TAME LOCI OF CERTAIN LOCAL COHOMOLOGY MODULES

MARKUS BRODMANN AND MARYAM JAHANGIRI

Abstract. Let \( M \) be a finitely generated graded module over a Noetherian homogeneous ring \( R = \bigoplus_{n \in \mathbb{N}_0} R_n \). For each \( i \in \mathbb{N}_0 \) let \( H^i_{R_+}(M) \) denote the \( i \)-th local cohomology module of \( M \) with respect to the irrelevant ideal \( R_+ = \bigoplus_{n > 0} R_n \) of \( R \), furnished with its natural grading. We study the tame loci \( T^i(M) \leq 3 \) at level \( i \in \mathbb{N}_0 \) in codimension \( \leq 3 \) of \( M \), that is the sets of all primes \( p_0 \subset R_0 \) of height \( \leq 3 \) such that the graded \( R_{p_0} \)-modules \( H^i_{R_+}(M)_{p_0} \) are tame.

1. Introduction

Throughout this note let \( R = \bigoplus_{n \geq 0} R_n \) be a homogeneous Noetherian ring. So, \( R \) is an \( \mathbb{N}_0 \)-graded \( R_0 \)-algebra and \( R = R_0[l_1,...,l_r] \) with finitely many elements \( l_1,...,l_r \in R_1 \). Moreover, let \( R_+ := \bigoplus_{n > 0} R_n \) denote the irrelevant ideal of \( R \) and let \( M \) be a finitely generated graded \( R \)-module. For each \( i \in \mathbb{N}_0 \) let \( H^i_{R_+}(M) \) denote the \( i \)-th local cohomology module of \( M \) with respect to \( R_+ \). It is well known, that the \( R \)-module \( H^i_{R_+}(M) \) carries a natural grading and that the graded components \( H^i_{R_+}(M)_n \) are finitely generated \( R_0 \)-modules which vanish for all \( n \gg 0 \) (s. [11], §15 for example). So, the \( R_0 \)-modules \( H^i_{R_+}(M)_n \) are asymptotically trivial if \( n \to +\infty \).

On the other hand a rich variety of phenomena occurs for the modules \( H^i_{R_+}(M)_n \) if \( i \in \mathbb{N}_0 \) is fixed and \( n \to -\infty \). So, it is quite natural to investigate the asymptotic behaviour of cohomology, e.g. the mentioned phenomena (s. [3]).

One basic question in this respect is to ask for the asymptotic stability of associated primes, more precisely the question, whether for given \( i \in \mathbb{N}_0 \) the set \( \text{Ass}_{R_0}(H^i_{R_+}(M)_n) \) (or some of its specified subsets) ultimately becomes independent of \( n \), if \( n \to -\infty \). In many particular cases this is indeed the case (s. [2], [5], [6], [7]), partly even in a more general setting (s. [16]). On the other hand it is known for quite a while, that the asymptotic stability of associated primes also may fail in many even surprisingly “nice” cases by various examples (s. [6], [8] and also [3]), which rely on the constructions given in [20] and [21].

Another related question is, whether for fixed \( i \in \mathbb{N}_0 \) certain numerical invariants of the \( R_0 \)-modules \( H^i_{R_+}(M)_n \) ultimately become constant if \( n \to -\infty \). A number of such asymptotic stability results for numerical invariants are indeed known (s. [4], [9], [10] and also [14]).

April 12, 2012.

The second author was in part supported by a grant from IPM (No. 89130115).
The oldest - and most challenging - question around the asymptotic behaviour of cohomology was the so-called tameness problem, that is the question, whether for fixed \( i \in \mathbb{N}_0 \) the \( R_0 \)-modules \( H^i_\mathfrak{p} (M)_n \) are either always vanishing for all \( n \ll 0 \) or always non-vanishing for all \( n \ll 0 \). This question seems to have raised already in relation with Marley’s paper [18]. In a number of cases, this tameness problem was shown to have an affirmative answer (s. [3], [7], [17], [19]).

Nevertheless by means of a duality result for bigraded modules given in [15], Cutkosky and Herzog [12] constructed an example which shows that the tameness-problem can have a negative answer also. In [13] an even more striking counter-example is given: a Rees- and Herzog [12] constructed an example which shows that the tameness-problem can have an affirmative answer (s. [3], [7], [17], [19]).

In a number of cases, this tameness problem was shown to have a non-vanishing for all \( \mathfrak{p} \in \text{Spec}(R_0) \) of all non-zero primes \( \mathfrak{p} \in \text{Spec}(R_0) \) that the \( \text{tame loci} \), that is the question, whether for fixed \( \mathfrak{p} \in \text{Spec}(R_0) \) of height 3 and have the property that all primes \( \mathfrak{p} \in \text{Spec}(R_0) \) of height 3 and have the property that all primes \( \mathfrak{p} \in \text{Spec}(R_0) \) which shall be used constantly in our arguments. In this section we also introduce the so called critical sets \( C^i(M) \subset \text{Spec}(R_0) \) which consist of primes of height 3 and have the property that all primes \( \mathfrak{p}_0 \notin C^i(M) \) of height \( \leq 3 \) belong to the tame locus \( \mathfrak{T}^i(M) \) (s. Proposition 2.8 (b)). Moreover the finiteness of the set \( C^i(M) \) has the particularly nice consequence that \( M \) is uniformly tame at level \( i \) in codimension \( \leq 3 \), e.g. there is an integer \( n_0 \) such that for each \( \mathfrak{p}_0 \in \mathfrak{T}^i(M) \leq 3 \) the \( (R_0)_{\mathfrak{p}_0} \)-module \( (H^i_{R_0} (M)_n)_{\mathfrak{p}_0} \) is either vanishing for all \( n \leq n_0 \) or non-vanishing for all \( n \leq n_0 \) (s. Proposition 2.8 (c)).

In Section 3 we give some finiteness criteria for the critical sets \( C^i(M) \). Here, we assume in addition that the base ring \( R_0 \) is a domain, so that the intersection \( \mathfrak{a}^i(M) \) of all non-zero primes \( \mathfrak{p}_0 \subset R_0 \) which are associated to \( H^i_{R_0} (M) \) is a non-zero ideal by a result of [5]. Our main result says, that the critical set \( C^i(M) \) is finite, if \( \mathfrak{a}^i(M) \) contains a quasi-non-zero divisor with respect to \( M \) (s. Theorem 3.4). This obviously
applies in particular to the case in which $M$ is torsion-free as an $R_0$-module in all large degrees or at all (s. Corollary 3.5 resp. Corollary 3.7). In order to force a situation as required in Theorem 3.4 one is tempted to replace $M$ by $M/\Gamma(x)(M)$ for some non-zero element $x \in R_0$. We therefore give a comparison result for the critical sets $C^i(M)$ and $C^i(M/\Gamma(x)(M))$ (s. Proposition 3.7). As an application we prove that the critical sets $C^i(M)$ are finite if $R_0$ is a domain and the $R_0$-module $M$ asymptotically satisfies some weak "unmixedness condition" (s. Corollary 3.8).

In our final Section 4 we give a few conditions for the tameness at level $i$ in codimension $\leq 3$ in terms of the "asymptotic smallness" of the graded $R$-modules $H_{R_+}^{i-1}(M)$ and $H_{R_+}^{-\infty}(M)$. We first prove that all primes $p_0 \subset R_0$ of height $\leq 3$ belong to the tame locus $\Sigma^i(M)$, provided that $\dim_{R_0}(H_{R_+}^{i-1}(M)_n) \leq 1$ and $\dim_{R_0}(H_{R_+}^{-\infty}(M)_n) \leq 2$ for all $n \ll 0$ (s. Theorem 4.2). In addition we show that $M$ is tame at almost all primes $p_0 \subset R_0$ of height $\leq 3$ provided that $R_0$ is a domain and $\dim_{R_0}(H_{R_+}^{i-1}(M)_n) \leq 0$ for all $n \ll 0$ (s. Theorem 4.4). We actually prove in both cases slightly sharper statements namely: the corresponding graded $R_{p_0}$-modules $H_{R_{p_0}}^{i-1}(M)$ are not only tame, but even what we call almost Artinian. Using this terminology we get in particular the following conclusion. If $R_0$ is a domain and the graded $R$-module $H_{R_+}^{i-1}(M)$ is almost Artinian, then for almost all primes $p_0 \in \text{Spec}(R_0)$ of height $\leq 3$ either the $(R_0)_{p_0}$-module $(H_{R_{p_0}}^{i-1}(M))_{p_0}$ is of dimension $> 0$ for all $n \ll 0$ or else the graded $R_{p_0}$-module $H_{R_+}^{i-1}(M)_{p_0}$ is almost Artinian (s. Corollary 4.5).

2. Tame Loci in Codimension $\leq 3$

We keep the previously introduced notations.

Convention and Notation 2.1. (A) Throughout this section we convene that the base ring $R_0$ of our Noetherian homogeneous ring $R = R_0 \oplus R_1 \oplus ...$ is essentially of finite type over some field. So, $R_0 = S^{-1}A$, where $A = K[a_1, \ldots, a_s]$ is a finitely generated algebra over some field $K$. $S \subset A$ is multiplicatively closed and there are finitely many elements $l_1, \ldots, l_r \in R_1$ such that $R = R_0[l_1, \ldots, l_r]$.

(B) If $n \in \mathbb{N}_0$ and $\mathfrak{P} \subseteq \text{Spec}(R_0)$ we write

$$\mathfrak{P}^=n := \{p_0 \in \mathfrak{P} \mid \text{height}(p_0) = n\}$$

$$\mathfrak{P}^\leq n := \{p_0 \in \mathfrak{P} \mid \text{height}(p_0) \leq n\}.$$

Reminder and Remark 2.2. (A) According to [1] for all $n \ll 0$ the set $\text{Ass}_{R_0}(M_n)$ is equal to the set $\{p \cap R_0 \mid p \in \text{Ass}_{R} \cap \text{Proj}(R)\}$ and hence asymptotically stable for $n \to \infty$, thus:

There is a least integer $m(M) \geq 0$ and a finite set $\text{Ass}_{R_0}(M) \subseteq \text{Spec}(R_0)$ such that $\text{Ass}_{R_0}(M_n) = \text{Ass}_{R_0}(M)$ for all $n > m(M)$.

(B) Let $f(M)$ denote the finiteness dimension of $M$ with respect to $R_+$, that is "the least integer" for which the $R$-module $H_{R_+}^i(M)$ is not finitely generated. Clearly we may write

$$f(M) = \inf\{i \in \mathbb{N}_0 \mid \exists \{n \in \mathbb{Z} \mid H_{R_+}^i(M)_n \neq 0\} = \infty\}.$$
(C) Keep in mind that \( f(M) > 0 \). According to [BH, Theorem 5.6] we know that the set \( \text{Ass}_{R_0}(H^{i(M)}_{R_+}(M)_n) \) is asymptotically stable for \( n \to -\infty \):

There is a largest integer \( n(M) \leq 0 \) and a finite set \( \mathfrak{U}(M) \subseteq \text{Spec}(R_0) \) such that \( \text{Ass}_{R_0}(H^{i(M)}_{R_+}(M)_n) = \mathfrak{U}(M) \) for all \( n \leq n(M) \).

In particular

\[
\text{Supp}_{R_0}(H^{i(M)}_{R_+}(M)_n) = \overline{\mathfrak{U}(M)}, \quad \forall n \leq n(M),
\]

where \( \overline{\cdot} \) denotes the formation of the topological closure in \( \text{Spec}(R_0) \).

(D) According to [B1, Theorem 4.1] we know that for each \( i \in \mathbb{N}_0 \) the set \( \text{Ass}_{R_0}(H^{i(M)}_{R_+}(M)_n) \) is asymptotically stable in codimension \( \leq 2 \) for \( n \to -\infty \):

For each \( i \in \mathbb{N}_0 \) there is a largest integer \( n^i(M) \leq 0 \) and a finite set \( \mathfrak{V}^i(M) \subseteq \text{Spec}(R_0)^{\leq 2} \) such that \( \text{Ass}_{R_0}(H^{i(M)}_{R_+}(M)_n)^{\leq 2} = \mathfrak{V}^i(M) \) for all \( n \leq n^i(M) \).

Now, combining this with the observations made in parts (B) and (C) we obtain:

(i) \( i < f(M) \Rightarrow \forall n \leq n^i(M) : H^{i}_{R_+}(M)_n = 0; \)

(ii) \( \forall n \leq n(M) : \text{Supp}_{R_0}(H^{i(M)}_{R_+}(M)_n) = \overline{\mathfrak{U}(M)}; \)

(iii) \( i > f(M) \Rightarrow \forall n \leq n^i(M) : \text{Supp}_{R_0}(H^{i(M)}_{R_+}(M)_n)^{\leq 2} = \overline{\mathfrak{V}^i(M)^{\leq 2}}. \)

**Definition and Remark 2.3.**

(A) Let \( i \in \mathbb{N}_0 \). We say that the finitely generated graded \( R \)-module \( M \) is (cohomologically) tame at level \( i \) if the graded \( R \)-module \( H^{i}_{R_+}(M) \) is tame, e.g.

\[
\exists n_0 \in \mathbb{Z} : (\forall n \leq n_0 : H^{i}_{R_+}(M)_n = 0) \lor (\forall n \leq n_0 : H^{i}_{R_+}(M)_n \neq 0).
\]

(B) Let \( p_0 \in \text{Spec}(R_0) \). We say that \( M \) is (cohomologically) tame at level \( i \) in \( p_0 \) if the graded \( R_{p_0} \)-module \( M_{p_0} \) is cohomologically tame at level \( i \). In view of the graded flat base change property of local cohomology it is equivalent to say that the graded \( R_{p_0} \)-module \( H^{i}_{R_+}(M)_{p_0} \) is tame.

(C) We define the \( i \)-th (cohomological) tame locus of \( M \) as the set \( \mathfrak{T}^i(M) \) of all primes \( p_0 \in \text{Spec}(R_0) \) such that \( M \) is (cohomologically) tame at level \( i \) in \( p_0 \). So, if \( p_0 \in \text{Spec}(R_0) \) we have

\[
p_0 \in \mathfrak{T}^i(M) \iff \exists n_0 \in \mathbb{Z} : \begin{cases} 
\forall n \leq n_0 : p_0 \in \text{Supp}_{R_0}(H^{i}_{R_+}(M)_n) \\
\forall n \leq n_0 : p_0 \notin \text{Supp}_{R_0}(H^{i}_{R_+}(M)_n)
\end{cases}
\]

If \( k \in \mathbb{N}_0 \), the set \( \mathfrak{T}^i(M)^{\leq k} \) is called the \( i \)-th (cohomological) tame locus of \( M \) in codimension \( \leq k \).

(D) Let \( \mathfrak{U} \subseteq \text{Spec}(R_0) \). We say that \( M \) is (cohomologically) tame at level \( i \) along \( \mathfrak{U} \), if \( \mathfrak{U} \subseteq \mathfrak{T}^i(M) \). We say that \( M \) is uniformly (cohomologically) tame at level \( i \) along \( \mathfrak{U} \) if there is an integer \( n_0 \) such that for all \( p_0 \in \mathfrak{U} \)

\[
(\forall n \leq n_0 : p_0 \in \text{Supp}_{R_0}(H^{i}_{R_+}(M)_n) \lor (\forall n \leq n_0 : p_0 \notin \text{Supp}_{R_0}(H^{i}_{R_+}(M)_n)).
\]

(E) If \( M \) is uniformly tame at level \( i \) along the set \( \mathfrak{U} \subseteq \text{Spec}(R_0) \), then it is tame along \( \mathfrak{U} \) at level \( i \).
Remark 2.4. (A) According to Reminder and Remark 2.2 (D) (i) and (ii) we have

M is uniformly tame along Spec(R₀) at all levels i ≤ f(M).

(B) Using the notation of Reminder and Remark 2.2 (A) we write \( \text{Supp}_R^*(M) := \text{Ass}_{R_0}(M) \) so that \( \text{Supp}_{R_0}(M_n) = \text{Supp}_M^*(M) \) for all \( n \geq m(M) \). Now, on use of Reminder and Remark 2.2 (D) it follows easily:

for all \( i > f(M) \), the module \( M \) is uniformly tame at level \( i \) along the set \( W^i(M) := (\text{Spec}(R_0) \setminus \text{Supp}_{R_0}^*(M)) \cup \text{Spec}(M) \cup \text{Spec}(R_0) \leq 2 \).

It follows in particular that \( W^i(M) \subseteq \mathfrak{T}_i(M) \), and moreover, for all \( i \in \mathbb{N}_0 \):

(i) \( M \) is uniformly tame at level \( i \) along the set \( \text{Spec}(R_0) \leq 2 \).

(ii) \( \mathfrak{T}_i(M) \leq 3 \) is stable under generalization.

If the graded \( R \)-module \( T = \bigoplus_{n \in \mathbb{Z}} T_n \) is tame, and \( p_0 \in \text{Spec}(R_0) \), then the graded \( R_{p_0} \)-module \( T_{p_0} \) need not to be tame any more. This hints that in general the loci \( \mathfrak{T}_i(M) \) could be non-stable under generalization. We now present such an example.

Example 2.5. Let \( K \) be algebraically closed. Then according to [CCHS], there exists a normal homogeneous Noetherian domain \( R' = \bigoplus_{n \geq 0} R'_n \) of dimension 4 such that \( (R_0, m'_0) \) is local, of dimension 3 with \( R'_0/m'_0 = K \) and such that for all negative integers \( n \) we have \( H^2_{R'_0}(R')_n = K^2 \) if \( n \) is even and \( H^2_{R'_0}(R')_n = 0 \) if \( n \) is odd.

Now, let \( l_1, \ldots, l_r \in R' \) be such that \( R'_1 = \sum_{i=1}^r R'_0 l_i \). Let \( x, x_1, \ldots, x_r \) be indeterminates, let \( R_0 \) denote the 4-dimensional local domain \( R_0[x, m'_0, x] \) with maximal ideal \( m_0 := (m'_0, x)R'_0 \), consider the homogeneous \( R_0 \)-algebras \( R := R_0[x_1, \ldots, x_r] \) and \( \overline{R} := R_0 \otimes_{R'_0} R' \) together with the surjective graded homomorphism of \( R_0 \)-algebras

\[
\Phi : R = R_0[x_1, \ldots, x_r] \to \overline{R}; \quad x_i \mapsto 1_{R_0} \otimes l_i.
\]

Now, let \( \alpha \in m'_0 \setminus \{0\} \), let \( t \) be a further indeterminate, consider the Rees algebra

\[
S = R_0[x, (x + \alpha)t] = \bigoplus_{n \geq 0}((x, x + \alpha)R_0)^n
\]

and the surjective graded homomorphism of \( R_0 \)-algebras

\[
\Psi : R \to S, \quad x_1 \mapsto xt, \quad x_2 \mapsto (x + \alpha)t, \quad x_i \mapsto 0 \text{ if } i \geq 3.
\]

We consider \( \overline{R} \) and \( S \) as graded \( R \)-modules by means of \( \Phi \) and \( \Psi \) respectively. Then \( M := \overline{R} \oplus S \) is a finitely generated graded \( R \)-module which is, in addition, torsion-free over \( R_0 \).

By the graded base ring independence and flat base change properties of local cohomology we get isomorphisms of graded \( R \)-modules

\[
H^2_{R_+}(\overline{R}) \cong R_0 \otimes_{R_0} H^2_{R'_+}(R'), \quad H^2_{R_+}(S) \cong H^2_{S_+}(S).
\]

As \( \text{cd}_{S_+}(S) = \dim(S/m_0S) = 2 \) we have \( H^2_{S_+}(S)_n \neq 0 \) for all \( n \ll 0 \). It follows that \( H^2_{R_+}(M)_n \cong H^2_{R_+}(\overline{R})_n \oplus H^2_{S_+}(S)_n \neq 0 \) for all \( n \ll 0 \) and so \( M \) is tame at level 2. In particular we have \( m_0 \in \mathfrak{T}^2(M) \).
Now, consider the prime $p_0 := m_0^2 R_0 \in \text{Spec}(R_0)^{=3}$. Then, for each $n < 0$ we have
\[(H^2_{R[p]}(R_n))_{p_0} \cong (R_0)_{m_0^2 R_0} \otimes_{R_0} H^2_{R[p]}(R^*) \cong \begin{cases} \mathbb{K}(x)^2, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}\]
Moreover $S_{p_0} = (R_0)_{p_0}[(x, x+\alpha)(R_0)_{p_0} t] = (R_0)_{p_0}[t]$ shows that $H^2_{(S[p_0])} (S_{p_0}) = 0$. It follows that $(H^2_{R[p]}(M)_n)_{p_0}$ vanishes precisely for all odd negative integers $n$. So $H^2_{R[p]}(M)p_0$ is not tame and hence $p_0 \notin \mathcal{T}^2(M)$.

Observe in particular that here $\mathcal{T}^2(M) = \mathcal{T}^2(M)^{=4}$ is not stable under generalization, and that $R_0$ is a domain and the graded $R$-module $M$ is torsion-free over $R_0$. On the other hand $\mathcal{T}^i(M)^{=3}$ is always stable under generalization, (cf. Remark 2.4 (B) (ii)).

One of our aims is to show that quite a lot can be said about the sets $\mathcal{T}^i(M)^{=3}$ if the base ring $R_0$ is a domain and $M$ is torsion-free over $R_0$. Indeed, we shall attack the problem in a more general context, beginning with the following result, in which $\mathfrak{P}^i(M)$ is defined according to Definition and Remark 2.2 (D).

**Lemma 2.6.** Let $i \in \mathbb{N}_0$ and let $n^i(M)$ be defined as in Reminder and Remark 2.2 (D). Then for all $n \leq n^i(M)$ we have
\[C_n^i(M) := (\text{Supp}_{R_0}(H^i_{R[p]}(M)_n) \setminus \mathfrak{P}^i(M))^{=3} = (\text{Ass}_{R_0}(H^i_{R[p]}(M)_n) \setminus \mathfrak{P}^i(M))^{=3}.\]

**Proof.** Let $n \leq n^i(M)$ and $p_0 \in ((\text{Supp}_{R_0}(H^i_{R[p]}(M)_n) \setminus \mathfrak{P}^i(M))^{=3}$. Then, there is some $q_0 \in \text{Ass}_{R_0}(H^i_{R[p]}(M)_n)$ with $q_0 \subseteq p_0$. As $p_0 \notin \mathfrak{P}^i(M)$ we have $q_0 \notin \mathfrak{P}^i(M) = \text{Ass}_{R_0}(H^i_{R[p]}(M)_n)^{=2}$. It follows that $\text{height}(q_0) \geq 3$, hence $q_0 = p_0$ and therefore
\[p_0 \in \text{Ass}_{R_0}(H^i_{R[p]}(M)_n)^{=3}.\]
This proves the inclusion "$\subseteq$". The converse inclusion is obvious. \(\square\)

**Definition 2.7.** Let $i \in \mathbb{N}_0$ and let $n^i(M)$ and $C_n^i(M)$ be as in Lemma 2.6. Then the set
\[C^i(M) := \bigcup_{n \leq n^i(M)} C_n^i(M)\]
is called the $i$th critical set of $M$.

**Proposition 2.8.** Let $i \in \mathbb{N}_0$. Then
(a) $M$ is uniformly tame at level $i$ along the set
\[\left(\text{Spec}(R_0) \setminus \text{Supp}_{R_0}(M) \setminus \mathfrak{P}^i(M) \setminus \text{Spec}(R_0)^{=3}\right) \cup \text{Spec}(R_0)^{=3} \setminus C^i(M).\]
(b) $\mathcal{T}^i(M)^{=3} \supseteq \text{Spec}(R_0)^{=3} \setminus C^i(M)$.
(c) The following statements are equivalent:
(i) $C^i(M)$ is a finite set;
(ii) $\mathcal{T}^i(M)^{=3}$ is open in $\text{Spec}(R_0)^{=3}$ and $M$ is uniformly tame at level $i$ along $\mathcal{T}^i(M)^{=3}$.
(iii) $\text{Spec}(R_0)^{=3} \setminus \mathcal{T}^i(M)$ is finite and $M$ is uniformly tame at level $i$ along $\mathcal{T}^i(M)^{=3}$. 
Proof. (a): This follows from Remark 2.4 (B) and the fact that
\[ \bigcup_{n \leq n^i(M)} \text{Supp}_{R_0} (H^i_{R^-}(M)_n) \supseteq \mathfrak{T}^i(M) = C^i(M). \]
(b): This is immediate by statement (a).
(c): "(i) \Rightarrow (ii)" : This follows easily by statements (a) and (b) and the fact that \( M \) is uniformly tame at level \( i \) along each finite subset \( V \subseteq \mathfrak{T}^i(M) \).
(ii) \Rightarrow (iii)" : Assume that statement (ii) holds. As \( \text{Spec}(R_0) \subseteq \mathfrak{T}^i(M) \) (s. Remark 2.4 (B) (i)) and as \( \mathfrak{T}^i(M) \) is open in \( \text{Spec}(R_0) \) it follows that \( \text{Spec}(R_0) \subseteq \mathfrak{T}^i(M) \) is a finite set, and this proves statement (iii).
(iii) \Rightarrow (i)" : Assume that statement (iii) holds so that \( \text{Spec}(R_0) \subseteq \mathfrak{T}^i(M) \) is finite and \( M \) is uniformly tame along \( \mathfrak{T}^i(M) \). By statement (b) we have \( \text{Spec}(R_0) \subseteq \mathfrak{T}^i(M) \subseteq C^i(M) \subseteq \text{Spec}(R_0) \). It thus suffices to show that the set \( F := C^i(M) \cap \mathfrak{T}^i(M) \) is finite.

By uniform tameness there is some integer \( n_0 \leq n^i(M) \) such that for each \( p_0 \in F \) either

(I) \( p_0 \in \text{Supp}_{R_0} (H^i_{R^-}(M)_n) \) for all \( n \leq n_0 \); or

(II) \( p_0 \notin \text{Supp}_{R_0} (H^i_{R^-}(M)_n) \) for all \( n \leq n_0 \).

Let \( F_I := \{ p_0 \in F \mid p_0 \text{ satisfies } (I) \} \) and \( F_{II} := \{ p_0 \in F \mid p_0 \text{ satisfies } (II) \} \). As \( F = F_I \cup F_{II} \) it suffices to show that \( F_I \) and \( F_{II} \) are finite.

If \( p_0 \in F_I \), we have \( p_0 \in (\text{Supp}_{R_0} (H^i_{R^-}(M)_n) \setminus \mathfrak{T}^i(M) \subseteq \mathfrak{T}^i(M) \). As \( n_0 \leq n^i(M) \) statement (a) implies \( p_0 \in \text{Ass}_{R_0} (H^i_{R^-}(M)_n) \). This proves that \( F_I \subseteq \text{Ass}_{R_0} (H^i_{R^-}(M)_n) \) and thus \( F_I \) is finite.

Clearly \( F_{II} \subseteq \bigcup_{n_0 \leq n \leq n^i(M)} \text{Supp}_{R_0} (H^i_{R^-}(M)_n) \setminus \mathfrak{T}^i(M) \) \( \supseteq \mathfrak{T}^i(M) \). So, by statement (a) we see that \( F_{II} \) is contained in the finite set \( \bigcup_{n_0 \leq n \leq n^i(M)} \text{Ass}_{R_0} (H^i_{R^-}(M)_n) \). \( \square \)

3. Finiteness of Critical sets

We keep all notations and hypotheses of the previous section. So \( R = \bigoplus_{n \in \mathbb{N}_0} R_n \) is a Noetherian homogeneous ring whose base ring \( R_0 \) is essentially of finite type over some field and \( M \) is a finitely generated graded \( R \)-module. By statement (c) of Proposition 2.8 it seems quite appealing to look for criteria which ensure that the critical sets \( C^i(M) \) are finite. This is precisely the aim of the present section.

Reminder 3.1. (A) Assume that \( R_0 \) is a domain. Then, according to [BFL, Theorem 2.5] there is an element \( s \in R_0 \setminus \{0\} \) such that the \( (R_0)_s \)-module \( (H^i_{R^-}(M))_s \) is torsion-free or 0 for all \( i \in \mathbb{N}_0 \). From this we conclude that (with the standard convention that \( \bigcap_{p_0 \in \mathfrak{p}_0} \mathfrak{p}_0 := R_0 \)):

If \( R_0 \) is a domain, the ideal
\[ \mathfrak{a}^i(M) := \bigcap_{p_0 \in \text{Ass}_{R_0} (H^i_{R^-}(M)) \setminus \{0\}} p_0 \]
is \( \neq 0 \) for all \( i \in \mathbb{N}_0 \).

(B) Keep the notations and hypotheses of part (A). Then:
If \( x \in a^i(M) \) and if \( N \) is a second finitely generated graded \( R \)-module such that the graded \( R_x \)-modules \( M_x \) and \( N_x \) are isomorphic, then \( x \in a^i(N) \).

This follows immediately from the fact, that for all \( n \in \mathbb{Z} \) there is an isomorphism of \((R_0)_x\)-modules \((H_{R^+_x}(M)_n)_x \cong (H_{R^+_x}(N)_n)_x \). For our purposes the most significant application of this observation is:

If \( x \in a^i(M) \) then \( x \in a^i(M/\Gamma(x)(M)) \).

**Notation 3.2.** An element \( x \in R_0 \) is called a quasi-non-zero divisor with respect to (the finitely generated graded \( R \)-module) \( M \) if \( x \) is a non-zero divisor on \( M_n \) for all \( n \gg 0 \). We denote the set of these quasi-non-zero divisors by \( \text{NZD}^*_R_0(M) \). Thus in the notation of Reminder and Remark 2.2 (A) we may write

\[
\text{NZD}^*_R_0(M) = R_0 \setminus \bigcup_{p_0 \in \text{Ass}^*_R_0(M)} p_0.
\]

**Lemma 3.3.** Let \( i, k \in \mathbb{N}_0 \) and assume that \( \text{height}(p_0) \geq k \) for all \( p_0 \in \text{Ass}^*_R_0(M) \). Then, the set \( \text{Ass}^*_R_0(H^i_{R^+_x}(M)_n)^{\leq k+2} \) is asymptotically stable for \( n \to -\infty \). In particular, if \( k > 0 \), then \( C^i(M) \) is finite.

**Proof.** There is some integer \( n_0 \in \mathbb{Z} \) such that \((0 :_{R_0} M_{\geq n_0}) \subseteq R_0 \) is of height \( \geq k \), where we use the notation \( M_{\geq n_0} := \bigoplus_{n \geq n_0} M_n \). As \( H^i_{R^+_x}(M) \) and \( H^i_{R^+_x}(M_{\geq n_0}) \) differ only in finitely many degrees we may replace \( M \) by \( M_{\geq n_0} \) and hence assume that \( a_0 M = 0 \) for some ideal \( a_0 \subseteq R_0 \) with \( \text{height}(a_0) \geq k \). As \( \text{height}(p_0/a_0) \leq \text{height}(p_0) - k \) for all \( p_0 \in \text{Var}(a_0) \) and in view of the natural isomorphisms of \( R_0 \)-modules \( H^i_{R^+_x}(M)_n \cong H^i_{(R/a_0)R^+_x}(M)_n \) we now get a canonical bijection

\[
\text{Ass}^*_R_0(H^i_{R^+_x}(M)_n)^{\leq k+2} \leftrightarrow \text{Ass}^*_R_0(H^i_{R^+_x}(M_{a_0})_n)^{\leq 2},
\]

for all \( n \in \mathbb{Z} \). So, by Reminder and Remark 2.2 (D) the left hand side set is asymptotically stable for \( n \to -\infty \). If \( k > 0 \) the finiteness of \( C^i(M) \) now follows easily from statement (a) of Lemma 2.6. \( \square \)

Let \( i \in \mathbb{N}_0 \). According to Remark 2.4 (B) we know that \( M \) is uniformly tame at level \( i \) in codimension \( \leq 2 \). We also know that \( M \) need not be tame at level \( i \) in codimension 3. It is natural to ask, whether there are only finitely many primes \( p_0 \) of height 3 in \( R_0 \) such that \( M \) is not tame at level \( i \) in \( p_0 \) and whether outside of these “bad” primes the module \( M \) is uniformly tame at level \( i \) in codimension \( \leq 3 \). We aim to give a few sufficient criteria for this behaviour. The following theorem plays a crucial rôle in this respect.

**Theorem 3.4.** Let \( i \in \mathbb{N}_0 \). Assume that \( R_0 \) is a domain and that \( \text{NZD}^*_R_0(M) \cap a^i(M) \neq \emptyset \). Then \( C^i(M) \) is a finite set. In particular the set \( \text{Spec}(R_0)^{\leq 3} \setminus \mathcal{X}^i(M) \) consists of finitely many primes of height 3 and \( M \) is uniformly tame at level \( i \) along \( \mathcal{X}^i(M) \)^\leq 3.

**Proof.** If \( i \leq f(M) \) our claim is clear by Remark 2.4 (A) and Proposition 2.8 (c). So, let \( i > f(M) \). Then in particular \( i > 1 \).

Now, let \( m(M) \in \mathbb{Z} \) be as in Reminder and Remark 2.2 (A) and set \( N := M_{\geq m(M)} := \bigoplus_{n \geq m(M)} M_n \). Then \( \text{NZD}^*_R_0(M) \) equals the set \( \text{NZD}^*_R_0(N) \) of non-zero divisors in \( R_0 \) on
As \( i > 1 \) we have \( H^i_{R_+}(N) = H^i_{R_+}(M) \) and hence \( \mathfrak{a}^i(M) = \mathfrak{a}^i(N) \) and \( C^i(M) = C^i(N) \). So, we may replace \( M \) by \( N \) and hence assume that \( \text{NZD}_{R_0}(M) \cap \mathfrak{a}^i(M) \neq \emptyset \).

Let \( x \in \text{NZD}_{R_0}(M) \cap \mathfrak{a}^i(M) \). Then, the short exact sequence \( 0 \to M \xrightarrow{x} M \to M/xM \to 0 \) implies exact sequences

\[
H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_n \to H^i_{R_+}(M/xM)_n
\]

for all \( n \in \mathbb{Z} \). Now, let \( p_0 \in C^i(M) \) so that \( \text{height}(p_0) = 3 \) (s. Lemma 2.6). Then, there is an integer \( n \leq n'(M) \) such that \( p_0 \) is a minimal associated prime of \( H^i_{R_+}(M)_n \). We thus get an exact sequence of \((R_0)_{p_0}\)-modules

\[
(H^i_{R_+}(M)_n)_{p_0} \xrightarrow{\varphi} (H^i_{R_+}(M)_n)_{p_0} \xrightarrow{\varphi} (H^i_{R_+}(M/xM)_n)_{p_0}
\]

in which the middle module is of finite length \( \neq 0 \). As \( x \in \mathfrak{a}^i(M) \subseteq p_0 \) it follows by Nakayama that \( \varphi \) is not the zero map. Therefore \((H^i_{R_+}(M/xM)_n)_{p_0}\) contains a non-zero \((R_0)_{p_0}\)-module of finite length. It follows that \( p_0 \in \text{Ass}_{R_0}(H^i_{R_+}(M/xM)_n)^{\neq 3} \). This shows that \( C^i(M) \subseteq \text{Ass}_{R_0}(H^i_{R_+}(M/xM)_n)^{\neq 3} \). So, by Lemma 3.3 the set \( C^i(M) \) is finite. \( \square \)

**Corollary 3.5.** Let \( i \in \mathbb{N}_0 \). Assume that \( R_0 \) is a domain and that \( M_n \) is a torsion-free \( R_0 \)-module for all \( n \gg 0 \). Then the set \( C^i(M) \) is finite. In particular, \( M \) is uniformly tame at level \( i \) along \( \mathfrak{F}^i(M)^{\leq 3} \) and the set \( \text{Spec}(R_0)^{\leq 3} \setminus \mathfrak{F}(M) \) is finite.

*Proof.* By our hypotheses we have \( \text{NZD}^*_R(M) = R_0 \setminus \{0\} \). By Reminder 3.1 (A) we have \( \mathfrak{a}^i(M) \neq 0 \). Now we conclude by Theorem 3.4. \( \square \)

**Corollary 3.6.** Let \( i \in \mathbb{N}_0 \) and assume that \( R_0 \) is a domain and \( M \) is torsion-free over \( R_0 \). Then \( M \) is uniformly tame at level \( i \) along a set which is obtained by removing finitely many primes of height 3 from \( \text{Spec}(R_0)^{\leq 3} \).

*Proof.* This is clear by Corollary 3.5. \( \square \)

Our next aim is to replace the requirement that \( M_n \) is \( R_0 \) torsion-free for all \( n \gg 0 \), which was used in Corollary 3.5 by a weaker condition. We begin with the following finiteness result for certain subsets of critical sets:

**Proposition 3.7.** Let \( R_0 \) be a domain, let \( i \in \mathbb{N} \) and let \( x \in R_0 \setminus \{0\} \) be such that \( xG_{(x)}(M) = 0 \). Then

(a) \( [C^i(M) \setminus C^i(M/G_{(x)}(M))] \cup [\overline{\mathfrak{F}^{i-1}(M/xM)} \cap \overline{\mathfrak{F}^{i+1}(G_{(x)}(M))}]^{\neq 3} \) is a finite set.

(b) If \( x \in \mathfrak{a}^i(M) \), then the set \( C^i(M/G_{(x)}(M)) \) and hence also the set

\[
C^i(M) \setminus [\overline{\mathfrak{F}^{i-1}(M/xM)} \cap \overline{\mathfrak{F}^{i+1}(G_{(x)}(M))}]^{\neq 3} \setminus C^i(M/G_{(x)}(M))
\]

is finite.

*Proof.* (a): Fix an integer \( n_0 \leq n'(M/xM), n'(G_{(x)}(M)), n'(M), n'(M/G_{(x)}(M)) \) and let \( p_0 \in C^i(M) \). Then \( p_0 \in \min \text{Ass}_{R_0}(H^i_{R_+}(M)_n) \) for some \( n \leq n'(M) \). If \( n_0 \leq n \), \( p_0 \) thus belongs to the finite set \( \bigcup_{m \geq n_0} \text{Ass}_{R_0}(H^i_{R_+}(M)_m) \). So, let \( n < n_0 \). The graded short exact sequences

\[
0 \to M/G_{(x)}(M) \to M \to M/xM \to 0
\]
and

\[ 0 \rightarrow \Gamma(x)(M) \rightarrow M \rightarrow M/\Gamma(x)(M) \rightarrow 0 \]

imply exact sequences

\[
(H_{R_+}^{i-1}(M/xM)_n)_{p_0} \rightarrow (H_{R_+}^{i}(M/\Gamma(x)(M))_n)_{p_0} \rightarrow (H_{R_+}^{i}(M)_n)_{p_0} \rightarrow (H_{R_+}^{i}(M/xM)_n)_{p_0}
\]

and

\[
(H_{R_+}^{i}(M)_n)_{p_0} \rightarrow (H_{R_+}^{i}(M/\Gamma(x)(M))_n)_{p_0} \rightarrow (H_{R_+}^{i+1}(\Gamma(x)(M))_n)_{p_0}.
\]

Assume that \( p_0 \notin C^i(M/\Gamma(x)(M)) \). Then \((H_{R_+}^{i}(M/\Gamma(x)(M))_n)_{p_0}\) either vanishes or is an \((R_0)_{p_0}\)-module of finite length. In the first case we have \((H_{R_+}^{i}(M)_n)_{p_0} \subseteq (H_{R_+}^{i}(M/xM)_n)_{p_0} \). As \((H_{R_+}^{i}(M)_n)_{p_0}\) is a non-zero \((R_0)_{p_0}\)-module of finite length it follows \( p_0 \in \text{Ass}_{R_0}(H_{R_+}^{i}(M/xM)_n) \). So \( p_0 \) belongs to the finite set \( \text{Ass}_{R_0}(H_{R_+}^{i}(M/xM)) \) (see Remark 3.3.2).

Assume now that \((H_{R_+}^{i}(M/\Gamma(x)(M))_n)_{p_0}\) is not of finite length. Then, by the above sequences \((H_{R_+}^{i-1}(M/xM)_n)_{p_0}\) and \((H_{R_+}^{i+1}(\Gamma(x)(M))_n)_{p_0}\) are both of infinite length, so that \( p_0 \in \mathfrak{P}^{-i}(M/xM) \) and \( p_0 \in \mathfrak{P}^{i+1}(\Gamma(x)(M)) \).

(b): According to Reminder 3.1 (B) we have \( x \in \mathfrak{a}'(M/\Gamma(x)(M)) \). As moreover it holds \( x \in \text{NZD}_{R_0}(M/\Gamma(x)(M)) \) our claim follows be Theorem 3.4.

**Corollary 3.8.** Let \( i \in \mathbb{N}_0 \), let \( R_0 \) be a domain and assume that \( \text{height}(p_0) \geq 3 \) for all \( p_0 \in \text{Ass}_{R_0}^*(M) \) \( \setminus \{0\} \cup \mathfrak{P}(M) \). Then \( C^i(M) \) is a finite set. In particular the set \( \text{Spec}(R_0)^{\leq 3} \setminus \mathfrak{P}(M) \) is finite and \( M \) is uniformly tame at level \( i \) along the set \( \mathfrak{P}(M)^{\leq 3} \).

**Proof.** Let \( m(M) \in \mathbb{Z} \) be as in Reminder and Remark 2.2 (A) so that \( \text{Ass}_{R_0}^*(M_n) = \text{Ass}_{R_0}(M) \) for all \( n \geq m(M) \). As \( H_{R_+}^{i}(M) \) and \( H_{R_+}^{i}(M/\Gamma(x)(M)) \) differ only in finitely many degrees we may replace \( M \) by \( M_{\geq m(M)} \) and hence assume that \( \text{Ass}_{R_0}^*(M) = \text{Ass}_{R_0}(M) \). If \( 0 \notin \text{Ass}_{R_0}(M) \) we get our claim by Lemma 3.3. So, let \( 0 \in \text{Ass}_{R_0}(M) \) and consider the non-zero ideal \( b_0 := \bigcap_{p_0 \in \text{Ass}_{R_0}(M) \setminus \{0\}} p_0 \). Then \( \text{Ass}_{R_0}(M/\Gamma(b_0)(M)) = \{0\} \) so that \( M/\Gamma(b_0)(M) \) is torsion-free over \( R_0 \). Let \( x \in b_0 \setminus \{0\} \) with \( x\Gamma(x)(M) = 0 \). Then it follows that \( \Gamma(b_0)(M) = \Gamma(x)(M) \). By Corollary 3.5 we therefore obtain that \( C^i(M/\Gamma(x)(M)) \) is finite. According to Proposition 3.7 (a) it thus suffices to show that \( C^i(M) \cap \mathfrak{P}^{i+1}(\Gamma(b_0)(M))^{\leq 3} \) is finite. So, let \( q_0 \) be an element of this latter set. Then \( \text{height}(q_0) = 3 \) and \( q_0 \notin \mathfrak{P}(M) \). Moreover, there is a minimal prime \( p_0 \) of \( b_0 \) with \( p_0 \subseteq q_0 \). In particular \( p_0 \in \text{Ass}_{R_0}(M) \) \( \setminus \{0\} \) and \( p_0 \notin \mathfrak{P}(M) \). So, by our hypothesis \( \text{height}(p_0) \geq 3 \), whence \( q_0 = p_0 \in \text{Ass}_{R_0}^*(M) \) \( \setminus \{0\} \).

This shows that \( C^i(M) \cap \mathfrak{P}^{i+1}(\Gamma(b_0)(M))^{\leq 3} \subseteq \text{Ass}_{R_0}^*(M) \) and hence proves our claim.

**Remark 3.9.** Clearly Corollary 3.6 applies to the domain \( R' \) constructed in [13] (s. Example 2.5), taken as a module over itself. In this example we have in particular \( \mathfrak{S}(R')^{\leq 3} = \text{Spec}(R'_0) \) \( \setminus \{m_0\} \). Moreover the uniform tameness of \( R' \) at level 2 along this set can be verified by a direct calculation.
4. Conditions on Neighbouring Cohomologies for Tameness in Codimensions $\leq 3$

We keep the hypotheses and notations of the previous sections. So $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a homogeneous Noetherian ring whose base ring $R_0$ is essentially of finite type over a field and $M$ is a finitely generated graded $R$-module.

Our first result says that $M$ is tame in codimension $\leq 3$ at a given level $i \in \mathbb{N}$, if the two neighbouring local cohomology modules $H^{i-1}_R(M)$ and $H^{i-2}_R(M)$ are “asymptotically sufficiently small”. (We set $H^k_{R_n}(\bullet) := 0$ for $k < 0$). We actually shall prove a more specific statement. To formulate it, we first introduce an appropriate notion.

**Definition and Remark 4.1.** (A) We say that a graded $R$-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is almost Artinian if there is some graded submodule $N = \bigoplus_{n\in\mathbb{Z}} N_n \subseteq T$ such that $N_n = 0$ for all $n < 0$ and such that the graded $R$-module $T/N$ is Artinian.

(B) A graded $R$-module $T$ which is the sum of an Artinian graded submodule and a Noetherian graded submodule clearly is almost Artinian. Moreover, the property of being almost Artinian passes over to graded subquotients.

(C) As $R_0$ is Noetherian and $R$ is homogeneous each graded almost Artinian $R$-module $T$ has the property that $\dim_{R_0}(T_n) \leq 0$ for all $n < 0$.

(D) Clearly an almost Artinian graded $R$-module is tame.

Now, we are ready to formulate and to prove the announced result.

**Theorem 4.2.** Let $i \in \mathbb{N}$ such that $\dim_{R_0}(H^{i-1}_R(M)_n) \leq 1$ and $\dim_{R_0}(H^{i-2}_R(M)_n) \leq 2$ for all $n \ll 0$. Then the following statements hold.

(a) The graded $R_{p_0}$-module $H^{i}_{R_{p_0}}(M)_{p_0}$ is almost Artinian for all $p_0 \in \text{Spec}(R_0)^{=3} \setminus \mathfrak{P}^{3}(M)$.

(b) $\mathfrak{T}^{i}(M)^{\leq 3} = \text{Spec}(R_0)^{=}^{3}$ and hence $M$ is tame at level $i$ in codimension $\leq 3$.

**Proof.** (a): Let $p_0 \in \text{Spec}(R_0)^{=}^{3} \setminus \mathfrak{P}^{3}(M)$. We consider the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{p_0}(H^q_{R_{p_0}}(M))_{p_0} \Rightarrow H^{p+q}_{p_0+R_{p_0}}(M)_{p_0}.$$  

By our assumption on the dimension of the $R_0$-modules $H^{i-1}_R(M)_n$ and $H^{i-2}_R(M)_n$, the $n$-th graded component $(E_2^{p,q})_n$ of the graded $R_{p_0}$-module $E_2^{p,q}$ vanishes for all $n \ll 0$ if $(p,q) = (2,i-1)$ or $(p,q) = (3,i-2)$. Therefore

$$(E_2^{0,i})_n \cong (E_\infty^{0,i})_n, \quad \forall n \ll 0.$$  

As the graded $R_{p_0}$-module $E_0^{0,i}$ is a subquotient of the Artinian $R_{p_0}$-module $H^{i}_{p_0+R_{p_0}}(M)_{p_0}$, it follows by Definition and Remark 4.1 (B) that the graded $R_{p_0}$-module

$$H^{0}_{p_0+R_{p_0}}(H^{i}_{R_{p_0}}(M)_{p_0}) \cong H^{0}_{p_0}(H^{i}_{R_{p_0}}(M)_{p_0}) = E_2^{0,i}$$

is almost Artinian. Now, since $p_0 \notin \mathfrak{P}^{3}(M)$ and $p_0$ is of height 3 we must have

$$\dim_{R_0}(H^{i}_{R_{p_0}}(M)_{p_0}) \leq 0, \quad \forall n \ll 0.$$  

and hence $H^{0}_{p_0+R_{p_0}}(H^{i}_{R_{p_0}}(M)_{p_0})$ and $H^{i}_{R_{p_0}}(M)_{p_0}$ coincide in all degrees $n \ll 0$. Therefore $H^{i}_{R_{p_0}}(M)_{p_0}$ is indeed almost Artinian.
(b): This follows immediately from statement (a), as $\overline{\mathfrak{F}}(M) \subseteq \mathfrak{T}'(M)$ (s. Remark 2.4 (B)).

**Remark 4.3.** The domain $R'$ constructed in [13] (s. Example 2.5), taken as a module over itself, clearly cannot satisfy the hypotheses of Theorem 4.1 with $i = 2$ as it does not fulfill the corresponding conclusion of this theorem. Indeed a direct calculation shows that $\dim R_0(H_{R_+}^1(R',n)) = 3$ for all $n < 0$.

Our next result says that the module $M$ is tame at level $i$ almost everywhere in codimension $\leq 3$ provided that $R_0$ is a domain and the local cohomology module $H_{R_+}^i(M)$ is “asymptotically very small”. Again, we aim to prove a more specific result.

**Theorem 4.4.** Let $R_0$ be a domain and $i \in \mathbb{N}$ such that $\dim R_0(H_{R_+}^{i-1}(M)) \leq 0$ for all $n \ll 0$. Then the following statements hold.

(a) There is a finite set $Z \subset \text{Spec}(R_0)^{=3}$ such that the graded $R_{p_0}$-module $H_{R_+}^i(M)_{p_0}$ is almost Artinian for all $p_0 \in \text{Spec}(R_0)^{=3} \setminus (Z \cup \overline{\mathfrak{F}}(M))$.

(b) $\text{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}'(M)$ is a finite subset of $\text{Spec}(R_0)^{=3}$.

**Proof.** (a): According to Reminder 3.1 (A) there is an element $x \in \mathfrak{a}'(M) \setminus \{0\}$ such that $x \Gamma(x)(M) = 0$. If we apply Lemma 3.3 with $k = 1$ to the $R$-module $M/xM$ (also with $i - 1$ instead of $i$) and to the $R$-module $\Gamma(x)(M)$ (with $i + 1$ instead of $i$) we see that the three sets

$$\text{Ass}_{R_0}(H_{R_+}^{i-1}(M/xM)_n)^{\leq 3}, \quad \text{Ass}_{R_0}(H_{R_+}^i(M/xM)_n)^{\leq 3}, \quad \text{Ass}_{R_0}(H_{R_+}^i(\Gamma(x)(M)_n)^{\leq 3}$$

are asymptotically stable for $n \to -\infty$. So, there is a finite set $Z \subset \text{Spec}(R_0)^{=3}$ such that

$$\text{Ass}_{R_0}(H_{R_+}^{i-1}(M/xM)_n)^{=3} \cup \text{Ass}_{R_0}(H_{R_+}^i(M/xM)_n)^{=3} \cup \text{Ass}_{R_0}(H_{R_+}^{i+1}(\Gamma(x)(M)_n)^{=3} = Z$$

for all $n \ll 0$. Let

$$p_0 \in \text{Spec}(R_0)^{=3} \setminus (Z \cup \overline{\mathfrak{F}}(M))$$

We aim to show that the graded $R_{p_0}$-module $H_{R_+}^i(M)_{p_0}$ is almost Artinian. As $p_0 \notin \overline{\mathfrak{F}}(M)$ and $\text{height}(p_0) = 3$ it follows

$$\text{length}_{(R_0)_{p_0}}(H_{R_+}^i(M)_{p_0}) < \infty$$

for all $n \ll 0$. As $\dim_{R_0}(H_{R_+}^{i-1}(M)_n) \leq 0$ for all $n \ll 0$ we also have

$$\text{length}_{(R_0)_{p_0}}(H_{R_+}^{i-1}(M)_{p_0}) < \infty$$

for all $n \ll 0$. As $p_0 \notin Z$ and $\text{height}(p_0) = 3$, we also can say

$$\Gamma_{p_0}(R_0)_{p_0}((H_{R_+}^{i-1}(M/xM)_n)_{p_0}) = \Gamma_{p_0}(R_0)_{p_0}((H_{R_+}^i(M/xM)_n)_{p_0}) =$$

$$\Gamma_{p_0}(R_0)_{p_0}((H_{R_+}^{i+1}(\Gamma(x)(M)_n)_{p_0}) = 0, \quad \forall n \ll 0.$$

Now, as in the proof of Proposition 3.8 (a), the canonical graded short exact sequences

$$0 \to M/\Gamma(x)(M) \overset{\phi}{\to} M \to M/xM \to 0$$
and
\[0 \to \Gamma_{(x)}(M) \to M \xrightarrow{\pi} M/\Gamma_{(x)}(M) \to 0\]
respectively imply exact sequences of \((R_0)_{p_0}\)-modules
\[(H_{R_+}^{i-1}(M)_{p_0} \to (H_{R_+}^{i-1}(M/xM)_{n})_{p_0} \to \]
\[(H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0} \to (H_{R_+}^i(M/xM)_{n})_{p_0} \]
and
\[(H_{R_+}^i(M)_{n})_{p_0} \to (H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0} \to (H_{R_+}^{i+1}(\Gamma_{(x)}(M))_{n})_{p_0}\]
for all \(n < 0\). Keep in mind, that in the first of these sequences the first and the second but last module are of finite length for all \(n < 0\), whereas the second and the last module are \(p_0(R_0)_{p_0}\)-torsion-free for all \(n < 0\). Observe further, that in the second sequence the first module is of finite length and the last module is \(p_0(R_0)_{p_0}\)-torsion-free for all \(n < 0\).
So there is an integer \(n(x)\) such that for each \(n \leq n(x)\) we have the exact sequence
\[0 \to (H_{R_+}^{i-1}(M/xM)_{n})_{p_0} \to (H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0} \to (H_{R_+}^i(M/xM)_{n})_{p_0} \to 0\]
and the relation
\[\text{Im}(H_{R_+}^i(\pi)_n)_{p_0} = \Gamma_{p_0(R_0)_{p_0}}(H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0}\].
Thus, for all \(n \leq n(x)\) the image of the composite map
\[(H_{R_+}^i(\pi)_n)_{p_0} \circ (H_{R_+}^i(\phi)_n)_{p_0} : (H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0} \to (H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0}\]
is the torsion module \(\Gamma_{p_0(R_0)_{p_0}}((H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0})\). As the composite map \(\pi \circ \phi : M/\Gamma_{(x)}(M) \to M/\Gamma_{(x)}(M)\) coincides with the multiplication map \(x = x\text{Id}_{M/\Gamma_{(x)}(M)}\) on \(M/\Gamma_{(x)}(M)\) we end up with
\[\Gamma_{p_0(R_0)_{p_0}}((H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0}) = x(H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0}, \quad \forall n \leq n(x).\]
Now, without affecting \(\Gamma_{(x)}(M)\) we may replace \(x\) by \(x^2\) and thus get the equalities
\[x(H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0} = x^2(H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0}\]
for all \(n \leq m(x) := \min\{n(x), n(x^2)\}\). Consequently, as \(x \in p_0\) and as the \((R_0)_{p_0}\)-modules \((H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0}\) are finitely generated, it follows by Nakayama that
\[\Gamma_{p_0(R_0)_{p_0}}((H_{R_+}^i(M/\Gamma_{(x)}(M))_{n})_{p_0}) = 0, \quad \forall n < 0.\]
Applying the functor \(\Gamma_{p_0(R_0)_{p_0}}(\bullet)\) to the above short exact sequences and keeping in mind that the right hand side module in these sequences is of finite length, we get the natural monomorphisms
\[0 \to (H_{R_+}^i(M)_{n})_{p_0} \to H_{p_0(R_0)_{p_0}}^1(H_{R_+}^{i-1}(M/xM)_{n})_{p_0}, \quad \forall n \leq m(x).\]
It is easy to see, that these monomorphisms are the graded parts of a homomorphism of graded \(R_{p_0}\)-modules. Moreover, as \(\dim((R_0/xR_0)_{p_0}) \leq 2\) the graded \(R_{p_0}\)-module
\[H_{p_0(R_0)_{p_0}}^1(H_{R_+}^{i-1}(M/xM)_{p_0}) \cong H_{p_0(R_0/xR_0)_{p_0}}^1(H_{(R/xR)_{p_0}}^{i-1}(M/xM)_{p_0})\]
is Artinian (s. [10] Theorem 5.10). In view of the observed monomorphisms and by Definition and Remark 4.1 (B), this implies immediately, that the graded $R_{p_0}$-module $(H^i_{R_+}(M))_{p_0}$ is almost Artinian.

(b): This follows immediately from statement (a), Reminder and Remark 4.1 (D) and Remark 2.4 (B).

This leads us immediately to the following observation.

**Corollary 4.5.** If $R_0$ is a domain and $i \in \mathbb{N}$ is such that the $R$-module $H^{i-1}_{R_+}(M)$ is almost Artinian, then the set of all primes $p_0 \in \text{Spec}(R_0) \leq 3 \setminus \mathfrak{T}(M)$ for which the graded $R_{p_0}$-module $H^i_{R_+}(M)_{p_0}$ is not almost almost Artinian as well as the set $\text{Spec}(R_0) \leq 3 \setminus \mathfrak{T}(M)$ are both finite subsets of $\text{Spec}(R_0) = 3$.

**Proof.** This is immediate by Theorem 4.4 and Definition and Remark 4.1 (C).

**References**


University of Zürich, Mathematics Institute, Winterthurerstrasse 190, 8057 Zürich.
E-mail address: brodmann@math.uzh.ch

Faculty of Mathematical Sciences and Computer, Tarbiat Moallem University, Tehran, Iran and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5746, Tehran, Iran.
E-mail address: jahangiri.maryam@gmail.com