

## BIAŁYNICKI-BIRULA DECOMPOSITION OF DELIGNE-MUMFORD STACKS

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ABSTRACT. This short note considers the Białynicki-Birula decomposition of Deligne-Mumford stacks under one-dimensional torus actions and extends a result of Oprea.

### 1. INTRODUCTION

In this short note, we consider actions of one-dimensional tori on tame Deligne-Mumford stacks which are smooth and proper over an algebraically closed field. We extend a result of Oprea [Opr06] to show that in the aforementioned case, if the stack has a scheme for a coarse moduli space, or if it is toric, then it admits a Białynicki-Birula decomposition and often a corresponding decomposition of cohomology. A history of the result can be found in [Bro05, Theorem 3.2].

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### 2. NOTATION AND TERMINOLOGY

Let  $k$  be an algebraically closed field of arbitrary characteristic, fixed in what follows.

In this note, an algebraic stack will be a stack  $X$  fibered over  $(\text{Sch}/k)$  in the étale topology, such that the diagonal mapping  $\Delta : X \rightarrow X \times X$  is representable, separated and quasi-compact, and such that there exists a smooth, surjective  $k$ -morphism  $U \rightarrow X$  from a  $k$ -scheme  $U$ , which will be called an atlas. Deligne-Mumford stacks are those admitting étale atlases. Proper Deligne-Mumford stacks are those admitting a finite, surjective morphism from a proper  $k$ -scheme. Tame Deligne-Mumford stacks are those with linearly reductive geometric stabilizer groups.

An affine fibration is a flat morphism  $p : E \rightarrow X$  which is étale locally a trivial bundle of affine spaces. This definition weakens the definition of a vector bundle by relaxing the requirement that the transition functions be linear.

Let  $T$  be a one-dimensional torus over  $k$  with an isomorphism to  $\mathbf{G}_m$ . Let the fixed points of an action of  $T$  on a stack  $X$  be denoted by  $X^T$  [Rom05].

If an algebraic stack has the form of a quotient  $[X/G]$  of a normal toric variety  $X$  over an algebraically closed field of characteristic zero by a subgroup  $G$  of the torus of  $X$ , then it is called a toric stack [GS11].

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## 3. BIALYNICKI-BIRULA DECOMPOSITION OF DELIGNE-MUMFORD STACKS

Oprea proved a form of the Bialynicki-Birula decomposition [BB73] for Deligne-Mumford stacks assuming there exists a  $T$ -equivariant, affine, étale atlas [Opr06, Proposition 5].

**Proposition 3.1** (Oprea). *Let  $X$  be a smooth, proper Deligne-Mumford stack over an algebraically closed field  $k$  with a  $T$ -action that admits a  $T$ -equivariant, affine, étale atlas  $U \rightarrow X$ . Let  $F = \coprod_i F_i$  be the decomposition of the fixed substack into connected components. Then  $X$  decomposes into disjoint, locally closed,  $T$ -equivariant substacks  $X_i$  which are  $T$ -equivariant affine fibrations over  $F_i$ .*

Oprea expected the existence of an atlas to be a general fact [Opr06, Section 2]. Here we show that the desired atlas exists under somewhat general conditions.

**Proposition 3.2.** *Let  $X$  be a tame, irreducible Deligne-Mumford stack, smooth and proper over  $k$ , whose generic stabilizer is trivial and whose coarse moduli space is a scheme. Furthermore, let an action of  $T$  on  $X$  be given such that  $T$  acts trivially on its fixed locus. Then there exists a  $T$ -equivariant, affine, étale atlas  $U \rightarrow X$ .*

*Remark 3.3.* If an algebraic group  $G$  acts algebraically on a Deligne-Mumford stack  $X$ , then the action descends to the coarse moduli space of  $X$  by the universal property of the coarse moduli space of  $G \times_k X$ .

*Remark 3.4.* If  $T$  does not act trivially on its fixed locus, it can be made to do so by a reparametrization of the action. Since  $T$ -invariance is not affected by this change, the decomposition for the new action will be a decomposition for the original action.

*Proof.* Since  $X$  is reduced and irreducible with a scheme for a coarse moduli space, the Keel-Mori theorem [KM97, Proposition 4.2] shows the coarse moduli space is in fact a normal, proper variety over  $k$ . It inherits a  $T$ -action by the remark 3.3. Since it is proper, any collection of open,  $T$ -invariant neighborhoods containing the fixed point locus covers it and contains a finite subcover. They can be chosen affine by Sumihiro's theorem [Sum74, Corollary 2]. So it suffices to find the desired atlas for an arbitrary fixed  $k$ -point  $x \in X$ . To this end, let  $Y$  be the pullback of  $X$  to an open, affine,  $T$ -invariant neighborhood of the image of  $x$  in the coarse moduli space.

Consider the frame bundle of  $Y$  with total space  $FY$ . Each  $k$ -point  $y \in Y$  lies in the image of an étale, representable morphism from a quotient stack of the form  $[U/G]$  for an affine, irreducible scheme  $U$  with an action of the stabilizer group  $G$  of  $y$  and containing a point  $u$  fixed by the  $G$ -action and mapping to  $y$  [KM97]. By the tameness hypothesis,  $G$  is linearly reductive. Since  $Y$  has trivial generic stabilizer,  $G$  acts faithfully on  $U$ . Applying [BB73, Theorem 2.4] to  $U$  and  $T_u U$  shows that  $G$  acts faithfully on  $T_y Y$ . So the total space  $FY$  is an algebraic space, and  $Y = [FY/GL_n]$ , where  $n$  is the dimension of  $X$ . But  $Y$  has an affine coarse moduli space, so  $FY$  is, in fact, an affine scheme (cf. [EHKV01, Remark 4.3]).

The action of  $T$  on  $Y$  induces an action on  $TY$  and hence an action on  $FY$ , which is an open,  $T$ -invariant substack of  $TY^{\oplus n}$ . Let  $p : FY \rightarrow Y$  be the projection. An atlas will be defined by finding a  $T$ -equivariant étale slice  $U \hookrightarrow FY$  over the fixed point  $y$ .

One may assume a fixed point  $f \in FY$  lies over  $y$  by modifying the action as follows. Choose a basis of the tangent space  $T_y Y$  which diagonalizes the  $T$ -action,

i.e., so  $t : (v_1, \dots, v_n) \mapsto (t^{a_1}v_1, \dots, t^{a_n}v_n)$ . Then the induced  $T^n$ -action on  $FY$  can be used to define a twisted  $T$ -action on  $FY$  by

$$\begin{aligned} T &\rightarrow T^n \\ t &\mapsto (t^{-a_1}, \dots, t^{-a_n}). \end{aligned}$$

The projection from the frame bundle remains  $T$ -equivariant after twisting the action, but now the action fixes the frame  $f$  formed by the basis vectors.

The proof finishes by arguing as in [Opr06, Lemma 3]. The torus  $T$  acts on the tangent space  $T_f FY$  at  $f$ , with  $T_f FY \rightarrow T_y Y$  surjective and  $T$ -equivariant. By the linear reductivity of  $T$ ,  $T_y Y$  may be identified with some  $T$ -invariant subspace  $N \subset T_y FY$ , compatibly with the  $T$ -actions. By a theorem of Białynicki-Birula [BB73, Theorem 2.1], there exists a reduced and irreducible, closed,  $T$ -invariant subscheme  $Z$  of  $FY$  containing  $f$  as a non-singular point such that  $T_f Z = N$ .

Taking  $U$  to be the largest open subscheme of  $Z$  on which the restriction of  $p$  is étale ensures that  $U$  is  $T$ -invariant. By applying Sumihiro's theorem again, one may shrink  $U$  to an affine,  $T$ -invariant neighborhood of  $f$  whose image in  $Y$  contains  $y$ .  $\square$

**Theorem 3.5.** *Let  $X$  be a tame Deligne-Mumford stack, smooth and proper over  $k$ , whose coarse moduli space is a scheme, and let an action of  $T$  on  $X$  be given. Then  $X$  admits a Białynicki-Birula decomposition.*

*Proof.* One first reduces to the case that  $X$  is irreducible by decomposing each irreducible component separately and combining to give a decomposition of all of  $X$ . Applying Remark 3.4, one may suppose  $T$  acts trivially on its fixed locus. By [Ols07, Proposition 2.1], there exists a rigidification  $\overline{X}$  of  $X$  which has trivial generic stabilizer. In particular, there is an étale, proper morphism  $X \rightarrow \overline{X}$  which forms a  $G$ -gerbe for a finite group  $G$ . The  $T$ -action descends to  $\overline{X}$  using the universal property of rigidification, so Proposition 3.2 guarantees the existence of a  $T$ -equivariant, affine, étale atlas of  $\overline{X}$ . Let the substacks  $\overline{F}_i$  and  $\overline{X}_i$ , together with  $T$ -equivariant affine fibrations  $\overline{X}_i \rightarrow \overline{F}_i$ , be defined according to Proposition 3.1.

Let the decomposition of  $X$  be defined by pulling back along  $X \rightarrow \overline{X}$ . Pulling back again by the affine fibration  $\overline{X}_i \rightarrow \overline{F}_i$  forms the diagram:

$$(3.1) \quad \begin{array}{ccccccc} \overline{X}_i \times_{\overline{F}_i} F_i & \longrightarrow & F_i & \longrightarrow & X_i & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{X}_i & \longrightarrow & \overline{F}_i & \longrightarrow & \overline{X}_i & \longrightarrow & \overline{X} \end{array}$$

So  $F = \coprod_i F_i$  is a decomposition of the fixed substack into connected components [Rom05]. Furthermore, all morphisms are  $T$ -equivariant.

In what follows, let  $i$  be fixed. It remains to prove the existence of an affine fibration, and diagram (3.1) shows it will suffice to supply a  $T$ -equivariant isomorphism  $\overline{X}_i \times_{\overline{F}_i} F_i \rightarrow X_i$  over  $\overline{X}_i$ . This can be done by specifying such an isomorphism, unique up to canonical 2-isomorphism, over an étale,  $T$ -equivariant atlas of  $\overline{X}_i$ , and then applying the descent property of the stack  $\underline{\mathrm{Hom}}_{\overline{X}_i}(\overline{X}_i \times_{\overline{F}_i} F_i, X_i)$  [Ols06].

First, étale,  $T$ -equivariant atlases forming the following pullback will be defined:

$$(3.2) \quad \begin{array}{ccc} W & \longrightarrow & P \\ \downarrow & & \downarrow \\ \overline{X}_i & \longrightarrow & \overline{F}_i \end{array}$$

There is an  $\text{Out}(G)$ -torsor on  $\overline{F}_i$  associated to the rigidification gerbe  $F_i \rightarrow \overline{F}_i$  whose class  $\alpha$  lies in  $H^1(\overline{F}_i, \text{Out}(G))$ . Base change to the total space of the torsor trivializes  $\alpha$  and gives an element of the étale cohomology  $\beta \in H^2(\overline{F}_i, Z)$  which classifies the gerbe, where  $Z$  is the center of  $G$  (cf. [EHKV01, Section 3.1]). Let  $P \rightarrow \overline{F}_i$  be an affine, étale atlas that trivializes  $\alpha$ ,  $\beta$ , and the affine fibration  $\overline{X}_i \rightarrow \overline{F}_i$ . Also, let  $W := P \times_{\overline{F}_i} \overline{X}_i$ , as in diagram (3.2).

By the Künneth formula for the algebraic fundamental group [Ray71, Proposition 4.6], the atlas  $W$  trivializes the  $\text{Out}(G)$ -torsor associated to the gerbe  $X_i \rightarrow \overline{X}_i$ . Then [Art73, Corollary 2.2] shows that the morphism  $P \times \mathbf{A}^n \cong W \rightarrow P$  induces an isomorphism  $H^2(P, Z) \xrightarrow{\sim} H^2(W, Z)$ , implying that the classifying element of  $X_i \times_{\overline{X}_i} W \rightarrow W$  vanishes. The isomorphism  $X_i \times_{\overline{X}_i} W \xrightarrow{\sim} F_i \times_{\overline{F}_i} W$  of trivial gerbes can be chosen, uniquely up to canonical 2-isomorphism, to be the isomorphism over  $W$  which extends the identity morphism of  $F_i \times_{\overline{F}_i} P$  over  $P$ . This follows from the triviality of the affine fibration  $W \rightarrow P$  and the Künneth formula. The isomorphism will also be  $T$ -equivariant by similar reasoning, since the identity is  $T$ -equivariant.  $\square$

*Remark 3.6.* If  $X$  is a tame Deligne-Mumford stack, smooth and proper over  $k$ , with a projective coarse moduli space and a  $T$ -action, then the induced decomposition forms a filtration, and a lemma of Oprea [Opr06, Lemma 6] implies that the Betti numbers of the stack are calculated by the Betti numbers of the fixed points.

In what follows,  $T$  may be a torus of arbitrary dimension.

**Proposition 3.7.** *Let  $\text{char } k = 0$ , and let  $X$  be a normal algebraic space, separated and of finite type over  $k$ , with an action of  $T$  which gives a dense, open embedding of  $T$  in  $X$ . Then  $X$  is a scheme, and hence a toric variety.*

*Proof.* First, let  $k = \mathbf{C}$ . The scheme locus of the normalized blow-up at the closure of any non-dense  $T$ -orbit forms a toric variety whose image includes the orbit. The associated fans give, in each  $T$ -orbit, a limit point of a  $\mathbf{G}_m$ -orbit of  $1 \in T \hookrightarrow X$  for a subtorus  $\mathbf{G}_m$  of  $T$ . Then  $X$  is a finite union of  $T$ -orbits of such points and hence a scheme [Hau00, Theorem 1].

For general  $k$ , one may immediately reduce to the case that  $k$  is a subfield of  $\mathbf{C}$ . Since the pullback to  $\mathbf{C}$  is toric, a theorem [GS11, Theorem, p.3] implies there exist a field extension  $L$  of  $k$  of finite transcendence degree and an étale, representable, surjective morphism to  $[X/T] \times_{\text{Spec } k} \text{Spec } L$  from a stack of the form  $[U/\mathbf{GL}_n]$  for a quasi-affine scheme  $U$  and an integer  $n$ . Writing  $L = k(V)$  for some affine variety  $V$ , one may suppose  $U$  with its  $\mathbf{GL}_n$ -action is defined over  $V$ , and  $p : [U/\mathbf{GL}_n] \rightarrow [X/T] \times V$  is étale and representable. The disjoint union of fibers of  $p$  over finitely many closed points of  $V$  forms an étale, representable, surjective morphism to  $[X/T]$ . Applying *loc. cit.* again, one deduces that  $X$  is a scheme.  $\square$

**Theorem 3.8.** *Assume  $\text{char } k = 0$ , and let  $X$  be a Deligne-Mumford stack, smooth and proper over  $k$ , with an action of  $T$  which gives a dense, open embedding of  $T$  in  $X$ . Then the induced action of any one-dimensional subtorus of  $T$  induces a Bialynicki-Birula decomposition of  $X$ .*

*Proof.* By remark 3.3, the action of  $T$  on  $X$  descends to the coarse moduli space. The theorem now follows from the previous two results.  $\square$

## REFERENCES

- Art73. M. Artin, *Morphismes acycliques*, Exp. XV, Théorie des topos et cohomologie étale des schémas. SGA 4, Tome 3, Lecture Notes in Math., Vol. 305, Springer-Verlag, Berlin, 1973.
- BB73. A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.
- Bro05. P. Brosnan, *On motivic decompositions arising from the method of Białynicki-Birula*, Invent. Math. **161** (2005), no. 1, 91–111.
- EHKV01. D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777.
- GS11. A. Geraschenko and M. Satriano, *Toric stacks I: The theory of stacky fans*, arXiv:1107.1906, 2011.
- Hau00. J. Hausen, *Algebraicity criteria for almost homogeneous complex spaces*, Arch. Math. (Basel) **74** (2000), no. 4, 317–320.
- KM97. S. Keel and S. Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213.
- Ols06. M. C. Olsson, *Hom-stacks and restriction of scalars*, Duke Math. J. **134** (2006), no. 1, 139–164.
- Ols07. M. Olsson, *A boundedness theorem for Hom-stacks*, Math. Res. Lett. **14** (2007), no. 6, 1009–1021.
- Opr06. D. Oprea, *Tautological classes on the moduli spaces of stable maps to  $\mathbb{P}^r$  via torus actions*, Adv. Math. **207** (2006), no. 2, 661–690.
- Ray71. M. Raynaud, *Propriété cohomologique des faisceaux d'ensembles et des faisceaux de groupes non commutatifs*, Exp. XIII, Revêtements étales et groupe fondamental. SGA 1, Lecture Notes in Math., 224, Springer-Verlag, Berlin, 1971.
- Rom05. M. Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236.
- Sum74. H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.

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