we can reformulate the Global Measure Formula due to [36], Theorem 2, to the following.

**Theorem 3.9.** There exist positive constants \( c_1, c_2 > 0 \) and \( t_0 > 0 \) such that for all \( \xi \in \Lambda \Gamma \) and for all \( t > t_0 \), we have that

\[
c_1 e^{-\delta t} \cdot e^{-(\delta - m)D_1(\xi)} \leq \mu(B_{d_0}(\xi, e^{-t})) \leq c_2 e^{-\delta t} \cdot e^{-(\delta - m)D_1(\xi)}.
\]

In particular, if \( \Gamma \) is convex-cocompact, then \( \mu \) satisfies a power law with respect to \( \delta \).

For the second case, let again \( o = 0 \) and note that, since \( \Gamma_2 \) is almost malnormal in \( \Gamma \), \( C_2 \) can be of dimension at most \( m \leq n \). Moreover, since \( C_2 \) is an \( m \)-dimensional hyperbolic subspace, the boundary \( \partial_{\infty}C_2 = \Lambda \Gamma_2 \subset \Lambda \Gamma \) of \( C_2 \) is an \( \Lambda_2 \)-dimensional sphere (with respect to \( d_0 \)). Hence, every image \( \varphi \Lambda \Gamma_2, \varphi \in \Gamma \), is contained in the set \( \mathcal{H}(\Gamma) \equiv \{ S \cap \Lambda \Gamma : S \text{ is a sphere in } S^n \text{ of codimension at least } 1 \} \). A finite Borel measure \( \mu \) on \( S^n \) is called \( \mathcal{H}(\Gamma) \)-friendly, if \( \mu \) is Federer and \( (\Lambda \Gamma \times (t_0, \infty), B_1, \mu) \) is absolutely \( (\tau, c_r) \)-decaying with respect to \( \mathcal{H}(\Gamma) \).

**Theorem 3.10 ([57], Theorem 2).** For every non-elementary convex-cocompact discrete group \( \Gamma \subset I(\mathbb{H}^{n+1}) \) (without elliptic elements), such that \( \Lambda \Gamma \) is not contained in a finite union of elements of \( \mathcal{H}(\Gamma) \), the Patterson-Sullivan measure \( \mu \) at \( o \) is \( \mathcal{H}(\Gamma) \)-friendly.

Note that if we consider only \( 0 \)-dimensional spheres, we can clearly choose \( \tau = \delta \).
3.3.4. The resonant sets. Let \( \vec{\Omega} = \Omega = \Lambda \Gamma \times (t_0, \infty) \), where \( t_0 \) is sufficiently large as in Theorem 3.9 and Theorem 3.10 above (as well as Lemma 3.7 and 3.15 below). We are given the discrete set of sizes \( \{ D_i([\varphi]) : [\varphi] \in \Gamma / \Gamma_i, D_i([\varphi]) > t_0 \} \) which we relabel to \( \{ s_m^i \}_{m \in \mathbb{N}} \subset \mathbb{R}^+ \) and reorder such that \( s_m^i \leq s_k^i \) for \( m \leq k \). For \( m \in \Lambda_i \equiv \mathbb{N} \) let

\[
R_m^i \equiv \left\{ \xi \in \varphi : \Gamma_i \setminus \{ [\varphi] \} \in \Gamma / \Gamma_i \text{ such that } t_0 < D_i([\varphi]) \leq s_m^i \right\} = \left\{ \xi \in \varphi : \Gamma_i \setminus \{ [\varphi] \} \in \Gamma / \Gamma_i \text{ such that } e^{-t_0} > e^{-D_i([\varphi])} \geq e^{-s_m^i} \right\}.
\]

Since \( \Gamma \) is discrete, for every metric ball \( B = B(\xi, e^{-t}) \), \( (\xi, t) \in \Omega \), only finitely many sets \( \varphi : \Gamma_i \setminus \{ [\varphi] \} \in \Gamma / \Gamma_i \) can intersect \( B \) and it is readily checked that \( (\Omega, B_1) \) is \( d^* \)-contracting with respect to \( \mathcal{F}_i \) where \( d_1 = \log(2) \). Moreover, since \( \Lambda \Gamma \) is compact, \( (\Omega, B_1) \) is \( \log(3) \)-separating. Also, \( d^c \leq \log(2) \) for all \( c > 0 \) and \( d^c \leq \log(2) \) for all \( c \geq \log(2) \).

For \( \mathcal{F}_i \equiv (\mathbb{N}, R_m^i, s_m^i) \), since \( \Lambda \Gamma \subset \mathbb{S}^n \) is closed (hence compact), we remark that

\[
\text{Bad}(D_1, e^{-c}) = \text{Bad}^{\Omega_1}(\mathcal{F}_i, c).
\]

3.3.5. The lower bound. For the lower bound, note that the following is shown in the author’s earlier work [40]. Subsection 3.6.5, using that \( C_i \) is \( (2\delta_0, T_0) \)-embedded: For two different cosets \( [\overline{\varphi}], [\varphi] \in \Gamma / \Gamma_i \), let \( \eta \in \varphi : \Gamma_i \), and \( \bar{\eta} \in \overline{\varphi : \Gamma_i} \). Then

\[
d_o(\eta, \bar{\eta}) \geq e^{-c_1} e^{-\max(D_i([\varphi]), D_i([\overline{\varphi}]))}, \tag{3.1}
\]

where

\[
c_1 \equiv \delta_0, \quad c_2 \equiv T(2\delta_0) + 2\delta_0,
\]

and \( \delta_0 \) is the hyperbolicity constant of \( \mathbb{H}^{n+1} \) (and \( i \) stands for the respective case).

For **Case 2**, we obtain that, for \( l_1 = c_2 + \log(3) \), for any formal ball \( (\xi, t) \in \Omega \) we have

\[
B(\xi, e^{-(t+\bar{\tau}_c)}) \cap R(t) = B(\xi, e^{-(t+\bar{\tau}_c)}) \cap S,
\]

where \( S \) is either empty or \( S = \varphi : \Gamma_2 \in \mathbb{S} \) for some \( [\varphi] \in \Gamma / \Gamma_2 \). Thus, (2.31) is satisfied with \( n_1 = 1 \). Proposition 2.7 and Theorem 3.10 show that \( (\Omega, B_1, \mu) \) is \( \tau_c \)-decaying with respect to \( \mathcal{F}_2 \), where \( \tau_c = c_2 e^{-\tau c - 2 \log(3)} \), for all \( c \leq 2 \log(3) \) and the parameters \( (c, l_c) \), \( l_c = T_0 + \delta_0 + 2 \log(3) \). We let \( c_0 \geq 2 \log(3) \) such that for all \( c \geq c_0 \) we have \( \tau_c < 1 \). Recall that \( (\Omega, B_1, \mu) \) satisfies a power law with respect to the parameters \( (\delta, c_1, c_2) \). Thus, remarking that \( \delta = 1 \) and using (2.10), (2.18) establishes the lower bound

\[
\dim(\text{Bad}(\mathcal{D}_2, e^{-(2c+L_c)})) \geq \delta \geq \frac{\log(2k_c) e^{-1} - \log(1 - \tau_c)}{c} \geq \delta \geq \frac{\log(2) + 2\delta log(\frac{c_2}{c_1}) + \log(3) + \log(1 - c_2 e^{2\tau \log(3)} \cdot e^{-\tau c})}{c}.
\]

**For Case 1**, we have that (2.33) is satisfied for \( l_1 = \delta_0 + \log(2) \) by (3.1). Using the Global Measure Formula, we can determine the required constants.

**Proposition 3.11.** For the parameters \( c, l_c = \delta_0 + \log(2) \) and \( d_c \leq \log(2) \) we have

\[
k_c \geq \frac{e^{2\delta_0} e^{-(2\delta-m)c}}{c_2} \equiv \overline{c}_1 e^{-(2\delta-m)c},
\]

\[
\tilde{k_c} \leq \frac{e^{2\delta (d_c + d_0)} e^{-m(d_c + d_0) c}}{c_1} \equiv \overline{c}_2 e^{-mc} \leq \overline{c}_2 e^{(2\delta-m)c}
\]

\[
\tau_c \leq \frac{e^{2\delta (d_c + d_0 + \delta_0)} e^{-m(d_0 + d_c + d_0) c}}{c_1} \equiv \overline{c}_3 e^{-(2\delta-m)c},
\]

in (2.12) and (2.13).
Proof. For any $\eta \in B(\xi, e^{-t})$ with $t$ sufficiently large, since $e^{-(\xi, \eta)_{\om}} = d_{\om}(\xi, \eta) \leq e^{-t}$ and $\mathbb{H}^{n+1}$ is a $\delta_0$-tripod-hyperbolic space, we have $d(\gamma_{0, \xi}(t), \gamma_{0, \eta}(t)) < \delta_0$. Hence $|D_{t}(\xi) - D_{t}(\eta)| \leq \delta_0$. Moreover, we have $|D_{h}(\eta) - D_{s}(\eta)| \leq |h - s|$ for all $h, s$. This shows that for $\eta \in B(\xi, e^{-t})$ and $s, t \geq 0$,

$$|D_{t+s}(\xi) - D_{t+h}(\eta)| \leq \delta_0 + s + h. \quad (3.2)$$

Recall that $t_k = s_k + k c + l_k$ and let $(\xi, t_k) \in \Omega$ be a given a formal ball. From the above (3.1), we know that $B(\xi, e^{-t_k}) \cap R(t_k - l_c)$ contains at most one point, say $\eta = \phi, \Lambda_{\Gamma}$. By (3.2), $D_{t_k + d_c}(\xi)$ and $D_{t_k + c - d_c}(\eta)$ can differ by at most $c + \delta_0 + d_c - d_\ast$. Moreover, since $D_1(\varphi) \leq t_k - l_c$, we have for the depth of $\eta$ that

$$D_{t_k + c - d_\ast}(\eta) = t_k + c - d_\ast - D(\varphi) \geq c + l_c - d_\ast.$$

Assuming that $c + l_c \geq c + \delta_0 + d_c$ (which is the case for $c \geq \log(2)$), we have $D_{t_k + c - d_\ast}(\eta) \geq D_{t_k + d_c}(\xi) + c + \delta_0 + d_c - d_\ast$. Using the Global Measure Formula, we obtain

$$\mu(B(\xi, e^{-(t_k + d_c)})) \geq c_1 e^{-\delta(t_k + d_c)} \cdot e^{-(\delta - \mu)D_{t_k + d_c}(\xi)} \geq c_2 e^{-\delta(t_k + c - d_\ast)} \cdot c_2^c e^{\delta(c - d_c - d_\ast)} e^{-(\delta - \mu)(D_{t_k + c - d_\ast}(\eta) - (c + \delta_0 + d_c - d_\ast))} \geq c_2 e^{-\delta(t_k + c - d_\ast)} \cdot c_2 e^{2\delta(c - d_c - d_\ast) - m(c + \delta_0 + d_c - d_\ast)} \geq \mu(B(\eta, e^{-(t_k + c - d_\ast)})) \cdot c_2 e^{-2\delta(d_c + d_\ast) + m(d_c + d_\ast) e^{(2\delta - m)c}} \geq \mu(B(\xi, e^{-(t_k + d_c)})) \cdot k_c^{-1}.$$

As above, using (3.2) for $\eta \in B(\xi, e^{-t_k})$ and the Global Measure Formula, we obtain

$$\mu(B(\xi, e^{-(t_k + 1)})) \geq c_1 e^{-\delta t_k + 1} \cdot e^{-(\delta - \mu)D_{t_k + d_c}(\xi)} \geq \mu(B(\xi, e^{-t_k})) \cdot c_2 e^{-\delta c e^{-(\delta - \mu)(c + \delta_0)} \equiv \mu(B(\xi, e^{-t_k})) \cdot k_c,$$

as well as

$$\mu(B(\xi, e^{-(t_k + d_c)})) \geq c_1 e^{-\delta(t_k + d_c)} \cdot e^{-(\delta - \mu)D_{t_k + d_c}(\xi)} \geq \mu(B(\eta, e^{-(t_k + 1 - d_\ast)})) \cdot c_2 e^{\delta(c - d_c - d_\ast) \cdot c_2 e^{-(\delta - \mu)(c + \delta_0 + d_c - d_\ast)} \geq \mu(B(\eta, e^{-(t_k + 1 - d_\ast)})) \cdot c_2 e^{-2\delta(d_c + d_\ast + \delta_0) + m(d_c + d_\ast) e^{m c}} \equiv \mu(B(\eta, e^{-(t_k + 1 - d_\ast)})) \cdot k_c^{-1}.$$

This finishes the proof. \hfill \square

Assuming that $c > c_0$, where $c_0$ is as in Lemma [3.7] and such that $\tau_{c_0} < 1$, the following Lemma will finish determining the parameters for the lower bound.

Lemma 3.12. For any $\xi \in \text{Bad}(\mathcal{F}, 2c + l_c)$ we have $d_\mu(\xi) \geq \delta$.

Proof. If $\xi \in \text{Bad}(\mathcal{F}, 2c + l_c)$, then $d_\mu(\xi, \phi, \varphi, \Lambda) > e^{-D_1((\varphi) + 2c + l_c)}$ for every $[\varphi] \in \Gamma / \Gamma_1$ with $D_1([\varphi]) > t_0$. Hence, Lemma [3.7] states that the length of $\gamma_{0, \xi}(\mathbb{R}^+ \cap \varphi(C_1)$ is bounded by $2(2c + l_c + 2 \log(\kappa_{\varphi}))$ for every $[\varphi] \in \Gamma / \Gamma_1$. In particular, the distance from $\gamma_{0, \xi}(t) \in \partial \varphi(C_1)$ is less than $2c + l_c + 2 \log(\kappa_{\varphi})$ for all $t > t_0$ and we see that $0 \leq D_{\mu}(\xi) \leq 2c + l_c + 2 \log(\kappa_{\varphi})$. The Global Measure Formula yields that $c_1 e^{-\delta t} C_{\epsilon} \leq \mu(B(\xi, e^{-t})) \leq c_2 e^{-\delta t} C_{\epsilon}$ for all $t > t_0$, for some $C = C(c) > 0$. In particular, $d_\mu(\xi) \geq \delta$. \hfill \square
Finally, using Proposition 3.11 (2.18) gives the lower bound
\[
\dim(\text{Bad}(D_1, e^{-(2c+\ell_c)})) \geq \delta - \frac{\log(2k_c k_c^{-1}) - \log(1 - \tau_c)}{c} \\
\geq \delta - \frac{\log(2\overline{c}c_i^{-1}) + |\log(1 - \overline{c}_3 e^{-(2\delta - m)c})|}{c}.
\]

The Standard Case. Let $\Lambda \Gamma = S^n$. Note that for any formal ball $(\xi, t_0)$, $\xi \in S^n$ we can take an isometry from the hyperbolic ball to the upper half space model (again denoted by $\mathbb{H}^{n+1}$) which maps $o$ to $(0, \ldots, 0, 1) \in \mathbb{H}^{n+1}$ and $\xi$ to $0 \in \mathbb{R}^n \subset \partial_{\infty} \mathbb{H}^{n+1}$. If $t_0 > 0$ is sufficiently large then $B(0, e^{-t_0})$ (with respect to the visual distance) is contained in the Euclidean unit ball $B \subset \mathbb{R}^n$ and we remark that the visual metric $d_o$ restricted to $B$ is bi-Lipschitz equivalent to the Euclidean metric on $B$; let $c_B \geq 1$ be the bi-Lipschitz constant.

We let $c = \log(m) > c_0$ for some $m \in \mathbb{N}$ sufficiently large (such that $\overline{c}_c < 1$ below). Up to modifying $\ell_c$ and $\tau_c$ to $\ell_c^i = \ell_c + \log(c_B)$ and $\tau_c^i = e_B^\alpha \tau_c$ respectively, we may use the same arguments as above and assume for any point $\xi \in B$ that (2.13) is satisfied with respect to the Lebesgue measure and the function $B_1$ (which is with respect to the Euclidean metric). Note also that we nowhere used the condition that $\xi \in L_k B^1(\ell)$ so that the condition becomes obsolete in this setting. Hence, Lemma 2.5 shows that (2.20) is satisfied for the parameters $\ell_c^i = \ell_c + a$, with $a \equiv 2\sqrt{n} + \log(2) + d_{\ast}$, and $\tau_c^i \leq e^{n(a - d_{\ast})} \tau_c$, where $i$ stands for the respective cases. Recalling that $\overline{c}_1 = \overline{c}_1 e^{-nc}$ and $\overline{c}_2 = \overline{c}_2 e^{-rc}$, (2.21) yields the lower bound
\[
\dim(\text{Bad}(D_i, e^{-(2c+\bar{l}_i)})) \geq \dim(\text{Bad} B^1(\mathcal{F}_i, 2c + \bar{l}_i) \cap B) \\
\geq \dim(\text{Bad} Q^1(\mathcal{F}_i, 2c + \bar{l}_i) \cap B) \geq n - \frac{|\log(1 - \overline{c}_c)|}{c}.
\]
Again, up to modifying $\overline{c}_c$ to $\overline{c}_c^i \equiv \overline{k}_1^i e^{-nc}$ and $\overline{c}_c^1 \equiv \overline{k}_1^i e^{-rc}$ for suitable constants $\overline{k}_1^i > 0$ (depending only on $c_0$), gives the result for sufficiently large general $c \geq c_0$.

3.3.6. The upper bound. We again distinguish between the cases and start with Case 2, by showing a Dirichlet-type Lemma. Recall that $D_0$ denotes the diameter of the compact set $K$ covering the convex core $\mathcal{C}M$.

Lemma 3.13. There exists a constant $\kappa_1 \geq 0$ such that for all $\xi \in \Lambda \Gamma$ and $t > t_0$, where $t_0 > 2D_0$, there exists $[\varphi] \in \Gamma/\Gamma_2$ with $D_2([\varphi]) \leq t$ such that
\[
d_o(\xi, \varphi.\Lambda_2) < e^{2D_0 + \kappa_1} e^{-t}.
\]
Proof. Let $\hat{K}$ be a lift of $K$ such that $o \in \hat{K}$. The geodesic ray $\gamma_{o,\xi}$ is contained in $C \Gamma$, which is covered by images $\varphi(\hat{K})$, $\varphi \in \Gamma$. Hence, let $\varphi \in \Gamma$ such that $\gamma_{o,\xi}(t - D_0) \in \varphi(\hat{K})$. Since $C_2 \subset C \Gamma$, some image of $C_2$ under $\Gamma$, say $C_2$ itself, intersects $K$. Thus, $\varphi(C_2)$ intersects $\varphi(\hat{K})$, and we see that
\[
D_2([\varphi]) = d(o, \varphi(C_2)) \leq d(o, \gamma_{o,\xi}(t - D_0)) + d(\gamma_{o,\xi}(t - D_0), \varphi(C_2)) \leq t.
\]
Moreover, there exists a geodesic line $\alpha$ contained in $\varphi(C_2)$ at distance at most $D_0$ to $\gamma_{o,\xi}(t - D_0)$. Let $H$ be the hyperbolic half-space such that $\gamma_{o,\xi}(t - 2D_0) \in \partial H$, $H$ orthogonal to $\gamma_{o,\xi}$ and $\xi \in \partial_{\infty} H$. Hence, one of the endpoints of $\alpha$ (which belongs to $\varphi.\Lambda_2$) must lie in the boundary $\partial_{\infty} H$ of $H$. Remarking that $\partial_{\infty} H$ is a subset of $B(\xi, e^{-(d(o,H) - \kappa_1)})$ for some universal constant $\kappa_1 > 0$, yields the claim. □
Setting \( u_* = 2D_0 + \kappa_1 \), we see that \( \mathcal{F} \) locally contains \( S \), which denotes the set of points of \( \Lambda \Gamma \). Moreover, since \( \Gamma \) is convex-cocompact, \((\Omega, B_1, \mu)\) satisfies a power law with respect to the parameters \((\delta, c_1, c_2)\). Proposition 2.9 shows that \((\Omega, B_1, \mu)\) is \( \tau^c \)-Dirichlet with respect to \( \mathcal{F} \) and the parameters \((c, u_*)\), where \( \tau^c = \frac{c_1}{c_2} e^{-\delta(d_* + d^* + \delta_0) - \delta} \).

Using \((2.10), (2.25)\) implies the upper bound
\[
\dim(\text{Bad}(D_2, e^{-c})) \leq \frac{\log(1 - \frac{c_1}{c_2} e^{-\delta(d_* + d^* + u_*) - \delta c}) - \log(\frac{\Lambda \Gamma}{c + u_*})}{c + u_*} + \frac{\log(\Lambda \Gamma)}{c + u_*}.
\]

We are left with Case 1. We start again with the following Dirichlet-type Lemma that follows from \([36] \), Theorem 1, which we reformulated in a version best suitable for us.

Lemma 3.14. There exists a \( t_0 \geq 0 \) and a constant \( \kappa_1 > 0 \) (we may assume \( \kappa_1 \geq 1 \)) such that for any \( \xi \in \Lambda \Gamma \), for any \( t > t_0 \) there exists \( [\varphi] \in \Gamma / \Gamma_1 \) with \( D_1([\varphi]) \leq t \), such that
\[
d_o(\varphi, \Lambda \Gamma_1) \leq \kappa_1 e^{-t/2} e^{-D_1([\varphi])/2}.
\]

Fix \( c > 0 \) and let \( u^c \equiv c + 2\log(\kappa_1) \). Recall that \( t_k = s_1^k + (k - 1)(c + u^c) \) and \( \ell_k = t_k - u^c \). We need the following refinement of the above lemma.

Lemma 3.15. For \( \xi \in \Lambda \Gamma \) with \( \xi \in U_{k-1}(c) \) and \( t_k > t_0 \), there exists \( [\varphi] \in \Gamma / \Gamma_1 \) with \( t_{k-1} < D_1([\varphi]) \leq t_k \) such that \( \varphi, \Lambda \Gamma_1 \in B(\xi, e^{-\ell_k}) \).

Proof. Let \( \xi \in \Lambda \Gamma \) and \( t_k > t_0 \). There exists \( [\varphi] \in \Gamma / \Gamma_1 \) with \( D([\varphi]) \leq t_k \) such that \((3.3)\) is satisfied. If \( D_1([\varphi]) \leq t_{k-1} = t_k - (c + u^c) \), then
\[
d_o(\varphi, \Lambda \Gamma_1) \leq \kappa_1 e^{-\ell_k/2} e^{-D_1([\varphi])/2} \leq \kappa_1 e^{-D_1([\varphi])+1/2(c+u^c)} \leq e^{-(D_1([\varphi])+1/2(c+u^c+2\log(\kappa_1)))} \leq e^{-(D_1([\varphi])+c)}.
\]

Thus, we see that
\[
\xi \in B(\varphi, \Lambda \Gamma_1, e^{-(D_1([\varphi])+c)}) \subset \bigcup_{s_n \leq \ell_{k-1}} \psi_n(R_n, s_n + c) = U_{k-1}(c),
\]
and we may assume that \( t_{k-1} < D_1([\varphi]) \leq t_k \). In this case, we have
\[
d_o(\varphi, \Lambda \Gamma_1) \leq \kappa_1 e^{-\ell_k/2} e^{-D_1([\varphi])/2} \leq \kappa_1 e^{-t_k-1/2(c+u^c)} = e^{-(t_k-1/2(c+u^c+2\log(\kappa_1)))} \leq e^{-(t_k-u^c)} = e^{-\ell_k}
\]
and hence, \( \varphi, \Lambda \Gamma_1 \in B(\xi, e^{-\ell_k}) \) which finishes the proof.

Combining the Global Measure Formula and the above lemma yields the parameters.

Proposition 3.16. For the parameters \( c, u^c \equiv c + 2\log(\kappa_1) \) and \( d^c \equiv \log(2) \) (independent of \( c \)) we have
\[
k^c \geq \frac{c_1}{c_2} e^{-(\delta-m)(2d_*+2d^*+\delta_0)} e^{-(2\delta-m)(c+u^c)} \equiv \tilde{c}_1 e^{-(2\delta-m)(c+u^c)}
\]
\[
\tau^c \geq \frac{c_1}{c_2} e^{-\delta(2c+u_*+2d_*+d^*)+m(c+d_*)} \equiv \tilde{c}_2 e^{-(3\delta-m)c}
\]
in \((2.22)\) and \((2.23)\).
Proof. Let \((\xi, \tilde{t}_k - d^c) \in \Omega\) be a given formal ball and \(\eta \in B(\xi, e^{-\tilde{t}_k}) \subset B(\xi, e^{-(\tilde{t}_k-d^c)})\). Using (3.2) we obtain
\[
D_{t_{k+1}+d_*}(\eta) \leq D_{t_k-d^c}(\xi) + c + u^c + d_* + d^c + \delta_0.
\]
The Global Measure Formula shows that
\[
\mu(B(\eta, e^{-(t_{k+1}+d_*)})) \geq c_1 e^{-\delta(t_{k+1}+d_*)} \cdot e^{-(\delta-m)D_{t_{k+1}+d_*}(\eta)} \geq \mu(B(\xi, e^{-(t_k-d^c)})) \cdot c_2 e^{-(\delta-m)(c+u^c+d_*+d^c+\delta_0)} \geq \mu(B(\xi, e^{-(t_k-d^c)})) \cdot e^{-(2\delta-m)(c+u^c)} \geq \mu(B(\xi, e^{-(t_k-d^c)})) \cdot k^c.
\]
Similarly, let \((\xi, \tilde{t}_k - d^c) \in \Omega\) be a given formal ball such that \(\xi \in U_{k-1}(c)\). By Lemma 3.15 there exists \([\varphi] \in \Gamma/\Gamma_1\) with \(t_{k-1} < D_1([\varphi]) \leq t_k\) and \(\eta \in B(\xi, e^{-\tilde{t}_k})\), where \(\eta \equiv \varphi.\Lambda \Gamma_1\). Moreover, since
\[
D_1([\varphi]) + c + d_* > t_{k-1} + c + d_* \geq \tilde{t}_k + d_*,
\]
the ball \(B(\eta, e^{-(D_1([\varphi])+c+d_*)}) \subset B(\eta, e^{-(\tilde{t}_k+d_*)})\) which in turn is contained in \(B(\xi, e^{-(\tilde{t}_k-d^c)})\) (since \(d^c = \log(2)\)). Finally, we have \(D_{t_k}([\varphi])+d_*(\eta) = c + d_*\) and \(D_1([\varphi]) \leq t_k = \tilde{t}_k + u^c\), the Global Measure Formula shows
\[
\mu(B(\xi, e^{-(t_k-d^c)}) \cap B(\eta, e^{-(D_1([\varphi])+c+d_*)})) \geq \mu(B(\eta, e^{-(D_1([\varphi])+c+d_*)})) \geq c_1 e^{-\delta(D_1([\varphi])+c+d_*)} \geq c_2 e^{-\delta(d^c-d^c)} \geq \mu(B(\xi, e^{-(t_k-d^c)})) \cdot e^{-\delta(2c+2d_*)+m(c+d_*)} \equiv \tau^c \mu(B(\xi, e^{-(t_k-d^c)})).
\]
This finishes the proof. □

Using Proposition 3.16, 2.25 gives the upper bound
\[
\dim(Bad(D_1, e^{-c})) \leq \frac{-\log(k^c) + \log(1 - \tau^c)}{c + u^c} (3.4)
\]
\[
\leq \frac{-\log(c) + (2\delta - m)(c + u^c) + \log(1 - \bar{c}^c)}{2c + 2\log(k_1)}.
\]

The Standard Case. Let again \(\Lambda \Gamma = S^n\) and, for \(c > 0\), let \(\tilde{c} = c + a\), where \(a = d^c + \tilde{d}_* + \log(2) + \sqrt{n}\) and \(\tilde{u}^c \geq u^c + a\) such that \(c + \tilde{u}^c = \log(m_i)\) (with \(m_i\) minimal). Let \(k_0 = 1\) and note that \(t_{k_0} \geq t_0\). Moreover, let \(\xi_j \in S^n\) be finitely many points such that \(B_j = B(\xi_j, \tilde{t}_j)\) cover \(S^n\). As for the lower bound, for each \(\xi_j\) we can take again an isometry to the upper half space model which maps \(o \in (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}\) and \(\xi \in 0 \in \mathbb{R}^n \subset \partial_\infty \mathbb{H}^{n+1}\) as well as \(B_j\) to a subset contained in the Euclidean unit ball \(B\). Up to modifying \(a\) to \(\tilde{a} = a + \log(c_B)\), we may even assume that the cube \(Q = Q_1(0, \tilde{t}_j) \supset B_j\) is contained in \(B\).

Recall that from Lemma 2.6 \(U_{k-1}^{Q_1} c + \tilde{d}_* \subset \bigcap_{s_n \leq \tilde{t}_k - \tilde{c} + d^c} B_1(R_n, s_n + \tilde{c})^{C}\). Moreover, we remark that in the above arguments for determining \(\tau^c\), it was in fact nowhere necessary to require \(t = t_k\) and we showed that, if \((\xi, \tilde{t}_k) \in \Omega\) with \(\xi \in U_{k-1}^{Q_1} c + \tilde{d}_*, k \geq k_0\) and \(\xi \in B\), then (2.29) is satisfied with respect to the Lebesgue measure and the function \(\psi_1\) (with respect to the visual metric \(d_o\)). Again, up to adding the constant \(\log(c_B)\) to \(\tilde{u}^c\)
as well as $\tau_k^* = c^{-n} \tau_k^i$, we see that $(\xi, t_k)$ satisfies (2.29) with respect to the Lebesgue measure and the function $B_1$ (with respect to the Euclidean metric). Hence, Lemma 2.6 implies (2.27) for all $k \geq k_0$ with respect to $\tau_k^* \geq c_B e^{-n \sqrt{\tau_k^*}} \equiv k_u^i e^{-n \tau_k^*}$. Finally, since $m_i$ above was chosen minimal there exists a constant $k_u^i \geq 0$ (independent on $c$) such that $\tilde{c} + u_i^c \leq n c + k_u^i$ with $n_1 = 2$ and $n_2 = 1$. Thus, (2.28) and (the remark after (2.28)) show

$$\dim(\text{Bad}(D_1, e^{-c+\sqrt{\tau}}) \cap B_j) \leq \dim(\text{Bad}_{\mathbb{R}^n}(F, c) \cap Q) \leq n - \frac{|\{ 1 \leq i \leq k \} + \log(1 - k_u^i e^{-n \tau_k^*})|}{n_i c + k_u^i}.$$  

3.4. Toral Endomorphisms. For the motivation of the following result, we refer to Bredertick, Fishman, Kleinbock [7] and references therein. For $n \in \mathbb{N}$, let $\mathcal{M} = (M_k)$ be a sequence of real matrices $M_k \in GL(n, \mathbb{R})$, with $t_k = ||M_k||_{op}$ (the operator norm), and $\mathcal{Z} = (Z_k)$ be a sequence of $\tau_k$-separated subsets of $\mathbb{R}^n$. Define

$$E_{\mathcal{M}, \mathcal{Z}} \equiv \{ x \in \mathbb{R}^n : \exists c = c(x) > 0 \text{ such that } d(M_k x, Z_k) \geq c \cdot \tau_k \text{ for all } k \in \mathbb{N} \},$$

where $d$ is the Euclidean distance. For $c > 0$, let $E_{\mathcal{M}, \mathcal{Z}}(c)$ be the elements $x \in E_{\mathcal{M}, \mathcal{Z}}$ with $c(x) \geq c$. We assume that, independently of $t \in \mathbb{R}^+$, for all $c > 0$ we have

$$|\{ k \in \mathbb{N} : \log(t_k/\tau_k) \in (t - c, t) \}| \leq f(c), \quad (3.5)$$

for some function $f : \mathbb{R}^+ \to \mathbb{R}^+$. The sequence $\mathcal{M}$ is lacunary, if $\inf_{k \in \mathbb{N}} \frac{t_{k+1}}{t_k} \equiv \lambda > 1$, and the sequence $\mathcal{Z}$ is uniformly discrete, if there exists $\tau_0 > 0$ such that every set $Z_k$ is $\tau_0$-separated. Note that if $\mathcal{M}$ is lacunary and $\mathcal{Z}$ is uniformly discrete, then (3.5) holds and $f$ is in fact bounded by $f(c) \leq c/\log(\lambda)$.

Let again $\mathcal{S}$ denote the set of affine hyperplanes in $\mathbb{R}^n$ and recall that the Lebesgue measure is absolutely $(1, c_0)$-decaying with respect to $\mathcal{S}$ and the function $\psi = B_1$. Using similar arguments for the proof as [7, 40], we want to show the following lower bounds.

Theorem 3.17. Let $X \subset \mathbb{R}^n$ be the support of an absolutely $(\tau, c_e)$-decaying measure $\mu$ (with respect to $\mathcal{S}$ and $\psi = B_1$) which also satisfies a power law with respect to the exponent $\delta$. Let $\mathcal{M}$ and $\mathcal{Z}$ be as above satisfying (3.5) with $f(c) \leq e^{\gamma c}$, where $0 < \tilde{\tau} < \tau$. Then, there exists $c_0 > 0$ such that for all $c > c_0$ we have

$$\dim(E_{\mathcal{M}, \mathcal{Z}}(e^{-(2c+\log(12))}) \cap X) \geq \frac{\log(2 e^2) + \log(2) + d_e + \log(1 - c_2^2 f(c)e^{-\tau c})}{c}.$$  

If $\mu$ denotes the Lebesgue measure, then there exist constants $k_l, \tilde{k}_l > 0$ and $c_0 > 0$, such that for all $c > c_0$ we have

$$\dim(E_{\mathcal{M}, \mathcal{Z}}(e^{-c})) \geq n - \frac{|\log(1 - \tilde{k}_l f(c/2)e^{-c/2})|}{c/2 - k_l}.$$  

Proof. Let $v_k \in \mathbb{R}^n$ be the unit vector such that $||M_k v_k|| = t_k$ and if $V_k \equiv \{ M_k v_k \}^\perp$ is the subspace orthogonal to $M_k v_k$, let $W_k \equiv M_k^{-1}(V_k)$. Then, for $k \in \mathbb{N}$ and $z \in Z_k$ we define the subsets

$$Y_k(z) \equiv (M_k^{-1}(z) + W_k) \cap M_k^{-1}(B(z, \tau_k/4)).$$

\footnote{That is, for every $y_1, y_2 \in Z_k$ we have $d(y_1, y_2) \geq \tau_k > 0$.}
Set $s_k \equiv \log(\tau_k/t_k)$, which we reorder such that $s_k \leq s_{k+1}$, so that we obtain a discrete set of sizes. For $k \in \Lambda \equiv \mathbb{N}$ let the resonant set $R_k$ be given by

$$R_k \equiv \{ x \in Y_t(z_t) : z_t \in Z_t \text{ and } \log(t_l/\tau_l) \leq s_k \}$$

which gives a nested and discrete family $\mathcal{F} = \{ \mathbb{N}, R_k, s_k \}$.

Note that for all $x \in \mathbb{R}^n$ we have $\|x\| \geq \|M_kx\|/t_k$. Hence, for distinct points $z_1$, $z_2 \in Z_k$, $Y_k(z_1)$ and $Y_k(z_2)$ are subsets of parallel affine hyperplanes and we have

$$\|Y_k(z_1) - Y_k(z_2)\| \geq \|M_k^{-1}(B(y_1, \tau_k/4)) - M_k^{-1}(B(y_2, \tau_k, /4))\| = \frac{\tau_k - 2\tau_k/4}{t_k} = \frac{\tau_k}{2t_k} \geq \frac{1}{2}e^{-s_k},$$

since $Z_k$ is $\tau_k$-separated. Let $l_c = \log(4) + \log(3)$. Given a closed ball $B = B(x, 2e^{-(t+l_c)}) \subset \mathbb{R}^n$ with $x \in X$, for every $k \in \mathbb{N}$ with $s_k \leq t$, it follows from (3.6) that at most one of the sets $Y_k(y)$, $y \in Z_k$, can intersect $B$. Moreover, for $c > 0$, the number of $k \in \mathbb{N}$ with $s_k \in (t-c, t]$ is bounded by $f(c)$ by (3.5). Recall that $R(t, c) \equiv R(t) - R(t - c)$. Thus, there exist at most $N = \lfloor f(c) \rfloor$ affine hyperplanes $L_1, \ldots, L_N \in S$ such that

$$B(x, 2e^{-(t+l_c)}) \cap R(t, c) \subset B(x, 2e^{-(t+l_c)}) \cap \bigcup_{i=1}^N L_i.$$
parameters $\tilde{l}_c = l_c + c_1$ and $\tilde{r}_c = c_2 r_c = \tilde{k}_l f(s)e^{-c},$ where $\tau = 1$, for some constants $c_1, c_2, \tilde{k}_l > 0$. Hence, (2.21) shows
\[
\dim(E_{M, Z}(e^{-(2c+\tilde{l}_c)})) \geq n - \frac{|\log(1 - \tilde{k}_l f(c)e^{-c})|}{c},
\]
since $\text{Bad}_{R^d}(2c + \tilde{l}_c) \subset \text{Bad}_{R^d}(2c + \tilde{l}_c)$. This finishes the proof. \hfill \square

For a nontrivial upper bound, we restrict to the following example. Let $Z = Z_k$ for all $k \in \mathbb{N}$ where $Z$ is a $\tau_0$-spanning set\footnote{That is, for any $x \in \mathbb{R}^2$, there exists $z \in Z$ such that $d(x, z) < \tau$.} of $\mathbb{R}^2$. Let $M = (M_k)$ with $M_k = M^k$, where $M \in GL(2, \mathbb{R})$ is a real diagonalizable matrix with eigenvalues $\lambda \geq \beta > 1$\footnote{Note that for $\beta = 1$ and $c$ sufficiently large, there might even exist $M$-invariant strips of $\mathbb{R}^2$, consisting of badly approximable elements $x$ with $c(x) \geq e^{-c}$.}. In fact, for simplicity, let $M = \text{diag}(\lambda, \beta)$\footnote{If $M = DAD^{-1}$ for $D \in GL(2, \mathbb{R})$, then consider $\tilde{\psi}(x, t) = x + D\psi(0, t)$, for $\psi$ as below, in the following.}, where $\lambda$ and $\beta$ are integers, and let $Z = Z^2$. For these assumptions, there exist constants $c_0 > 0$ and $k_u, \tilde{k}_u > 0$ such that for $c > c_0$ we have
\[
\dim(E_{M, Z}(e^{-c}) \cap [0, 1]^2) \leq 2 - \frac{\log(\beta)}{\log(\lambda)} \log(1 - \tilde{k}_u e^{-\gamma}) \times B(0, e^{-\sigma t}).
\]
\[(3.7)\]

Sketch of the proof of (3.7). We let $\sigma = \log(\beta)$ and $\bar{\sigma} = \log(\lambda)$. On $\Omega = \mathbb{R}^2 \times \mathbb{R}^+$, define the monotonic function $\psi = \psi(\lambda, \beta)$ by the rectangle centered at $x \in \mathbb{R}^2$,
\[
\psi(x, t) = x + B(0, e^{-\sigma t}) \times B(0, e^{-\sigma t}).
\]
Since $\lambda, \beta \in \mathbb{N}_{\geq 2}$, there exist parameters $\tilde{c} > 1$\footnote{In particular true for every integer $\tilde{c} \in \mathbb{N}$.} such that $\lambda^{\tilde{c}} = p, \beta^{\tilde{c}} = q$ with $p, q \in \mathbb{N}_{\geq 2}$, and we can partition $\psi(x, t)$ into $pq$ rectangles $\psi(x \pm \tilde{c})$ as in (2.19). Note that we have $\mu(\psi(x \pm \tilde{c})) = e^{-(\beta+\gamma)c} \mu(\psi(x \pm \tilde{c}))$, where $\mu$ denotes the Lebesgue measure.

Fix a parameter $c > 0$ sufficiently large and let $\tilde{c} = c/\sigma + \log(2)/\sigma + 1$ be the minimal parameter such that a partition as above is possible. Let $Q_0 = Q_1(x_0, 0) = [0, 1]^2$. Now, assume we are given a rectangle $Q = Q_{i_0...i_k} = \psi(x_{i_0...i_k}, k\tilde{c})$. Let $\tilde{k} \in \mathbb{N}$ be the minimal integer such that $\tilde{k} \geq k \log(2)/\sigma$. Thus, we have
\[
R = M^\tilde{k}Q = M^\tilde{k}x_{i_0...i_k} + B(0, k\big)^{\tilde{k}e^{-\sigma t}} \times B(0, e^{-\sigma t})
\]
which is a rectangle of edge lengths in $[2, 2^{2/\sigma} \lambda] \times [2, 2\beta]$ since $\tilde{k}$ was chosen minimal. In particular, there exists an integer point $z \in Z^2$ such that $Q_1(z, c) = \text{diag}(\lambda, \beta)$. Hence, $Q = M^{-\tilde{k}}R \supset M^{-\tilde{k}}Q_1(z, c)$
\[
= M^{-\tilde{k}}z + B(0, e^{-\sigma t}) \times B(0, e^{-\sigma t}) \supset \psi(M^{-\tilde{k}}z, \tilde{k} + c/\sigma).
\]
This in particular shows that $Q \cap E_{M, Z}(e^{-c}) \subset Q \cap \psi(M^{-\tilde{k}}z, \tilde{k} + c/\sigma)^C$ and it suffices to cover the sup set. Again since $\tilde{k}$ is minimal, $t_k + c/\sigma + \log(2)/\sigma + 1 \geq k + c/\sigma$, so that a rectangle $\psi(x, t_k + \tilde{c}) \subset Q$ that intersects $Q \cap \psi(M^{-\tilde{k}}z, \tilde{k} + c/\sigma)^C$ does not intersect $\psi(M^{-\tilde{k}}z, \tilde{k} + c/\sigma + \log(3)/\sigma)$. Moreover, we have
\[
\mu(Q \cap \psi(M^{-\tilde{k}}z, \tilde{k} + c/\sigma + \log(3)/\sigma) \geq \tilde{k}_u e^{-(\beta+\gamma)c} \mu(Q) \equiv \tau^c \mu(Q),
\]
for some constant $\bar{k}_u > 0$. Thus, if $Q_{i_0...i_{k}i_{k+1}} = \psi(x_{i_0...i_{k}i_{k+1}}, t_k + \bar{c})$ are the rectangles from the partition of $Q$, then we can bound the number of rectangles $Q_{i_0...i_{k}i_{k+1}}$ not intersecting $\psi(M^{-k}z, k + c/\sigma + \log(3)/\sigma)$ by $pq(1 - \tau^c) = e(\bar{c} + 1)(1 - \tau^c)$. In particular, we can bound the number of rectangles of the covering constructed this way at stage $k$ by

$$N_k \leq (e(\bar{c} + 1)(1 - \tau^c))^k.$$ 

Thus we obtain a covering of $E_{M,z}(e^{-c})$ by $N_k$ rectangles of diameter at most $2e^{-\sigma k^1}$.

The argument used to obtain (2.26) actually shows that we can cover each rectangle $Q_{i_0...i_{k}i_{k+1}}$ with $Z_{k+1}$ cubes $Q_{\psi}(y_{i_0...i_{k}i_{k+1}}, t_k + 1)$, hence of diameter at most $2e^{-\sigma k^1}$, where $Z_{k+1} \leq c_2 e^{(2\sigma - (\sigma + \sigma)) t_k^1}$ for some constant $c_2 > 0$. Finally, as in (2.26), we obtain

$$\dim(E_{M,z}(e^{-c}) \cap Q_0) \leq \liminf_{k \to \infty} \frac{\log(Z_{k+1} N_k)}{-\log(2e^{-\sigma k^1})} \leq 2 - \frac{\bar{c}}{\sigma} + \frac{\sigma}{\bar{c}} - \frac{1}{\sigma} \log(1 - \tau^c - (\bar{c} + 1) e^{-\bar{c}}) \leq 2 - \frac{|\log(1 - \bar{k}_u e^{-(\bar{c} + 1)})|}{\sigma \bar{c}} \leq 2 - \frac{|\log(1 - \bar{k}_u e^{-(\bar{c} + 1)})|}{\bar{c} + \bar{k}_u},$$

where we used that $\bar{c}$ is chosen minimally, so that $\bar{c} \leq c/\sigma + \log(6)/\sigma + 2 \equiv c/\sigma + k_u/\sigma$. This finishes the proof. □

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S. WEIL, INSTITUT FÜR MATHEMATIK DER UNIVERSITÄT ZÜRICH, E-mail address: steffen.weil@math.uzh.ch