Birkhoff coordinates for KdV on phase spaces of distributions

T. Kappeler[*], C. Möhr, P. Topalov[†]

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Abstract

The purpose of this paper is to extend the construction of Birkhoff coordinates for the KdV equation from the phase space of square integrable 1-periodic functions with mean value zero to the phase space $H_0^{-1}(\mathbb{T}^1)$ of mean value zero distributions from the Sobolev space $H^{-1}(\mathbb{T}^1)$ endowed with the symplectic structure $(\partial/\partial x)^{-1}$. More precisely we construct a globally defined real analytic symplectomorphism $\Omega : H_0^{-1}(\mathbb{T}^1) \to \mathfrak{h}^{-1/2}$ where $\mathfrak{h}^{-1/2}$ is a weighted Hilbert space of sequences $(x_n, y_n)_{n\geq 1}$ supplied with the canonical Poisson structure so that the KdV Hamiltonian for potentials in $H_0^1(\mathbb{T}^1)$ is a function of the actions $((x_n^2 + y_n^2)/2)_{n\geq 1}$ alone.

1 Introduction

The aim of this paper is to construct Birkhoff coordinates for the Korteweg-de Vries equation (KdV)

$$q_t = -q_{xxx} + 6qq_x \tag{1.1}$$

on the subspace $H_0^{-1}(\mathbb{T}^1)$ of the Sobolev space of 1-periodic real valued distributions $H^{-1}(\mathbb{T}^1)$ with mean value zero. More precisely, we show that there

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are canonical coordinates $(x_n, y_n)_{n\geq 1}$ on $H_0^{-1}(\mathbb{T}^1)$ in which the KdV equation for sufficiently regular initial data takes the form

$$\begin{aligned} \dot{x}_n &= \omega_n(I) y_n \\ \dot{y}_n &= -\omega_n(I) x_n \end{aligned} \tag{1.2}$$

with frequencies $(\omega_n(I))_{n\geq 1}$ which only depend on the actions $I_1, I_2, \ldots$ given by $I_k = \frac{1}{2}(x_k^2 + y_k^2)$ $(k \geq 1)$. Recall that the KdV equation on the circle $\mathbb{T}^1$ can be expressed as a Hamiltonian system on the phase space $H^\alpha(\mathbb{T}^1)$ – for $\alpha \in \mathbb{Z}_{\geq 0}$ sufficiently large – of real 1-periodic functions in the Sobolev space H^α endowed with the Poisson structure $\frac{d}{dx}$,

$$q_t = \frac{d}{dx}\frac{\partial H}{\partial q}. \tag{1.3}$$

Here H denotes the KdV Hamiltonian

$$H(q) \stackrel{\text{Def}}{=} \int_{\mathbb{T}^1} (\frac{q_x^2}{2} + q^3) dx \tag{1.4}$$

and $\frac{\partial H}{\partial q}$ the L^2-gradient of H – see e.g. [13] for more details.
As the mean value functional $M(q) \stackrel{\text{Def}}{=} [q]$ of an element $q \in H^\alpha(\mathbb{T}^1)$ has gradient $\frac{\partial M}{\partial q} = 1$ one has

$$\frac{d}{dx}\frac{\partial M}{\partial q} = 0.$$

Therefore M is a Casimir functional for the Poisson structure $\frac{d}{dx}$. In particular, the space $H_0^\alpha(\mathbb{T}^1)$ of potentials q in $H^\alpha(\mathbb{T}^1)$ with mean value $[q] = 0$ is an invariant subspace for every Hamiltonian flow defined in terms of the Poisson structure $\frac{d}{dx}$ with α chosen appropriately. Hence in the following, we restrict our attention to the phase space $H_0^\alpha(\mathbb{T}^1)$. Although the Hamiltonian vector field $\frac{d}{dx}\frac{\partial H}{\partial q} = -q_{xxx} + 6qq_x$ of the KdV Hamiltonian H seems only be well defined if q is at least in $H^3(\mathbb{T}^1)$, it has been shown in [13] (cf. also [1]) that there exist closed formulas for the frequencies ω_n in (1.2) for smooth potentials which continue to be well defined for certain classes of distributions. This approach has been used in [15] to prove that KdV is globally wellposed in $H^{-\alpha}(\mathbb{T}^1, \mathbb{R})$ for any $\alpha \leq 1$ – see [1] as well as [3], [4], [5], [9], [17] for related results. In particular, our results will apply to $H_0^{-1/2}(\mathbb{T}^1)$ which is, in the terminology of Kuksin [21] (cf. also [4]), the symplectic Hilbert space

of KdV. In such a space, typically, interesting nonsqueezing phenomena for solutions can be established.
To state our main result let us introduce some notations. Denote for $0 \leq \alpha \leq 1$ by $H_0^{-\alpha}(\mathbb{T}^1)$ the Sobolev space

$$H_0^{-\alpha}(\mathbb{T}^1) \stackrel{\text{Def}}{=} \{f = \sum_{k \neq 0} \hat{f}(k) e^{2\pi i k x} |\ \hat{f}(-k) = \overline{\hat{f}(k)}, ||f||_{H^{-\alpha}(\mathbb{T}^1)} < \infty\}$$

where

$$||f||_{H^{-\alpha}(\mathbb{T}^1)} \stackrel{\text{Def}}{=} \Big(|\hat{f}(0)|^2 + \sum_{k \in \mathbb{Z} \setminus 0} |k|^{-2\alpha} |\hat{f}(k)|^2 \Big)^{1/2}.$$

If $\alpha = 0$ we write simply $||f||$. For any $s \in \mathbb{R}$, denote by $h^s \stackrel{\text{Def}}{=} h^s(\mathbb{N}, \mathbb{R})$ the weighted l^2-sequence space

$$h^s \stackrel{\text{Def}}{=} \{x = (x_n)_{n \geq 1} |\ ||x||_s < \infty\}$$

where

$$||x||_{h^s} \stackrel{\text{Def}}{=} \Big(\sum_{n \geq 1} n^{2s} |x_n|^2 \Big)^{1/2}.$$

Endow $H_0^{-\alpha}(\mathbb{T}^1)$ with the Poisson bracket $\{F, G\} \stackrel{\text{Def}}{=} \int_{\mathbb{T}^1} \frac{\partial F}{\partial q} \frac{d}{dx} \frac{\partial G}{\partial q} dx$ – see explanations below – and the Hilbert space

$$\mathfrak{h}^s \stackrel{\text{Def}}{=} \{(x_n, y_n)_{n \geq 1} |\ (x_n)_{n \geq 1}, (y_n)_{n \geq 1} \in h^s\}$$

with the standard Poisson bracket for which $\{x_n, y_m\} = \delta_{nm}$ while all other brackets vanish.

Theorem 1.1. *There exists a diffeomorphism $\Omega : H_0^{-1}(\mathbb{T}^1) \to \mathfrak{h}^{-1/2}$ with the following properties:*

(i) Ω is one-to-one, onto, bi-analytic and preserves the Poisson bracket.

(ii) For any $0 \leq \alpha < 1$, the restriction $\Omega_{-\alpha}$ of Ω to $H_0^{-\alpha}(\mathbb{T}^1)$ is a map $\Omega_{-\alpha} : H_0^{-\alpha}(\mathbb{T}^1) \to \mathfrak{h}^{-\alpha+1/2}$ which is one-to-one, onto, and bi-analytic as well.

(iii) Ω_0 coincides with the diffeomorphism constructed in [13, Theorem 5.1]. In particular, the coordinates (x, y) in $\mathfrak{h}^{3/2}$ are global Birkhoff coordinates for the KdV equation. That is, the transformed KdV Hamiltonian $H \circ \Omega^{-1}$ depends only on $x_n^2 + y_n^2$, $n \geq 1$, with (x, y) being canonical coordinates in $\mathfrak{h}^{3/2}$.

Remark 1.2. *Theorem 1.1 improves and completes results in [24] where by a different approach the maps* $\Omega_{-\alpha}$ *have been constructed for* $0 \leq \alpha < 1$ *leaving open the case* $\alpha = 1$ *and the problem of the ontoness of* $\Omega_{-\alpha}$.

Let us give a brief overview of the proof of Theorem 1.1. Following an approach developed in [11] and expanded on in [13], we extend the Birkhoff map established there for potentials in L^2_0 to potentials in $H^{-1}_0(\mathbb{T}^1)$, using the spectral results of the Schrödinger operator $-\frac{d^2}{dx^2} + q$ for $q \in H^{-1}_0(\mathbb{T}^1)$ from [14] (cf. also [20]). The stated results for $\Omega_{-\alpha}$ with $0 \leq \alpha < 1$ are then proved by using the spectral results for $-\frac{d^2}{dx^2} + q$ with q in $H^{-\alpha}_0(\mathbb{T}^1)$ established in [12]. The construction of $\Omega \equiv \Omega_{-1}$ goes as follows. In Section 2, we define action variables by formulas due to Flaschka and McLaughlin [10], prove their analyticity and derive a formula for their gradients. In Section 3 we define angle variables θ_n in terms of the Abel map with the help of holomorphic differentials, constructed in Appendix B, and the Dirichlet eigenvalues $\mu_n = \mu_n(q)$ $(n \geq 1)$ of $-\frac{d^2}{dx^2} + q$ studied in [14]. For any $n \geq 1$, the angle θ_n will be defined on the dense subset $\mathcal{W} \setminus D_n$ of $\mathcal{W}$ where $\mathcal{W}$ is a complex neighborhood of $H^{-1}_0(\mathbb{T}^1)$ in $H^{-1}_0(\mathbb{T}^1, \mathbb{C})$ independent of n and D_n denotes the set of potentials with collapsed n-th gap,

$$D_n \overset{\text{Def}}{=} \{q \in \mathcal{W} | \ \gamma_n(q) = 0\}$$

(cf. Section 3 for details) [1]. In Section 4, the cartesian coordinates x_n and y_n, associated to the actions and angles, are introduced. We show that x_n and y_n, initially defined only on $\mathcal{W} \setminus D_n$, can be extended real analytically to $\mathcal{W}$. Using suitable asymptotic estimates, we obtain the analytic map

$$\Omega : q \mapsto (x_n(q), y_n(q))_{n \geq 1}$$

from $\mathcal{W}$ into the sequence space $\mathfrak{h}^{-1/2}$. In Section 5, we derive canonical relations among the coordinates. These relations will follow by continuity and density from the corresponding relations for L^2-potentials established in [13] (cf. also [23]). With the help of these relations, we show in Section 6 that the map Ω is a local diffeomorphism. In Section 7 we prove that Ω is bijective, using a priori estimates established in [14] and show the remaining statements of Theorem 1.1. Note that for the proof of the ontoness of Ω a

[1] In the sequel we may need to shrink this neighborhood several times, but nevertheless will denote it by the same symbol throughout.

novel approach is needed as the approach used for Ω_0 relied on the identity $\frac{1}{2}||q||^2 = \sum_{j\geq 1} 2\pi j I_j$ which no longer makes sense for $q \in H_0^{-1}(\mathbb{T}^1)$.
Let us introduce some more notation which will be used in the sequel – see also [13, 24, 14]. Given a C^1-functional $F : \mathcal{W} \to \mathbb{C}$, defined on an open subset of $H^s_{\mathbb{C}} = H^s(\mathbb{T}^1, \mathbb{C})$ for some $s \in \mathbb{R}$, the L^2-*gradient*

$$\partial_q F \equiv \frac{\partial F}{\partial q} \in H^{-s}_{\mathbb{C}}$$

of F at the point $q \in \mathcal{W}$ is the unique element in $H^{-s}_{\mathbb{C}}$ such that

$$d_q F(h) = \left\langle h, \overline{\partial_q F} \right\rangle \ \forall h \in H^s_{\mathbb{C}}.$$

Here, $d_q F \in (H^s_{\mathbb{C}})'$ denotes the derivative of F at the point q, and the pairing $\langle \cdot, \cdot \rangle$ is the sesquilinear pairing of $H^s_{\mathbb{C}}$ and $H^{-s}_{\mathbb{C}}$ extending the L^2-inner product. Note that the complex conjugation $\overline{\partial_q F}$ is used in the above definition of the gradient in order to ensure that the correspondence

$$q \mapsto \partial_q F, \quad q \in \mathcal{W}, \tag{1.5}$$

is analytic from $\mathcal{W}$ into $H^{-s}_{\mathbb{C}}$. The L^2_0-gradient of a functional, given on $\mathcal{W} \cap H^s_0(\mathbb{T}^1, \mathbb{C})$, is defined similarly.
Functionals, such as the actions, naturally extend from a (sufficiently small) complex neighborhood of H^s_0 in $H^s_{0,\mathbb{C}}$ to a complex neighborhood of H^s in $H^s_{\mathbb{C}}$, and their gradients turn out to be elements in $H^{-s}_{\mathbb{C}}$ with mean value 0. Whenever it is not necessary we will not distinguish between the L^2- and the L^2_0-gradient.
Given two C^1-functionals $F, G : \mathcal{W} \to \mathbb{C}$, we define their *bracket* $[F, G]$ to be

$$[F, G] \stackrel{\text{Def}}{=} \left\langle \partial F, \overline{\partial_x \partial G} \right\rangle, \ \partial_x \stackrel{\text{Def}}{=} d/dx, \ \partial \stackrel{\text{Def}}{=} \partial_q$$

provided that the latter pairing is well defined. As $\left\langle \partial F, \overline{\partial_x \partial G} \right\rangle = \left\langle \partial_x \partial G, \overline{\partial F} \right\rangle$ the bracket can be written as

$$[F, G] = dF(\partial_x \partial G).$$

If both brackets $[F, G]$ and $[G, F]$ are well defined we write

$$\{F, G\} := [F, G]$$

and refer to $\{F, G\}$ as the *Poisson bracket* between F and G. As a particular case note that for a functional F which is analytic on a complex neighborhood $\mathcal{W}$ of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ the gradient ∂F is in $H_0^1(\mathbb{T}^1, \mathbb{C})$, hence the derivative $\partial_x \partial F$ is in $L_0^2(\mathbb{T}^1, \mathbb{C})$. Consequently, given two such functionals F and G, their bracket $[F, G]$ is always defined and, moreover, an analytic function of q. This fact helps to overcome the complications in [13] arising from the fact that the brackets are not per-se well defined if the functions F and G are only known to be analytic on a complex neighborhood of L_0^2 instead of $H_0^{-1}(\mathbb{T}^1)$.
Given a linear operator $A \in \mathcal{L}(H_{\mathbb{C}}^s)$, the *dual operator* $A^* \in \mathcal{L}(H_{\mathbb{C}}^{-s})$ is defined to be the adjoint of A with respect to the pairing $\langle \cdot, \cdot \rangle$, i.e.

$$\langle f, A^* g \rangle = \langle Af, g \rangle \text{ for } f \in H_{\mathbb{C}}^s,\ g \in H_{\mathbb{C}}^{-s}.$$

Finally we introduce the following useful notation.

Definition 1.3. *For any $\epsilon > 0$ and any elements a, b in a Banach space with norm $\| \cdot \|$, we write*

$$a = b + \mathtt{le}(\epsilon)$$

if $||a - b|| \leq \epsilon$. This notation is also used when $b = 0$.

2 Actions

Let $\mathcal{W}$ be a complex neighborhood of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ chosen as in Corollary 10.2 (Appendix C, Section 10.1). Hence, for any $p \in \mathcal{W}$, we can choose a neighborhood $V(p) \subset \mathcal{W}$ of p and mutually disjoint convex open neighborhoods $U_n \subset \mathbb{C}$ $(n \geq 1)$ containing for any $q \in V(p)$ a circuit Γ_n (independent of q and with counterclockwise orientation) around the gap interval

$$G_n(q) = \{(1 - t)\lambda_{2n-1}(q) + t\lambda_{2n}(q) \mid 0 \leq t \leq 1\}.$$

Such a system of neighborhoods is refered to as a *system of isolating neighborhoods* (cf. Section 10.1 and Definition 10.3 for the details). Following [13], we then define the action variables $I_n(q)$ for $q \in V(p)$ by

$$I_n(q) \overset{\mathtt{Def}}{=} \frac{1}{\pi} \int_{\Gamma_n} \lambda \frac{\dot{\Delta}(\lambda, q)}{\sqrt[c]{\Delta^2(\lambda, q) - 4}}\, d\lambda, \quad n \geq 1 \tag{2.1}$$

where $\dot{\Delta}(\lambda, q)$ is the λ-derivative of the discriminant $\Delta(\lambda, q)$ and $\sqrt[c]{\Delta^2 - 4}$ the *canonical root* – see Appendix C, Section 10.2 for the definition of the discriminant and page 50 for the definition of the canonical root. By Proposition 10.5, the integrand in the definition of $I_n(q)$ is analytic on an open neighborhood of $\Gamma_n \times V(p)$, hence $I_n(q)$ is an analytic function on $V(p)$. In view of the definition (2.1), the action variable $I_n(q)$ is independent of the choice of the contour Γ_n as long as it stays inside U_n, hence for any $n \geq 1$ the actions I_n are analytic functions on $\mathcal{W}$.
Note that the action variable $I_n(q)$ is defined by (2.1) for a complex neighborhood of $H^{-1}(\mathbb{T}^1)$ and as in [13, Theorem 6.1] one proves that the L^2-gradient of I_n is given by

$$\partial I_n(q) = -\frac{1}{\pi} \int_{\Gamma_n} \frac{\partial \Delta(\lambda, q)}{\sqrt[c]{\Delta^2(\lambda, q) - 4}} \, d\lambda. \tag{2.2}$$

The L^2-gradient ∂I_n belongs to $H^1(\mathbb{T}^1, \mathbb{C})$ and has mean value zero,

$$[\partial I_n] = \int_0^1 \partial I_n dx = \langle \partial I_n, 1 \rangle = \frac{d}{d\epsilon}|_{\epsilon=0} \, I_n(q + \epsilon) = 0$$

where the bracket $\langle \cdot, \cdot \rangle$ denotes the sesquilinear pairing between $H^{-1}(\mathbb{T}^1, \mathbb{C})$ and $H^1(\mathbb{T}^1, \mathbb{C})$. Arguing as in [13, Theorem 6.1] we find that for any $n \geq 1$ on the real space $H_0^{-1}(\mathbb{T}^1)$,

$$I_n \geq 0, \quad \text{and} \quad I_n = 0 \Leftrightarrow \gamma_n = 0.$$

Note that the actions I_n are compact in the sense of Proposition 10.7.
Let D_n be the subvariety of potentials in $\mathcal{W}$ with collapsed n-th gap,

$$D_n \overset{\texttt{Def}}{=} \{q \in \mathcal{W} \mid \gamma_n(q) = 0\}. \tag{2.3}$$

As I_n and γ_n^2 are analytic on $\mathcal{W}$, the quotient I_n/γ_n^2 is analytic on $\mathcal{W} \setminus D_n$. The following proposition is an extension of Theorem 6.3 in [13] for potentials in $H_0^{-1}(\mathbb{T}^1)$.

Proposition 2.1. *There exists a complex neighborhood $\mathcal{W}$ of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ such that the quotient I_n/γ_n^2 extends analytically to $\mathcal{W}$ for all n and satisfies the asymptotic formula*

$$8\pi n \frac{I_n}{\gamma_n^2} = 1 + \texttt{le}(\epsilon)$$

locally uniformly on $\mathcal{W}$ and uniformly in $n \geq n_0$ (with n_0 depending on $\epsilon > 0$ and the local neighborhood in $\mathcal{W}$) for arbitrary chosen $\epsilon > 0$ with $\mathtt{le}(\epsilon)$ as in Definition 1.3. Moreover,

$$\xi_n \overset{\mathtt{Def}}{=} \sqrt[+]{\frac{8I_n}{\gamma_n^2}}$$

is well-defined as a real analytic, non-vanishing function on $\mathcal{W}$ and satisfies for $n \geq n_0$

$$\xi_n = \frac{1}{\sqrt{\pi n}}\,(1 + \mathtt{le}(\epsilon))$$

locally uniformly on $\mathcal{W}$. For $q = 0$, we have $\xi_n = \frac{1}{\sqrt{\pi n}}$ for all $n \geq 1$.

Proof. Using the asymptotic estimates established in [14] (cf. Proposition 10.1 in the the present paper) and Propositions 10.9 and 10.10 as well as the product representation (10.9) of $\dot{\Delta}(\lambda)$, the result can be obtained in the same way as Theorem 6.3 in [13]. □

3 Angles

In order to define angle variables on $\mathcal{W}$ we associate to a Dirichlet eigenvalue $\mu_n(q)$ of the Hill operator $L_q \overset{\mathtt{Def}}{=} -\frac{d^2}{dx^2} + q$ (cf. [14]) a point

$$\mu_n^*(q) = \left(\mu_n(q), \sqrt[*]{\Delta^2(\mu_n(q), q) - 4}\right), \tag{3.1}$$

refered to as *Dirichlet divisor* [2], on the Riemann surface

$$\Sigma_q \overset{\mathtt{Def}}{=} \{(\lambda, w)|\ w^2 = \Delta^2(\lambda, q) - 4\}.$$

The square root $\sqrt[*]{\Delta^2(\mu_n(q), q) - 4}$ is defined by

$$\sqrt[*]{\Delta^2(\mu_n(q), q) - 4} = \tilde{y}_1(1, \tilde{\mu}_n, r) - \tilde{y}_2'(1, \tilde{\mu}_n, r)$$

where $\tilde{\mu}_n = \mu_n + ||r||^2$, $r \overset{\mathtt{Def}}{=} R^{-1}(q)$ with R denoting the Riccati map (cf. [14] and Appendix C), and $\tilde{y}_i(x, \tilde{\lambda}, r)$ $(i = 1, 2)$ are the fundamental solutions of the equation $-u'' - 2ru' = \tilde{\lambda}u$.

[2] It is not hard to see (cf. the definition of the discriminant $\Delta(\lambda, q)$ in Appendix C) that $\Delta^2(\mu_n(q), q) - 4 = (\tilde{y}_1(1, \tilde{\mu}_n, r) - \tilde{y}_2'(1, \tilde{\mu}_n, r))^2$ hence indeed $\mu_n^*(q) \in \Sigma(q)$.

The angles θ_n are defined using the holomorphic differentials ψ_n – see Section 9 (Appendix B). Let $n \geq 1$ and recall that $D_n = \{q \in \mathcal{W} \mid \gamma_n(q) = 0\}$. For q in $\mathcal{W} \setminus D_n$, define

$$\theta_n(q) \overset{\mathtt{Def}}{=} \eta_n(q) + \sum_{\substack{k \geq 1 \\ k \neq n}} \beta_k^n(q), \tag{3.2}$$

where

$$\eta_n(q) \overset{\mathtt{Def}}{=} \int_{\lambda_{2n-1}(q)}^{\mu_n^*(q)} \frac{\psi_n(\lambda, q)}{\sqrt{\Delta^2(\lambda, q) - 4}}\, d\lambda \quad (\bmod 2\pi), \tag{3.3}$$

and, for $k \neq n$,

$$\beta_k^n(q) \overset{\mathtt{Def}}{=} \int_{\lambda_{2k-1}(q)}^{\mu_k^*(q)} \frac{\psi_n(\lambda, q)}{\sqrt{\Delta^2(\lambda, q) - 4}}\, d\lambda. \tag{3.4}$$

We call a path of integration from $\lambda_{2k-1}(q)$ to $\mu_k^*(q)$ on the Riemann surface Σ_q *admissible* if its projection by the map $\pi(\lambda, z) = \lambda$ onto $\mathbb{C}$ stays inside an isolating neighborhood U_k. As in [13], one shows that the functions η_n and β_k^n are well-defined in the sense that they are independent of the choice of an admissible path of integration.

More precisely, one shows that, for any $k \neq n$, the function β_k^n is well-defined on all of $\mathcal{W}$ as well as real analytic, and vanishes at $q = 0$. For $n \geq 1$, the function η_n is well-defined on $\mathcal{W} \setminus D_n$ and real analytic on $\mathcal{W} \setminus D_n$ if taken modulo π. In order to ensure its analyticity the function η_n has to be taken modulo π due to the discontinuities of the periodic eigenvalues as functions of q when q is not real, see [13]. On the real space $H_0^{-1}(\mathbb{T}^1) \setminus D_n$, the function η_n is real analytic when taken modulo 2π.

To show that the series in (3.2) converges, one uses the asymptotic estimates from Section 9 (in particular, $\sigma_k^n = \tau_k + O(|\gamma_k|^2/k)$ for $k \neq n$) to prove the following analogue of Lemma 7.4 in [13].

Lemma 3.1.

$$\beta_k^n = O\left(\frac{|\gamma_k| + |\mu_k - \tau_k|}{|k^2 - n^2|} \cdot \frac{n}{k} \right)$$

locally uniformly on $\mathcal{W}$ and uniformly in k and n with $k \neq n$.

Proof. By Theorem 9.1,

$$\beta_k^n(q) \overset{\mathtt{Def}}{=} \int_{\lambda_{2k-1}(q)}^{\mu_k^*(q)} \frac{\psi_n(\lambda, q)}{\sqrt{\Delta^2(\lambda, q) - 4}}\, d\lambda = \int_{\lambda_{2k}(q)}^{\mu_k^*(q)} \frac{\psi_n(\lambda, q)}{\sqrt{\Delta^2(\lambda, q) - 4}}\, d\lambda.$$

Without loss of generality assume that $|\mu_k(q) - \lambda_{2k-1}(q)| \le |\mu_k(q) - \lambda_{2k}(q)|$ otherwise interchange the role of $\lambda_{2k-1}(q)$ and $\lambda_{2k}(q)$ in the following argument. Further we may assume that $\lambda_{2k-1}(q) \neq \mu_k(q)$ as $\beta_k^n(q) = 0$ otherwise. By the product representation $\psi_n(\lambda, q) = \frac{2}{n\pi} \prod_{k\in\mathbb{N}\setminus\{n\}} \frac{\sigma_k^n(q)-\lambda}{k^2\pi^2}$ (cf. Theorem 9.1) we obtain for λ near G_k

$$\frac{\psi_n(\lambda,q)}{\sqrt[c]{\Delta^2(\lambda,q)-4}} = \frac{\sigma_k^n(q)-\lambda}{\sqrt[s]{(\lambda_{2k}(q)-\lambda)(\lambda-\lambda_{2k-1}(q))}}\,\zeta_k^n(\lambda,q) \tag{3.5}$$

where

$$\zeta_k^n(\lambda,q) \overset{\text{Def}}{=} \frac{(-1)^{k-1}\pi n}{(\sigma_n^n(q)-\lambda)\sqrt[+]{\lambda-\lambda_0(q)}} \prod_{m\in\mathbb{N}\setminus\{k\}} \frac{\sigma_m^n(q)-\lambda}{\sqrt[+]{(\lambda_{2m}(q)-\lambda)(\lambda_{2m-1}(q)-\lambda)}} \tag{3.6}$$

with $\sigma_n^n = \tau_n (\overset{\text{Def}}{=} (\lambda_{2n-1}+\lambda_{2n})/2)$. By Theorem 9.1, $\sigma_m^n = \tau_m + O(|\gamma_m|^2/m)$ uniformly for $n \in \mathbb{N}$ as well as locally uniformly in $\mathcal{W}$ and by Proposition 10.1, $\tau_n - n^2\pi^2 = h^{-1}(n)$ locally uniformly on $\mathcal{W}$. By Lemma 8.7

$$|\zeta_k^n(\lambda,q)| = O\Big(\frac{n}{k|k^2-n^2|}\Big) \tag{3.7}$$

uniformly for $k \neq n$, $\lambda \in U_k$ and locally uniformly in $\mathcal{W}$.
Let $\gamma_k(q) \neq 0$. For $\lambda \overset{\text{Def}}{=} \lambda_{2k-1} + (\mu_k - \lambda_{2k-1})t$ with $0 \le t \le 1$ one has, as $|\lambda - \lambda_{2k-1}| \le |\lambda - \lambda_{2k}|$ and $|\lambda_{2k} - \lambda| \ge |\gamma_k|/2$

$$\begin{aligned} \left|\frac{\sigma_k^n - \lambda}{\lambda_{2k}-\lambda}\right| &\le \frac{|\sigma_k^n - \lambda_{2k-1}|}{|\lambda_{2k}-\lambda|} + \frac{|\lambda-\lambda_{2k-1}|}{|\lambda_{2k}-\lambda|} \\ &\le \frac{|(\sigma_k^n - \tau_k) + \frac{\gamma_k}{2}|}{|\gamma_k|/2} + 1 \\ &= O(1) \end{aligned} \tag{3.8}$$

locally uniformly on $\mathcal{W}$. Estimate (3.8) is obviously true also when $\gamma_k(q) = 0$ as by Theorem 9.1 $\sigma_k^n = \tau_k = \lambda_{2k}$. Now, it follows from (3.5), (3.7) and (3.8) that

$$|\beta_k^n(q)| = O\Big(\frac{n}{k|k^2-n^2|}\Big) \left|\int_{\lambda_{2k-1}(q)}^{\mu_k(q)} \sqrt{\frac{\lambda_{2k}-\lambda}{\lambda-\lambda_{2k-1}}}\, d\lambda.\right|$$

By the change of variable $\lambda = \lambda_{2k-1} + t(\mu_k - \lambda_{2k-1})$ in the latter integral we obtain with $\delta \overset{\text{Def}}{=} |\mu_k - \lambda_{2k-1}|$

$$\begin{aligned} \left| \int_{\lambda_{2k-1}(q)}^{\mu_k(q)} \sqrt{\frac{\lambda_{2k} - \lambda}{\lambda - \lambda_{2k-1}}}\, d\lambda \right| &\leq \int_0^1 \frac{\sqrt{|\gamma_k| + t\delta}}{\sqrt{t\delta}} \delta\, dt \\ &\leq 2\sqrt{\delta}\sqrt{|\gamma_k| + \delta} \leq \delta + (|\gamma_k| + \delta) \end{aligned}$$

As $\delta \leq |\mu_k - \tau_k| + |\gamma_k|/2 \leq |\mu_k - \tau_k| + |\gamma_k|$ the claimed statement of the lemma follows. □

We can now formulate the main result of this section.

Proposition 3.2. *There exists a complex neighborhood* $\mathcal{W} \subset H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ *of* $H_0^{-1}(\mathbb{T}^1)$ *with the following properties. The functions*

$$\beta_n \overset{\text{Def}}{=} \sum_{k \neq n} \beta_k^n, \quad n \geq 1$$

are analytic on $\mathcal{W}$ *and locally bounded on* $\mathcal{W}$ *uniformly in* n.
The angle function

$$\theta_n = \eta_n + \beta_n = \eta_n + \sum_{k \neq n} \beta_k^n$$

is real analytic on the real space $H_0^{-1}(\mathbb{T}^1) \setminus D_n$ *and extends to an analytic function on* $\mathcal{W} \setminus D_n$ *when taken modulo* π. *Its gradient has mean value zero.*

Proof. The estimate of β_k^n, $k \neq n$, stated in Lemma 3.1 ensures that the sum $\sum_{k \neq n} \beta_k^n$ converges absolutely, locally uniformly on $\mathcal{W}$. In fact, using the Cauchy-Schwartz inequality, we have, locally uniformly on $\mathcal{W}$,

$$\begin{aligned} \sum_{k \neq n} |\beta_k^n| &\leq C \sum_{k \neq n} \frac{n}{|k^2 - n^2|} \frac{|\gamma_k| + |\mu_k - \tau_k|}{k} \\ &\leq \sqrt{2} C n \left(\sum_{k \neq n} \frac{1}{|k^2 - n^2|^2} \right)^{1/2} (||\gamma||_{h^{-1}} + ||\mu - \tau||_{h^{-1}}) \\ &= O(1) \end{aligned}$$

in view of Lemma 8.2 and the fact that the h^{-1}-norms of the sequences (γ_k), $(\tau_k - k^2\pi^2)$, and $(\mu_k - k^2\pi^2)$ are locally bounded on $\mathcal{W}$. As θ_n is invariant under translations of the potential when considered on $H^{-1}(\mathbb{T}^1) \setminus D_n$ its gradient has mean value zero. □

4 Cartesian coordinates

Given the actions I_n and the angles θ_n introduced in sections 2 and 3 respectively, the associated rectangular coordinates x_n and y_n are defined on $H_0^{-1}(\mathbb{T}^1) \setminus D_n$ by

$$x_n = \sqrt{2I_n}\cos(\theta_n), \qquad y_n = \sqrt{2I_n}\sin(\theta_n),$$

where the choice of sin and cos is made so that $dx_n \wedge dy_n = dI_n \wedge d\theta_n$. We extend this definition to the complex domain $\mathcal{W} \setminus D_n$ by setting

$$x_n \overset{\text{Def}}{=} \xi_n \gamma_n \frac{e^{i\theta_n} + e^{-i\theta_n}}{4}, \qquad y_n \overset{\text{Def}}{=} \xi_n \gamma_n \frac{e^{i\theta_n} - e^{-i\theta_n}}{4i}$$

where $\mathcal{W}$ is given so that Propositions 2.1 and 3.2 hold.
The main result of this section is that these functions are in fact well-defined and real analytic on all of $\mathcal{W}$.
As the functions ξ_n and $\beta_n = \theta_n - \eta_n$ have already been shown to be real analytic on $\mathcal{W}$, we focus our attention on the complex functions

$$z_n^{\pm} \overset{\text{Def}}{=} \gamma_n e^{\pm i\eta_n},$$

which so far are defined on $\mathcal{W} \setminus D_n$. As in [13], Lemma 8.1, one shows that they are in fact analytic on $\mathcal{W} \setminus D_n$. The aim now is to show that the functions $z_n^{\pm}$ can be analytically extended to all of $\mathcal{W}$. To this end, we investigate the limiting behavior of $z_n^{\pm}(q)$ as q approaches a point p in D_n, i.e. as the n-th gap collapses. We will see that this limit is zero if $\mu_n(p)$ coincides with $\lambda_{2n-1}(p) = \lambda_{2n}(p)$, but different from zero if $p \in X_n \cap D_n$ where X_n denotes the open set

$$X_n \overset{\text{Def}}{=} \{q \in \mathcal{W} |\ \mu_n(q) \notin G_n(q)\}.$$

Since, for any real q, we have $\mu_n(q) \in G_n(q)$, this set does not intersect the real space $H_0^{-1}(\mathbb{T}^1)$. For λ near G_n, we write

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} = \frac{\zeta_n(\lambda)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}}, \tag{4.1}$$

where

$$\zeta_n(\lambda) = (-1)^{n+1} \frac{n\pi}{\sqrt[+]{\lambda - \lambda_0}} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\sqrt[+]{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}}.$$

Observe that ζ_n is analytic for $q \in \mathcal{W}$ and $\lambda \in U_n$, and bounded locally uniformly on $\mathcal{W}$ as well as uniformly for $n \geq 1$ and $\lambda \in U_n$. The boundedness is a consequence of Lemma 8.7. Moreover, we have

Lemma 4.1. *For $\mu \in G_n$,*

$$\zeta_n(\mu) = 1 + O\left(\frac{|\gamma_n|}{n}\right)$$

locally uniformly on $\mathcal{W}$ and uniformly in $n \geq 1$.

Proof. We start with a real valued potential $q \in \mathcal{W} \setminus D_n$. By Theorem 9.1

$$1 = \frac{1}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda)}{\sqrt[+]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \, d\lambda.$$

Hence

$$\zeta_n(\mu) = 1 - \frac{1}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda) - \zeta_n(\mu)}{\sqrt[+]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \, d\lambda \tag{4.2}$$

for $\mu \in G_n$. Upon the substitution $\lambda(t) = \tau_n + t\frac{\gamma_n}{2}$, the integral term in (4.2) takes the form

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\zeta_n(\lambda(t)) - \zeta_n(\mu)}{\sqrt[+]{1 - t^2}} \, dt, \tag{4.3}$$

hence it is bounded by $2 \sup_{\lambda \in G_n} |\zeta_n(\lambda) - \zeta_n(\mu)|$. We want to estimate this latter expression. By Corollary 10.2 for any $q_0 \in \mathcal{W}$ there exist $n_0 \in \mathbb{N}$ and a neighborhood $U(q_0) \subset \mathcal{W}$ of q_0 such that, for any $n \geq n_0$ and $q \in U(q_0)$, the straight interval $[\lambda_{2n-1}(q), \lambda_{2n}(q)]$ is inside the disk in $\mathbb{C}$ of radius n around the point $n^2\pi^2$ and the distance between $[\lambda_{2n-1}(q), \lambda_{2n}(q)]$ and $\Gamma_n \overset{\text{Def}}{=} \{|\lambda - n^2\pi^2| = n\}$ is greater than $n/2$. In particular, for $\mu, \lambda \in G_n$, by the Cauchy formula and the uniform boundedness of ζ_n on Γ_n, one has for any $n \geq n_0$

$$\begin{aligned} |\zeta_n(\lambda) - \zeta_n(\mu)| &= \frac{1}{2\pi} \left| \int_{\Gamma_n} \frac{\zeta_n(z)}{z - \lambda} \, dz - \int_{\Gamma_n} \frac{\zeta_n(z)}{z - \mu} \, dz \right| \\ &= \frac{|\lambda - \mu|}{2\pi} \left| \int_{\Gamma_n} \frac{\zeta_n(z)}{(z - \lambda)(z - \mu)} \, dz \right| \\ &= O\Big(|\lambda - \mu|/n\Big) = O\Big(|\gamma_n|/n\Big) \end{aligned} \tag{4.4}$$

locally uniformly on $U(q_0)$. Estimate (4.4) is obviously true also for $n = 1, 2, ..., n_0 - 1$. It holds trivially for $\gamma_n = 0$. Combining (4.4) with (4.2) and (4.3) we prove the statement of the lemma for real q.
For complex $q \in \mathcal{W}$ the preceding identities remain true at least up to a sign. Hence by the continuity of ζ_n in λ and q, the claimed statement must also be valid for complex q. □

Let

$$\chi_n(q) \stackrel{\text{Def}}{=} \int_{\tau_n(q)}^{\mu_n(q)} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n(q))}{\lambda - \tau_n(q)} \, d\lambda.$$

Note that the latter integral is analytic in q. For a potential $q \in X_n$ define a sign $\varepsilon_n = \pm 1$ by

$$\frac{\psi_n(\mu_n(q))}{\sqrt[*]{\Delta^2(\mu_n(q), q) - 4}} = \frac{\varepsilon_n \zeta_n(\mu_n(q))}{\sqrt[s]{(\lambda_{2n}(q) - \mu_n(q))(\mu_n(q) - \lambda_{2n-1}(q))}}.$$

As in [13], Lemma 8.3, one shows that, as $q \notin D_n$ tends to $p \in D_n \cap X_n$, the function $z_n^{\pm} = \gamma_n e^{\pm i\eta_n}$ tends to $-2(1 \pm \varepsilon_n)(\mu_n(p) - \tau_n(p))e^{\pm\varepsilon_n \chi_n(p)}$. We then extend the function $z_n^{\pm}$ to D_n by defining

$$z_n^{\pm} \stackrel{\text{Def}}{=} \begin{cases} -2(1 \pm \varepsilon_n)(\mu_n - \tau_n)e^{\pm\varepsilon_n \chi_n} & \text{on } D_n \cap X_n, \\ 0 & \text{on } D_n \setminus X_n. \end{cases} \tag{4.5}$$

The main result is the following

Proposition 4.2. *The functions $z_n^{\pm} = \gamma_n e^{\pm i\eta_n}$, as extended above, are analytic on $\mathcal{W}$. Moreover, they satisfy the asymptotics*

$$z_n^{\pm} = O\left(|\gamma_n| + |\mu_n - \tau_n|\right)$$

locally uniformly on $\mathcal{W}$.

Proof. Analyticity is shown as in Theorem 8.4 in [13]. To prove the estimate for z_n^+, consider first potentials in $X_n \setminus D_n$. It follows from the proof of Lemma 8.3 in [13] that, on $X_n \setminus D_n$, we have the following representation of $z_n^+ = \gamma_n e^{i\eta_n}$,

$$\gamma_n e^{i\eta_n} = \gamma_n \left(-\rho_n + i\varepsilon_n \sqrt[s]{1 - \rho_n^2}\right) e^{i\varepsilon_n \phi(\rho_n)\hat{\zeta}_n} e^{i\eta_n''}, \tag{4.6}$$

where

$$\rho_n \stackrel{\text{Def}}{=} \frac{\mu_n - \tau_n}{\gamma_n/2}, \quad \phi(w) \stackrel{\text{Def}}{=} \int_{-1}^{w} \frac{dz}{\sqrt[s]{1-z^2}}, \quad \hat{\zeta}_n = \zeta_n(\lambda_{2n-1}) - 1,$$

and

$$\eta_n'' = \varepsilon_n \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}}\, d\lambda \mod 2\pi.$$

The factors on the right hand side of (4.6) are examined separately. First we claim that

$$e^{i\varepsilon_n \phi(\rho_n)\hat{\zeta}_n} = O(1)$$

locally uniformly on $\mathcal{W}$ and uniformly in n. Indeed, for $|\rho_n| \geq 1$ and $C \stackrel{\text{Def}}{=} \max_{|w|\leq 1} |\phi(w)|$, one has

$$\begin{aligned} |\phi(\rho_n)| &\leq C + \int_1^{|\rho_n|} \frac{1}{\sqrt{t+1}} \frac{dt}{\sqrt{t-1}} \\ &\leq C + 2\sqrt{t-1}\Big|_1^{|\rho_n|} = O(|\rho_n|). \end{aligned}$$

Using Lemma 4.1 and the definition of ρ_n, we then get

$$|i\varepsilon_n \phi(\rho_n)\hat{\zeta}_n| = O\Big(1 + 2\frac{|\mu_n - \tau_n|}{|\gamma_n|}\Big)\frac{|\gamma_n|}{n} = O(1)$$

in view of the asymptotics of μ_n, τ_n and γ_n – see Proposition 10.1 (Appendix C).

We claim that $e^{i\eta_n''} = O(1)$ locally uniformly on $\mathcal{W}$ and uniformly in n. To this end, assume without loss of generality that

$$|\mu_n - \lambda_{2n-1}| \leq |\mu_n - \lambda_{2n}|. \tag{4.7}$$

If this does not hold, reverse the roles of λ_{2n-1} and λ_{2n} in the representation (4.6) and in the following calculations. This is possible as (see Theorem 9.1)

$$\eta_n \stackrel{\text{Def}}{=} \int_{\lambda_{2n-1}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}}\, d\lambda = \int_{\lambda_{2n}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}}\, d\lambda + \pi \mod 2\pi.$$

In view of the assumption (4.7), we have $|\lambda - \lambda_{2n-1}| \leq |\lambda - \lambda_{2n}|$ for any λ on the straight interval $[\lambda_{2n-1}, \mu_n]$. We then obtain from the definition of η_n''

(integrating along $[\lambda_{2n-1}, \mu_n]$)

$$\begin{aligned} |\eta_n''| &\stackrel{\text{Def}}{=} \left| \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})}{\lambda - \lambda_{2n-1}} \sqrt{\frac{\lambda - \lambda_{2n-1}}{\lambda_{2n} - \lambda}}\, d\lambda \right| \\ &\le |\mu_n - \lambda_{2n-1}| \sup_{\lambda \in (\lambda_{2n-1}, \mu_n]} \left| \frac{\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})}{\lambda - \lambda_{2n-1}} \right|. \end{aligned}$$

Using a decomposition as in (4.4) and arguing similarly, one finds that the sup-term above is, locally uniformly on $\mathcal{W}$, of order $O(1/n)$, and we arrive at

$$|\eta_n''| = |\mu_n - \lambda_{2n-1}|\, O(1/n) = O(1),$$

so indeed $e^{i\eta_n''} = O(1)$ locally uniformly on $\mathcal{W}$ and uniformly in n.
The remaining term in (4.6) can be estimated by

$$\left| \gamma_n \left(-\rho_n + i\varepsilon_n \sqrt[s]{1 - \rho_n^2} \right) \right| \le 6\, (|\gamma_n| + |\mu_n - \tau_n|).$$

This follows by bounding, for $|\rho_n| \le 1$, the square root $\sqrt[s]{1-\rho_n^2}$ by $\sqrt{2}$, and, for $|\rho_n| > 1$, using that $\sqrt[s]{1-\rho_n^2} = i\rho_n \sqrt[+]{1-\rho_n^{-2}}$ as well as $\gamma_n \rho_n = 2(\mu_n - \tau_n)$. Consequently, the estimate of $z_n^+ = \gamma_n e^{i\eta_n}$ is established for potentials in $X_n \setminus D_n$. By continuity, it extends in a locally uniform fashion to all of $\mathcal{W}$, since all estimates involved are valid locally uniformly on $\mathcal{W}$. The proof for z_n^- is completely analogous. □

We now define cartesian coordinates on $\mathcal{W}$ by $(n \ge 1)$

$$x_n \stackrel{\text{Def}}{=} \frac{\xi_n}{4} \left(z_n^+ e^{i\beta_n} + z_n^- e^{-i\beta_n} \right), \qquad y_n \stackrel{\text{Def}}{=} \frac{\xi_n}{4i} \left(z_n^+ e^{i\beta_n} - z_n^- e^{-i\beta_n} \right),$$

where $z_n^\pm$ are the functions extended to D_n by (4.5). The functions x_n and y_n are real analytic in view of Proposition 2.1, Proposition 3.2, and Proposition 4.2. By continuity it follows from the corresponding results in [13] that the L^2-gradients of $x_n(q)$ and $y_n(q)$ have mean value zero. With any potential q in $\mathcal{W}$ we may then associate coordinates

$$\Omega(q) = (\mathbf{x}(q), \mathbf{y}(q)),$$

where

$$\mathbf{x}(q) = (x_1(q), x_2(q), \dots), \qquad \mathbf{y}(q) = (y_1(q), y_2(q), \dots).$$

In view of the asymptotic estimates of Proposition 2.1, Proposition 3.2, and Proposition 4.2,

$$|x_n| + |y_n| = O\left(\frac{|\gamma_n| + |\mu_n - \tau_n|}{\sqrt{n}}\right)$$

locally uniformly on $\mathcal{W}$. Consequently, in view of the asymptotic estimates for the periodic and Dirichlet eigenvalues (cf. Appendix C), Ω maps the complex neighborhood $\mathcal{W}$ of $H_0^{-1}(\mathbb{T}^1)$ into the space $h^{-1/2}(\mathbb{N},\mathbb{C}) \times h^{-1/2}(\mathbb{N},\mathbb{C})$ and is locally bounded. Moreover, it maps the real space $H_0^{-1}(\mathbb{T}^1)$ into the real space

$$\mathfrak{h}^{-1/2} \overset{\mathtt{Def}}{=} h^{-1/2}(\mathbb{N},\mathbb{R}) \times h^{-1/2}(\mathbb{N},\mathbb{R}).$$

Note that the diffeomorphism constructed in [13] coincides with the restriction Ω_0 of $\Omega : H_0^{-1}(\mathbb{T}^1) \to \mathfrak{h}^{-1/2}$ to $L_0^2(\mathbb{T}^1)$,

$$\Omega_0 = \Omega|_{L_0^2(\mathbb{T}^1)} : L_0^2(\mathbb{T}^1) \to \mathfrak{h}^{1/2}. \tag{4.8}$$

Indeed, the periodic and Dirichlet eigenvalues of $q \in L_0^2(\mathbb{T}^1)$ are the same if we consider q as an element in $H_0^{-1}(\mathbb{T}^1)$. Let us summarize the main results of this section.

Proposition 4.3. *The map*

$$\Omega \,:\, H_0^{-1}(\mathbb{T}^1) \to \mathfrak{h}^{-1/2}, \quad q \mapsto \Omega(q) = (\mathbf{x}(q), \mathbf{y}(q))$$

is real analytic and extends analytically to all of $\mathcal{W}$. Its Jacobian at $q = 0$ is boundedly invertible, and its inverse is the weighted Fourier transform

$$d_0\Omega^{-1} : \mathfrak{h}^{-1/2} \to H_0^{-1}(\mathbb{T}^1), \quad (\mathbf{x}, \mathbf{y}) \mapsto \sum_{n \geq 1} \sqrt{2\pi n}\,(x_n\, e_n + y_n\, e_{-n}),$$

where $e_n = \sqrt{2}\cos(2\pi n x)$ and $e_{-n} = \sqrt{2}\sin(2\pi n x)$ for $n \geq 1$. In particular, Ω is a local diffeomorphism at $q = 0$.

Proof. The analyticity of Ω on $\mathcal{W}$ follows from Theorem A.6 in [13] in view of the local boundedness of Ω and the analyticity of each of the components x_n and y_n. The statement for $q = 0$ follows by continuity from the corresponding formulas for $d_0\Omega_0^{-1}$ in [13]. $\square$

5 Orthogonality relations

Recall that, in view of the previous analyticity results, any bracket $\{\cdot,\cdot\}$ between x_n, y_n, and I_n is a well defined analytic function on $\mathcal{W}$, and, for brackets involving the angle variable θ_n, or θ_n and θ_m, this is true on $\mathcal{W} \setminus D_n$, or $\mathcal{W} \setminus (D_n \cup D_m)$, respectively. Therefore any orthogonality relation established for $L^2_0(\mathbb{T}^1)$ holds by density on $H^{-1}_0(\mathbb{T}^1)$. On $H^{-1}_0(\mathbb{T}^1)$, we thus obtain from [13], for any $n, m \geq 1$,

$$\begin{aligned} \{x_n, x_m\} = 0, \quad \{y_n, y_m\} = 0, \qquad \{x_n, y_m\} = \delta_{nm}, \\ \{I_n, I_m\} = 0, \quad \{x_n, I_m\} = \delta_{nm} y_n, \quad \{y_n, I_m\} = -\delta_{nm} x_n. \end{aligned}$$

Moreover,

$$\{I_n, \theta_m\} = \delta_{nm} \text{ on } H^{-1}_0(\mathbb{T}^1) \setminus D_m, \qquad \{\theta_n, \theta_m\} = 0 \text{ on } H^{-1}_0(\mathbb{T}^1) \setminus (D_n \cup D_m).$$

6 Local diffeomorphism

In this section we show

Proposition 6.1. *The map*

$$\Omega : H^{-1}_0(\mathbb{T}^1) \to \mathfrak{h}^{-1/2}, \quad q \mapsto (\mathbf{x}(q), \mathbf{y}(q))$$

is a local diffeomorphism everywhere.

To prove this proposition, recall that Ω is a local diffeomorphism at $q = 0$ by Proposition 4.3. Instead of Ω, it is more convenient to consider the map

$$\Phi \stackrel{\text{Def}}{=} (d_0 \Omega)^{-1} \circ \Omega : H^{-1}_0(\mathbb{T}^1) \to H^{-1}_0(\mathbb{T}^1),$$

which is given by

$$\Phi(q) = \sum_{n \geq 1} \sqrt{2\pi n}\, (x_n(q) e_n + y_n(q) e_{-n})$$

with

$$e_n = \sqrt{2} \cos(2\pi n x), \quad e_{-n} = \sqrt{2} \sin(2\pi n x).$$

Clearly, Ω is a local diffeomorphism if and only if Φ is a local diffeomorphism. The Jacobian $d_q\Phi \in \mathcal{L}(H_0^{-1}(\mathbb{T}^1))$ of Φ at q is given by $(h \in H_0^{-1}(\mathbb{T}^1))$

$$d_q\Phi(h) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \langle h, d_n(q)\rangle\, e_n \tag{6.1}$$

where, for $n \geq 1$,

$$d_n(q) \overset{\text{Def}}{=} \sqrt{2\pi n}\partial_q x_n, \quad d_{-n}(q) \overset{\text{Def}}{=} \sqrt{2\pi n}\partial_q y_n.$$

We show that $d_q\Phi$ is a linear isomorphism by proving that its dual map $(d_q\Phi)^* \in \mathcal{L}(H_0^1(\mathbb{T}^1))$ is a linear isomorphism. Observe that the dual map is given by $(h \in H_0^1(\mathbb{T}^1))$

$$(d_q\Phi)^*(h) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \langle h, e_n\rangle\, d_n(q). \tag{6.2}$$

In order to show that $(d_q\Phi)^*$ is an isomorphism, we proceed as follows. By Lemma 6.2 below, $(d_q\Phi)^*$ is a compact perturbation of the identity. In Lemma 6.4 we show that $(d_q\Phi)^*$ is injective. Hence, by the Fredholm alternative, $(d_q\Phi)^*$ is a linear isomorphism of $H_0^1(\mathbb{T}^1)$.

Lemma 6.2. *For $q \in H_0^{-1}(\mathbb{T}^1)$, the map $(d_q\Phi)^* - Id \in \mathcal{L}(H_0^1(\mathbb{T}^1))$ is compact.*

Proof. A potential q is called a *finite gap potential* if $\gamma_n(q) \neq 0$ only for finitely many n. The finite gap potentials form a dense subset in L_0^2 (cf. [13], Appendix B), and hence in $H_0^{-1}(\mathbb{T}^1)$. We show the claimed statement first for finite gap potentials. The general statement then follows from the fact that the map $q \mapsto (d_q\Phi)^* - Id$ is continuous, as Φ is analytic.
Let $T \equiv T(q) \in \mathcal{L}(H_0^1(\mathbb{T}^1))$ denote the map $(d_q\Phi)^* - Id$, which by (6.2) is given by $(h \in H_0^1(\mathbb{T}^1),\ d_n \equiv d_n(q))$

$$Th = \sum_{n\neq 0} \langle h, e_n\rangle\,(d_n - e_n).$$

For $N \geq 1$, define the map $T_N \in \mathcal{L}(H_0^1(\mathbb{T}^1))$,

$$T_N h = \sum_{0\neq|n|\leq N} \langle h, e_n\rangle\,(d_n - e_n).$$

The maps T_N are bounded operators of finite rank and hence compact, so T is compact if $T_N \to T$ in $\mathcal{L}(H_0^1(\mathbb{T}^1))$. But, for $h \in H_0^1(\mathbb{T}^1)$, using the Cauchy-Schwartz inequality,

$$\begin{aligned}\|(T-T_N)h\|_{H_0^1} &\leq \sum_{|n|>N} \frac{|\langle h, e_n\rangle|}{|n|^{-1}} \frac{\|d_n - e_n\|_{H_0^1}}{|n|} \\ &\leq \Big(\sum_{n\neq 0} |n|^2 |\langle h, e_n\rangle|^2\Big)^{1/2} \Big(\sum_{|n|>N} \frac{\|d_n - e_n\|_{H_0^1}^2}{|n|^2}\Big)^{1/2}.\end{aligned}$$

The first term on the latter line is equal to $\|h\|_{H_0^1}$. For a finite gap potential q, the second term tends to 0 as $N \to \infty$ by Lemma 6.3 below. □

Lemma 6.3. *At a finite gap potential with* $[q] = 0$*,*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\|d_n - e_n\|_{H_0^1}^2}{|n|^2} < \infty.$$

Proof. Let $q \in H_0^{-1}(\mathbb{T}^1)$ be a finite gap potential, and let $\mathcal{N} \equiv \mathcal{N}(q) := \{n \in \mathbb{N} \mid \gamma_n(q) = 0\}$. By assumption, the complement of the set $\mathcal{N}$ is finite. Hence it suffices to show that

$$\sum_{\pm n\in\mathcal{N}} \frac{\|d_n - e_n\|_{H_0^1}^2}{|n|^2} < \infty.$$

Since q is a finite gap potential it is smooth, hence in L_0^2, and we can use the following results from [13]. By the proof of Theorem 8.7 in [13], for $n \in \mathcal{N}$,

$$\partial x_n + i\partial y_n = \frac{\xi_n}{2} e^{i\beta_n} \partial z_n^+, \tag{6.3}$$

where

$$\xi_n = \frac{1}{\sqrt{n\pi}} \left(1 + O\left((\log n)/n\right)\right), \ \beta_n = O\left(1/n\right), \tag{6.4}$$

and, by Theorem 8.5 in [13], ∂z_n^+ is given by

$$\partial z_n^+ = h_n^2 - g_n^2 + i2g_nh_n \tag{6.5}$$

where g_n denotes the normalized eigenfunction for the Dirichlet eigenvalue μ_n ($\int_0^1 g_n^2(x)dx = 1$, $g_n'(0) > 0$) and h_n the periodic eigenfunction corresponding to the eigenvalue $\lambda_{2n}(= \lambda_{2n-1} = \mu_n)$ which is orthogonal to g_n and satisfies $\int_0^1 h_n^2(x)dx = 1$ as well as $h_n(0) > 0$.
We now argue similarly as in the proof of Lemma 10.12 in [13]. Consider the operator $Q = -\frac{1}{2}\partial_x^3 + q\partial_x + \partial_x q$. One verifies that the product $Y \overset{\text{Def}}{=} y_i y_j$ of any two fundamental solutions of $-y'' + qy = \lambda y$ satisfies the differential equation

$$QY = 2\lambda \partial_x Y.$$

By (6.3), (6.5) and as the mean values of d_n and d_{-n} vanish for any $n \in \mathcal{N}$ we obtain that

$$d_{\pm n} = \frac{1}{2\lambda_{2n}} D_0^{-1} Q d_{\pm n}$$

where we have assumed without loss of generality that $\lambda_{2n} \neq 0$ for any $n \in \mathcal{N}$ and where D_0^{-1} denotes the inverse of the restriction of ∂_x to $H_0^1(\mathbb{T}^1)$. For $f \in H_0^1(\mathbb{T}^1)$ one has

$$\begin{aligned} \langle d_{\pm n}, f \rangle &= \frac{1}{2\lambda_{2n}} \left\langle D_0^{-1} Q d_{\pm n}, f \right\rangle \\ &= \frac{1}{2\lambda_{2n}} \left\langle d_{\pm n}, Q D_0^{-1} f \right\rangle \\ &= -\frac{1}{4\lambda_{2n}} \langle d_{\pm n}, f^* \rangle \end{aligned}$$

with $f^* \overset{\text{Def}}{=} f'' - 4qf - 2q' D_0^{-1} f \in H^{-1}(\mathbb{T}^1)$. Therefore

$$\begin{aligned} \langle d_{\pm n}, f'' \rangle &= \langle d_{\pm n}, f^* \rangle - \langle d_{\pm n}, f^* - f'' \rangle \\ &= -4\lambda_{2n} \langle d_{\pm n}, f \rangle - \left\langle d_{\pm n}, -4qf - 2q' D_0^{-1} f \right\rangle \end{aligned}$$

Combining this with

$$\langle e_{\pm n}, f'' \rangle = -4\pi^2 n^2 \langle e_{\pm n}, f \rangle$$

one obtains

$$\begin{aligned} \langle d_{\pm n} - e_{\pm n}, f'' \rangle &= 4(\pi^2 n^2 - \lambda_{2n}) \langle e_{\pm n}, f \rangle \\ &- 4\lambda_{2n} \langle d_{\pm n} - e_{\pm n}, f \rangle \\ &+ \left\langle d_{\pm n}, 4qf + 2q' D_0^{-1} f \right\rangle \end{aligned} \tag{6.6}$$

By standard estimates one has for f in $H_0^1(\mathbb{T}^1)$

$$|\langle e_{\pm n}, f\rangle| = O(1/n)||f||_{H_0^1}$$

$$\lambda_{2n} = n^2\pi^2 + O(1/n)$$

$$\begin{aligned} \left|\left\langle d_{\pm n}, 4qf + 2q' D_0^{-1} f\right\rangle\right| &\leq ||d_{\pm n}|| \cdot ||4qf + 2q' D_0^{-1} f|| \\ &\leq C||f||_{H_0^1}, \end{aligned} \tag{6.7}$$

where we used that $||d_{\pm n}|| = O(1)$ by Theorem 8.5 in [13], and from Lemma 10.12 in [13] one gets

$$|\langle d_{\pm n} - e_{\pm n}, f\rangle| = O((\log n)/n^2)||f||_{H_0^1}.$$

Note that every element in $H_0^{-1}(\mathbb{T}^1)$ is of the form f'' for some $f \in H_0^1(\mathbb{T}^1)$ with $||f''||_{H_0^{-1}} = 4\pi^2||f||_{H_0^1}$.
In view of (6.6) we then conclude

$$||d_{\pm n} - e_{\pm n}||_{H_0^1} \leq \sup_{||f||_{H_0^1} \leq 1} |\langle d_{\pm} - e_{\pm}, f''\rangle| = O(\log n)$$

which leads to the claimed estimate. □

Lemma 6.4. *For any $q \in H_0^{-1}(\mathbb{T}^1)$, the map $(d_q\Phi)^* \in \mathcal{L}(H_0^1(\mathbb{T}^1))$ is injective.*

Proof. Assume that there exist real numbers a_k, $k \neq 0$, such that

$$g = \sum_{k \neq 0} a_k d_k = 0,$$

where the sum converges in $H_0^1(\mathbb{T}^1)$. It is to show that all a_k vanish. As $\partial_x \partial x_n, \partial_x \partial y_n \in L_0^2(\mathbb{T}^1)$, the pairing of g with $\partial_x \partial x_n, \partial_x \partial y_n$ is well defined. In view of the orthogonality relations established in Section 5, we obtain, for $n \geq 1$

$$\begin{aligned} 0 = \langle g, \partial_x \partial x_n\rangle &= \sum_{k \neq 0} a_k \langle d_k, \partial_x \partial x_n\rangle \\ &= \sum_{k \geq 1} \sqrt{2\pi k}\, \left(a_k\{x_k, x_n\} + a_{-k}\{y_k, x_n\}\right) = -\sqrt{2\pi n} a_{-n} \end{aligned}$$

and, similarly, $0 = \langle g, \partial_x \partial y_n\rangle = \sqrt{2\pi n} a_n$. So, indeed, all a_n vanish. □

7 Diffeomorphism property

In this section we state and prove the main theorem of this paper. Recall that $\mathfrak{h}^s \overset{\mathtt{Def}}{=} h^s(\mathbb{N},\mathbb{R}) \times h^s(\mathbb{N},\mathbb{R})$ for $s \in \mathbb{R}$. Endow this sequence space with the standard Poisson bracket, for which

$$\{x_n, y_m\} = \delta_{nm},$$

while all other brackets vanish.

Theorem 7.1. *The map*

$$\Omega \overset{\mathtt{Def}}{=} \Omega_{-1} : H_0^{-1}(\mathbb{T}^1) \to \mathfrak{h}^{-1/2}, \ q \mapsto (\mathbf{x}(q), \mathbf{y}(q))$$

has the following properties:

(i) Ω *is a bi-analytic diffeomorphism and preserves the Poisson bracket.*

(ii) $\Omega_0 \overset{\mathtt{Def}}{=} \Omega|_{L_0^2(\mathbb{T}^1)}$ *agrees with the map constructed in [13].*

(iii) For any $0 < \alpha < 1$

$$\Omega_{-\alpha} \overset{\mathtt{Def}}{=} \Omega|_{H_0^{-\alpha}(\mathbb{T}^1)} : H_0^{-\alpha}(\mathbb{T}^1) \to \mathfrak{h}^{1/2-\alpha}$$

is a bi-analytic diffeomorphism as well.

Remark 7.2. *In view of the results from [13], it follows from (ii) that the coordinates* $(\mathbf{x}, \mathbf{y})$ *in* $\mathfrak{h}^{3/2}$ *are Birkhoff coordinates for the KdV equation. That is, for* $\Omega_1 \overset{\mathtt{Def}}{=} \Omega|_{H_0^1}$, *the transformed KdV Hamiltonian* $H \circ \Omega_1^{-1}$ *depends only on* $x_n^2 + y_n^2$, $n \geq 1$, *with* $(\mathbf{x}, \mathbf{y})$ *being canonical coordinates in* $\mathfrak{h}^{3/2}$.

Proof of Theorem 7.1. By Proposition 6.1, the map Ω is a real analytic, local diffeomorphism and in view of Section 5 preserves the Poisson bracket. Statement *(ii)* follows from the considerations before Proposition 4.3. With regard to statement *(iii)* it is easy to see that $\Omega_{-\alpha}$ is an analytic local diffeomorphism which is one-to-one. The fact that $\Omega_{-\alpha}$ is onto follows from Corollary 2.6 in [12].
To prove the injectivity of Ω we argue by contradiction and assume that there exist points $q_1 \neq q_2$ in $H_0^{-1}(\mathbb{T}^1)$ so that

$$\Omega(q_1) = \Omega(q_2) = a \in \mathfrak{h}^{-1/2}.$$

Since $\Omega : H_0^{-1}(\mathbb{T}^1) \to \mathfrak{h}^{-1/2}$ is a local diffeomorphism, there exist disjoint open neighborhoods $\mathcal{V}_1$, $\mathcal{V}_2$ in $H_0^{-1}(\mathbb{T}^1)$ of q_1, q_2, respectively, and an open neighborhood U of a in $\mathfrak{h}^{-1/2}$ such that the restrictions

$$\Omega\big|_{\mathcal{V}_i} : \mathcal{V}_i \to U \quad (i = 1, 2)$$

are diffeomorphisms. Since L_0^2 is dense in $H_0^{-1}(\mathbb{T}^1)$, and $\mathfrak{h}^{1/2}$ is dense in $\mathfrak{h}^{-1/2}$, where both inclusion maps are continuous, the sets $\tilde{\mathcal{V}}_i \stackrel{\text{Def}}{=} \mathcal{V}_i \cap L_0^2$ $(i = 1, 2)$, and $\tilde{U} \stackrel{\text{Def}}{=} U \cap \mathfrak{h}^{1/2}$ are non-empty and open in L_0^2, and $\mathfrak{h}_{1/2}$, respectively. It then follows from (ii) and Theorem 10.8 in [13] that the restrictions of the map $\Omega_0 \equiv \Omega|_{L_0^2} : L_0^2 \to \mathfrak{h}^{1/2}$ to $\tilde{\mathcal{V}}_i$,

$$\Omega\big|_{\tilde{\mathcal{V}}_i} : \tilde{\mathcal{V}}_i \to \tilde{U} \quad (i = 1, 2)$$

are diffeomorphisms. Since $\tilde{\mathcal{V}}_1$ and $\tilde{\mathcal{V}}_2$ are disjoint, this contradicts Theorem 10.8 in [13] stating that the map $\Omega_0 : L_0^2 \to \mathfrak{h}^{1/2}$ is a global diffeomorphism. The ontoness is proved in Lemma 7.3 bellow. $\square$

Lemma 7.3. *Ω is onto.*

Proof. Take an arbitrary element $(\mathbf{x}, \mathbf{y}) \in \mathfrak{h}^{-1/2}$ with $\mathbf{x} \stackrel{\text{Def}}{=} (x_k)_{k\geq 1}$ and $\mathbf{y} \stackrel{\text{Def}}{=} (y_k)_{k\geq 1}$. We will prove the lemma by constructing a potential $u \in H_0^{-1}(\mathbb{T}^1)$ such that $\Omega(u) = (\mathbf{x}, \mathbf{y})$. The proof is divided into two steps. In a first step we prove the existence of an auxiliary potential $q \in H_0^{-1}(\mathbb{T}^1)$ such that $I_k(q) = (x_k^2 + y_k^2)/2$ for any $k \geq 1$. Then, in a second step, we use q to construct the potential $u \in H_0^{-1}(\mathbb{T}^1)$ with the help of the orthogonal relations established in Section 5.

Construction of the auxiliary potential q: By [13], $\Omega(L_0^2(\mathbb{T}^1)) = \mathfrak{h}^{1/2}$ and as $\mathfrak{h}^{1/2}$ is dense in $\mathfrak{h}^{-1/2}$ we see that $\Omega(L_0^2(\mathbb{T}^1))$ is dense in $\mathfrak{h}^{-1/2}$. Hence, there exists a sequence $(q_n)_{n\geq 1}$ in $L_0^2(\mathbb{T}^1)$ so that $(\Omega(q_n))_{n\geq 1}$ converges in $\mathfrak{h}^{-1/2}$ and $\lim_{n\to\infty} \Omega(q_n) = (\mathbf{x}, \mathbf{y})$.
By [2, Lemma 2.2], for any $k \geq 1$,

$$I_k(q_n) \geq \frac{1}{c} \min(k\gamma_k(q_n), \gamma_k(q_n)^2/k) \tag{7.1}$$

where $c = 2^6\pi^6$. We claim that $\gamma(q_n) \stackrel{\text{Def}}{=} (\gamma_k(q_n))_{k\geq 1}$ is a bounded sequence in h^{-1}: As $(\Omega(q_n))_{n\geq 1}$ converges in $\mathfrak{h}^{-1/2}$ there exists $k_0 \geq 1$ so that, for any $n \geq 1$

$$\sum_{k=k_0+1}^{\infty} k^{-1} 2I_k(q_n) \leq 1/c \tag{7.2}$$

where we used that $x_k(q_n)^2 + y_k(q_n)^2 = 2I_k(q_n)$. With regard to (7.1) note that for any $k \geq 1$ with $k\gamma_k(q_n) < \gamma_k(q_n)^2/k$ one has $k^2 < \gamma_k(q_n)$ and hence by (7.1), $k^3 < cI_k(q_n)$ or $k^{-1}2I_k(q_n) > 2k^2/c$. In view of (7.2) this implies that k satisfies $1 \leq k \leq k_0$. As a consequence, for any $n \geq 1$ and $k \geq k_0 + 1$,

$$k\gamma_k(q_n) \geq \gamma_k(q_n)^2/k$$

and hence by (7.1) and (7.2)

$$\sum_{k=k_0+1}^{\infty} k^{-2}\gamma_k(q_n)^2 \leq c \sum_{k=k_0+1}^{\infty} k^{-1} I_k(q_n) \leq 1/2. \tag{7.3}$$

Further, for any $1 \leq k \leq k_0$,

$$\min(\gamma_k(q_n), \gamma_k(q_n)^2/k^2) \leq ck^{-1}I_k(q_n) \leq ck^{-1}||\Omega_k(q_n)||^2/2$$

which implies

$$\gamma_k(q_n)^2 \leq (c/2)^2 k||\Omega_k(q_n)||^2(1 + k^{-1}||\Omega_k(q_n)||^2),$$

where $\Omega_k(p) \stackrel{\text{Def}}{=} (x_k(p), y_k(p))$ and $||\Omega_k(p)||^2 \stackrel{\text{Def}}{=} x_k^2(p) + y_k^2(p)$. This estimate leads to the bound

$$\sum_{k=1}^{k_0} k^{-2}\gamma_k(q_n)^2 \leq (c/2)^2||\Omega(q_n)||^2_{\mathfrak{h}^{-1/2}}(1 + ||\Omega(q_n)||^2_{\mathfrak{h}^{-1/2}})$$

which combined with (7.3) shows that $(\gamma(q_n))_{n\geq 1}$ is a bounded sequence in h^{-1}. By [14, Theorem 5] (cf. also [20]),

$$||q_n||_{H_0^{-1}} \leq M||\gamma(q_n)||_{h^{-1}}(1 + M||\gamma(q_n)||_{h^{-1}})^3$$

where M is an absolute constant. Hence $(q_n)_{n\geq 1}$ is a bounded sequence in $H_0^{-1}(\mathbb{T}^1)$. Without loss of generality we may assume that $(q_n)_{n\geq 1}$ is weakly convergent in $H_0^{-1}(\mathbb{T}^1)$. It follows from Proposition 10.7 (cf. Appendix C) that there exist an element $q \in H_0^{-1}(\mathbb{T}^1)$ and a subsequence, again denoted by $(q_n)_{n\geq 1}$, so that for any $k \geq 1$,

$$I_k(q_n) \to I_k(q) = (x_k^2(q) + y_k^2(q))/2 \quad (n \to \infty).$$

By assumption, $\Omega_k(q_n) \to (x_k, y_k)$ as $n \to \infty$ which implies that $I_k(q_n) = (x_k^2(q_n) + y_k^2(q_n))/2 \to (x_k^2 + y_k^2)/2$ as $n \to \infty$. Hence, for every $k \in \mathbb{N}$

$$I_k(q) = (x_k^2(q) + y_k^2(q))/2 = (x_k^2 + y_k^2)/2. \tag{7.4}$$

Construction of the potential u: Let us fix $l \in \mathbb{N}$. If $r_l \overset{\texttt{Def}}{=} \sqrt[+]{x_l^2 + y_l^2} = 0$ then $x_l(q) = 0 = x_l$ and $y_l(q) = 0 = y_l$ by formula (7.4). In the case $r_l \neq 0$ one has

$$(x_l, y_l) = r_l(\cos\theta_l, \sin\theta_l), \quad (x_l(q), y_l(q)) = r_l(\cos\theta_l(q), \sin\theta_l(q)) \tag{7.5}$$

where $\theta_l = \theta_l(q) + \theta_l^0$. If $\theta_l^0 \equiv 0 \mod 2\pi$ we have that $x_l(q) = x_l$ and $y_l(q) = y_l$. If $\theta_l^0 \not\equiv 0 \mod 2\pi$ consider the solution $p : J \to H_0^{-1}(\mathbb{T}^1)$ of the initial value problem

$$\dot{p}(t) = X_l(p(t)), \quad p(0) = q \tag{7.6}$$

where $X_l(q) \overset{\texttt{Def}}{=} \frac{d}{dx}\frac{\partial I_l}{\partial q}$ and J denotes the maximal interval of existence of the solution p. As I_l is a real analytic functional on $H_0^{-1}(\mathbb{T}^1)$ the L^2-gradient $\frac{\partial I_l}{\partial q}$ is in $H_0^1(\mathbb{T}^1)$. Thus X_l is an analytic vector field on $H_0^{-1}(\mathbb{T}^1)$ and hence (7.6) admits a unique solution $p(t)$ on J (cf. [6]).

Lemma 7.4. *For any $l \in \mathbb{N}$, the vector field X_l is complete.*

Proof. By continuity arguments, as in Section 5, one concludes from Lemma 9.4 (ii) in [13] that for any $l \in \mathbb{N}$, the Lie derivative $\mathcal{L}_{X_l}\Delta_\lambda$ of Δ_λ satisfies

$$\mathcal{L}_{X_l}\Delta_\lambda \overset{\texttt{Def}}{=} \langle \partial\Delta_\lambda, X_l \rangle = \{\Delta_\lambda, I_l\} = 0, \ \forall \lambda \in \mathbb{C}$$

where $\Delta_\lambda(q) \equiv \Delta(\lambda, q)$ is the discriminant of the Hill operator $L_q \overset{\texttt{Def}}{=} -\frac{d^2}{dx^2} + q$. Hence, for any $t \in J$, $\Delta_\lambda(p(t))$ is independent of t. Thus $p(t) \in \texttt{Iso}(L_q) \overset{\texttt{Def}}{=} \{p \in H_0^{-1}(\mathbb{T}^1) \mid \texttt{spec}(L_p) = \texttt{spec}(L_q)\}$. On the other side, the isospectral set $\texttt{Iso}(L_q)$ is compact by Corollary 3 in [14]. Hence, a standard result from the theory of ordinary differential equations in Banach spaces says that $J = \mathbb{R}$. □

We are now in a position to finish the proof of Lemma 7.3. Instead of the potential q, consider $q_{,l} \overset{\texttt{Def}}{=} p(\theta_l^0)$. It is clear from the orthogonality relation $\{I_l, \theta_l\} = 1$ in Section 5 that $\theta_l(q_{,l}) = \theta_l(q) + \theta_l^0 = \theta_l$. Therefore, $x_l(q_{,l}) = x_l$ and $y_l(q_{,l}) = y_l$. The orthogonality relations $\{x_k, I_l\} = 0$ and $\{y_k, I_l\} = 0$ $(k \neq l)$ imply that for any $k \neq l$, $(x_k(q_{,l}), y_k(q_{,l})) = (x_k(q), y_k(q))$. Now define the sequence of potentials $(u_k)_{k\geq 1}$ in $H_0^{-1}(\mathbb{T}^1)$ by the recursive formula

$$u_k \overset{\texttt{Def}}{=} \begin{cases} u_{k-1} & \text{if } \Omega_k(u_{k-1}) = (x_k, y_k) \\ (u_{k-1})_{,k} & \text{otherwise} \end{cases}$$

with $u_0 \overset{\texttt{Def}}{=} q$. As in the proof of Lemma 7.4 one sees that $(u_k)_{k\geq 1}$ is contained in the set $\texttt{Iso}(L_q)$. By Corollary 3 in [14] $\texttt{Iso}(L_q)$ is compact, hence there exists a subsequence of $(u_k)_{k\geq 1}$, again denoted by $(u_k)_{k\geq 1}$, and an element $u \in \texttt{Iso}(L_q)$ so that $u_k \to u$ as $k \to \infty$ in $H_0^{-1}(\mathbb{T}^1)$. As $\Omega(u_k) \to (\mathbf{x}, \mathbf{y})$ for $k \to \infty$ by construction and Ω is continuous one concludes that $\Omega(u) = (\mathbf{x}, \mathbf{y})$. □

8 Appendix A: Estimates on products

Denote by $h_{\mathbb{C}}^{\alpha}$ the Hilbert space of weighted l^2-sequences $a = (a_k)_{k\geq 1} \subset \mathbb{C}$ satisfying

$$||a||_{h^\alpha} \overset{\texttt{Def}}{=} \Big(\sum_{k\geq 1} |k|^{2\alpha} |a_k|^2 \Big)^{1/2} < \infty.$$

For $a \in h_{\mathbb{C}}^{\alpha}$ and $n > 1$ recall that

$$||a||_{h^\alpha}^{[n]} \overset{\texttt{Def}}{=} \Big(\sum_{k \geq [\frac{n}{2}]} |k|^{2\alpha} \, |a_k|^2 \Big)^{1/2}. \tag{8.1}$$

Lemma 8.1. *Let* $0 \leq \alpha \leq 1$ *and* $a \in h_{\mathbb{C}}^{-\alpha}$. *Then*

$$\sum_{k \in \mathbb{N}\setminus\{n\}} \frac{|a_k|}{|k^2 - n^2|} = O(n^{\alpha - 3/2}) ||a||_{h^{-\alpha}} + O(n^{\alpha-1}) ||a||_{h^{-\alpha}}^{[n]}$$

where the terms $O(n^{\alpha-3/2})$ *and* $O(n^{\alpha-1})$ *are independent of* a.

Proof. By the Cauchy-Schwartz inequality one obtains

$$\begin{aligned}\sum_{k\in\mathbb{N}\setminus\{n\}} \frac{|a_k|}{|k^2-n^2|} &\leq \Big(\sum_{k\leq[\frac{n}{2}]} \frac{|k|^{2\alpha}}{|k^2-n^2|^2}\Big)^{1/2} ||a||_{h^{-\alpha}} \\ &+ \Big(\sum_{k>[\frac{n}{2}]} \frac{|k|^{2\alpha}}{|k^2-n^2|^2}\Big)^{1/2} ||a||_{h^{-\alpha}}^{[n]}.\end{aligned}$$

The claimed statement then follows from the estimate

$$\sum_{k\leq[\frac{n}{2}]} \frac{|k|^{2\alpha}}{|k^2-n^2|^2} = O(n^{2\alpha-3})$$

together with Lemma 8.2 below. □

Lemma 8.2. *For any* $0\leq\alpha\leq 1$ *there exists a constant* $C=C(\alpha)>0$ *such that*

$$\sum_{k\in\mathbb{N}\setminus\{n\}} \frac{|k|^{2\alpha}}{|k^2-n^2|^2} \leq Cn^{2\alpha-2}.$$

Proof. The latter sum is split up into two parts which are estimated separately. For the first part we obtain

$$\begin{aligned}\sum_{1\leq k\leq 2n, k\neq n} \frac{|k|^{2\alpha}}{|k^2-n^2|^2} &\leq 2^{2\alpha}n^{2\alpha} \sum_{1\leq k\leq 2n, k\neq n} \frac{1}{n^2|k-n|^2} \\ &\leq 2^{2\alpha}n^{2\alpha-2}\sum_{k\geq 1}\frac{2}{k^2}\end{aligned}$$

and for the second part

$$\sum_{k\geq 2n+1} \frac{|k|^{2\alpha}}{|k^2-n^2|^2} \leq \frac{16}{9}\sum_{k\geq 2n+1} k^{2\alpha-4} \leq \frac{16}{9}\int_{2n}^{\infty} x^{2\alpha-4}\,dx \leq C_1 n^{2\alpha-3} \quad (8.2)$$

as $|k^2-n^2|\geq 3k^2/4$ for $k\geq 2n+1$. □

The following technical lemma is used frequently.

Lemma 8.3. *Consider a sequence* $(b_k(s))_{k\geq 1}\subset\mathbb{C}$ *depending on a parameter* s *from a parameter set* X. *Assume that* $|b_k(s)|\leq 1/2$ *for any* $s\in X$ *and* $k\geq 1$ *and that there exists a constant* $\varkappa>0$ *such that* $\sum_{k\geq 1}|b_k(s)|<\varkappa$ *for any* s. *Then there exists a constant* $C=C(\varkappa)>0$ *such that*

$$\Big|\prod_{k\geq 1}(1+b_k(s))\Big|\leq 1+C\sum_{k\geq 1}|b_k(s)|$$

uniformly in $s\in X$.

The proof of Lemma 8.3 is straightforward and hence omitted.

Lemma 8.4. *Let* $0\leq\alpha\leq 1$ *and* $a=(a_k)_{k\geq 1}\in h_{\mathbb{C}}^{-\alpha}$. *Then there exists* $n_0\in\mathbb{N}$ *such that uniformly on the circles* $|\lambda|=(n+1/2)^2\pi^2$, $n\geq n_0$,

$$\prod_{k\geq 1}\frac{k^2\pi^2+a_k-\lambda}{k^2\pi^2}=\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Big(1+O(||a||_{h^{-\alpha}}n^{\alpha-3/2}+||a||_{h^{-\alpha}}^{[n]})\Big). \tag{8.3}$$

Moreover, n_0 *can be chosen locally uniformly in* a.

Proof. Taking $n_0\in\mathbb{N}$ sufficiently big and applying Lemma 8.3 together with Lemma 8.1 and the product representation $\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}=\prod_{k\geq 1}\frac{k^2\pi^2-\lambda}{k^2\pi^2}$ (cf. [26]) one obtains for $n\geq n_0$

$$\begin{aligned}\Big(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Big)^{-1}\prod_{k\geq 1}\frac{k^2\pi^2+a_k-\lambda}{k^2\pi^2} &= \Big(1+\frac{a_n}{n^2\pi^2-\lambda}\Big)\prod_{k\in\mathbb{N}\setminus\{n\}}\Big(1+\frac{a_k}{k^2\pi^2-\lambda}\Big)\\ &= (1+O(|a_n|/n))\Big(1+O\Big(\sum_{k\in\mathbb{N}\setminus\{n\}}\frac{|a_k|}{|k^2-n^2|}\Big)\Big)\\ &= 1+O(||a||_{h^{-\alpha}}n^{\alpha-3/2}+||a||_{h^{-\alpha}}^{[n]}).\end{aligned}$$

□

Lemma 8.5. *Let* $0\leq\alpha\leq 1$, $a^0\in h_{\mathbb{C}}^{-\alpha}$, *and* $\epsilon>0$. *Then there exist a neighborhood* $U(a^0)\subset h_{\mathbb{C}}^{-1}$ *of* a^0, $n_0\in\mathbb{N}$ *and a constant* $0<\delta<\pi/2$ *such that for any* $a\in U(a^0)$, $n\geq n_0$, *and* $\sqrt{\lambda}=n\pi+\mathtt{le}(\delta)$

$$\prod_{k\in\mathbb{N}\setminus\{n\}}\frac{k^2\pi^2+a_k-\lambda}{k^2\pi^2}=\frac{(-1)^{n+1}}{2}(1+\mathtt{le}(\epsilon)).$$

Proof. For any $0 < \beta < \pi/2$ there exists a constant $\rho > 0$ such that for any $n \geq 1$, $k \neq n$ and $\sqrt{\lambda} = n\pi + \mathtt{le}(\beta)$ one has $|k^2\pi^2 - \lambda| \geq \rho|k^2 - n^2|$. Taking $n_0 \geq 1$ sufficiently big and applying Lemma 8.3 together with Lemma 8.1 one obtains for $n \geq n_0$

$$\begin{aligned} \prod_{k\in\mathbb{N}\setminus\{n\}} \frac{k^2\pi^2 + a_k - \lambda}{k^2\pi^2 - \lambda} &= \prod_{k\in\mathbb{N}\setminus\{n\}} \Big(1 + \frac{a_k}{k^2\pi^2 - \lambda}\Big) \\ &= 1 + O\Big(\sum_{k\in\mathbb{N}\setminus\{n\}} \frac{|a_k|}{|k^2 - n^2|}\Big) \\ &= 1 + O(n^{\alpha-3/2})||a||_{h^{-\alpha}} + O(n^{\alpha-1})||a||^{[n]}_{h^{-\alpha}}. \quad (8.4) \end{aligned}$$

On the other side one easily gets from the formula $\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = \prod_{k\geq 1} \frac{k^2\pi^2 - \lambda}{k^2\pi^2}$ that

$$\left| \prod_{k\in\mathbb{N}\setminus\{n\}} \frac{k^2\pi^2 - \lambda}{k^2\pi^2} - \frac{(-1)^{n+1}}{2} \right| = O(\beta) + O(1/n) \quad (8.5)$$

uniformly for $\sqrt{\lambda} = n\pi + \mathtt{le}(\beta)$. Indeed, for $\sqrt{\lambda} = n\pi + \mu$, $|\mu| \leq \beta$ we obtain

$$\begin{aligned} \prod_{k\in\mathbb{N}\setminus\{n\}} \frac{k^2\pi^2 - \lambda}{k^2\pi^2} &= \frac{n^2\pi^2}{n^2\pi^2 - \lambda} \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \\ &= \frac{(-1)^{n+1}}{2}(1 + \mu/2n\pi)^{-1}(1 + \mu/n\pi)^{-1}\frac{\sin\mu}{\mu} \\ &= \frac{(-1)^{n+1}}{2}(1 + O(\beta) + O(1/n)). \end{aligned}$$

It then follows from (8.4) and (8.5) together with the estimate

$$||a||^{[n]}_{h^{-\alpha}} \leq ||a^0||^{[n]}_{h^{-\alpha}} + ||a - a^0||_{h^{-\alpha}} \quad (8.6)$$

that there exist $U(a^0)$, n_0, and δ as claimed. □

Let $\mathcal{W}$ be the neighborhood of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ given by Corollary 10.2.

Lemma 8.6. *Let $(\sigma^0, q_0) \in h_{\mathbb{C}}^{-1} \times \mathcal{W}$ and $\epsilon > 0$. Then there exist a neighborhood U of (σ^0, q_0) in $h_{\mathbb{C}}^{-1} \times \mathcal{W}$ and $n_0 \in \mathbb{N}$ such that for any $(\sigma, q) \in U$, $n \geq n_0$, and $\sqrt{\lambda} = n\pi + \mathtt{le}(\pi/2)$*

$$\prod_{k\in\mathbb{N}\setminus\{n\}} \frac{k^2\pi^2 + \sigma_k - \lambda}{\sqrt[+]{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}} = (-1)^{n+1}(1 + \mathtt{le}(\epsilon)).$$

Proof. In view of Lemma 3, Appendix L in [25] it suffices to prove that

$$\Big(\prod_{k\in\mathbb{N}\setminus\{n\}} \frac{k^2\pi^2+\sigma_k-\lambda}{\sqrt[+]{(\lambda_{2k}-\lambda)(\lambda_{2k-1}-\lambda)}}\Big)^2 \quad = \quad \prod_{k\in\mathbb{N}\setminus\{n\}}\Big(1+\frac{(\bar\sigma_k-\lambda)^2-(\tau_k-\lambda)^2+(\gamma_k/2)^2}{(\lambda_{2k}-\lambda)(\lambda_{2k-1}-\lambda)}\Big)$$

can be estimated as $1+\mathtt{le}(\epsilon)$ for n sufficiently large. Hence by Lemma 8.3 we need to estimate the sum $\sum_{k\in\mathbb{N}\setminus\{n\}} b_k(\sigma,\lambda,q)$ where

$$b_k(\sigma,\lambda,q) \overset{\mathtt{Def}}{=} \frac{|(\bar\sigma_k-\lambda)^2-(\tau_k-\lambda)^2+(\gamma_k/2)^2|}{|\lambda_{2k}-\lambda||\lambda_{2k-1}-\lambda|}$$

and $\bar\sigma_k \overset{\mathtt{Def}}{=} k^2\pi^2+\sigma_k$. As in Lemma 10.4 there exist constants $r_1, r_2>0$, a neighborhood U of (σ^0,q_0) in $h^{-1}_{\mathbb{C}}\times\mathcal{W}$ and a constant $m_0>0$ so that for any $n\ge m_0$, $\sqrt{\lambda}=n\pi+\mathtt{le}(\pi/2)$ and any $k\ne n$, $|\bar\sigma_k-\lambda|\le r_1|k^2-n^2|$, $|\tau_k-\lambda|\le r_1|k^2-n^2|$, $|\lambda_{2k}-\lambda|\ge r_2|k^2-n^2|$, and $|\lambda_{2k-1}-\lambda|\ge r_2|k^2-n^2|$ uniformly in U. In particular, for $(\sigma,q)\in U$ and $\sqrt{\lambda}=n\pi+\mathtt{le}(\pi/2)$ the sum $\sum_{k\in\mathbb{N}\setminus\{n\}} b_k(\sigma,\lambda,q)$ is estimated by

$$\sum_{k\in\mathbb{N}\setminus\{n\}}\frac{|\bar\sigma_k-\tau_k|}{|k^2-n^2|}+\sum_{k\in\mathbb{N}\setminus\{n\}}\frac{|\gamma_k|^2}{|k^2-n^2|^2}. \tag{8.7}$$

One has

$$\begin{aligned}\sum_{k\in\mathbb{N}\setminus\{n\}}\frac{|\gamma_k|^2}{|k^2-n^2|^2} \quad &\le \quad \Big(\sum_{k\le[\frac{n}{2}]}\frac{|k|^4}{|k^2-n^2|^4}\Big)^{1/2}||(\gamma_k^2/|k|^2)||_{h^0}\\ &+ \quad \Big(\sum_{k>[\frac{n}{2}],k\ne n}\frac{|k|^4}{|k^2-n^2|^4}\Big)^{1/2}||(\gamma_k^2/|k|^2)||_{h^0}^{[n]}\end{aligned}$$

where $\Big(\sum_{k\le[\frac{n}{2}]}\frac{|k|^4}{|k^2-n^2|^4}\Big)^{\frac12}=O(n^{-\frac32})$ and $\Big(\sum_{k>[\frac{n}{2}]}\frac{|k|^4}{|k^2-n^2|^4}\Big)^{\frac12}=O(1)$ are estimated similarly as in the proof of Lemma 8.2. By estimate (10.3) and Proposition 10.1 the quantity $||(\gamma_k^2/k^2)||_{h^0}^{[n]}$ is locally uniformly small in q. Applying Lemma 8.1 to the first sum in (8.7) and estimating the h^{-1}-norms using Proposition 10.1 and estimate (8.6) we conclude that for any $\epsilon>0$ there exist $n_0\ge m_0$ and a neighborhood U of (σ^0,q_0) in $h^{-1}_{\mathbb{C}}\times\mathcal{W}$ such that for any $n\ge n_0$, $(\sigma,q)\in U$, and $\sqrt{\lambda}=n\pi+\mathtt{le}(\pi/2)$, one has $b_k(\sigma,\lambda,q)\le 1/2$ for any $k\ne n$ and $\prod_{k\in\mathbb{N}\setminus\{n\}}(1+b_k)=1+\mathtt{le}(\epsilon)$. □

Lemma 8.7. *For any* $(\sigma^0, q_0) \in h^{-1}_{\mathbb{C}} \times \mathcal{W}$ *there exists a neighborhood* U *of* (σ^0, q_0) *in* $h^{-1}_{\mathbb{C}} \times \mathcal{W}$ *such that for any* $(\sigma, q) \in U$ *and any* $n \geq 1$*,*

$$\prod_{k\in\mathbb{N}\setminus\{n\}} \frac{k^2\pi^2 + \sigma_k - \lambda}{\sqrt[+]{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}} = O(1) \tag{8.8}$$

uniformly for $\lambda \in U_n$ *where* $\{U_k\}_{k\geq 1}$ *is a system of isolating neighborhoods.*

Proof. We split the index set in the product in (8.8) into a finite part $A \overset{\mathtt{Def}}{=} \{k \in \mathbb{N} \setminus \{n\} | 1 \leq k \leq N\}$ and its complement $A^c \overset{\mathtt{Def}}{=} \{k \in \mathbb{N} \setminus \{n\} | k > N\}$ and estimate the corresponding products separately. Choosing N sufficiently large it follows as in the proof of Lemma 8.6 that there exists a neighborhood U of (σ^0, q_0) in $h^{-1}_{\mathbb{C}} \times \mathcal{W}$ so that the product over the index set A^c is of size $O(1)$. Shrinking if necessary the neighborhood U, the same holds for the product over the index set A. $\square$

Combining Lemma 8.5 and 8.6 one obtains

Lemma 8.8. *Let* $q_0 \in H^{-1}_0(\mathbb{T}^1, \mathbb{C})$*,* $\epsilon > 0$*. Then there exists a neighborhood* $U(q_0) \subset H^{-1}_0(\mathbb{T}^1, \mathbb{C})$ *of* q_0*,* $n_0 \in \mathbb{N}$ *and a constant* $0 < \beta < \pi/2$ *such that for any* $q \in U(q_0)$*,* $n \geq n_0$*, and* $\sqrt{\lambda} = n\pi + \mathtt{le}(\beta)$

$$\prod_{k\in\mathbb{N}\setminus\{n\}} \frac{k^2\pi^2}{\sqrt[+]{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}} = 2(1 + \mathtt{le}(\epsilon)).$$

9 Appendix B: Holomorphic differentials

In this section we prove the following theorem concerning holomorphic differentials. It extends the corresponding theorem [13, Theorem D.1] stated for L^2-potentials. For notions such as Γ_m, $\Delta(\lambda, q)$ or $\sqrt[c]{\Delta(\lambda, q)^2 - 4}$ see Section 10.

Theorem 9.1. *There exists a complex neighborhood* $\mathcal{W}$ *of* $H^{-1}_0(\mathbb{T}^1)$ *such that for each* q *in* $\mathcal{W}$ *there exist entire functions* ψ_n*,* $n \geq 1$*, satisfying*

$$\frac{1}{2\pi} \int_{\Gamma_m} \frac{\psi_n(\lambda, q)}{\sqrt[c]{\Delta^2(\lambda, q) - 4}} \, d\lambda = \delta_{mn}$$

for all $m \geq 1$. These functions depend analytically on λ, q and admit a product representation

$$\psi_n(\lambda) = \frac{2}{\pi n} \prod_{m \neq n} \frac{\sigma_m^n(q) - \lambda}{m^2 \pi^2},$$

whose complex coefficients σ_m^n depend real analytically on q and satisfy, for any $m \geq 1$,

$$|\sigma_m^n - \tau_m| \leq C \, \frac{|\gamma_m^2|}{m}$$

where $C > 0$ can be chosen locally uniformly on $\mathcal{W}$ and uniformly in n, and $\tau_m \overset{\text{Def}}{=} (\lambda_{2m} + \lambda_{2m-1})/2$.

It turns out that the same approach used in [13] also works for potentials in $H_0^{-1}(\mathbb{T}^1)$, taking into account the spectral results in [14] and the results on infinite products stated in Appendix A. As in [13] we prove this theorem with the help of the implicit function theorem. Following [13] we reformulate the statement in terms of a functional equation.

In the following, it is convenient to denote σ_m^n as $\overline{\sigma}_m^n$, and to use the former symbol for general $h_{\mathbb{C}}^{-1}$-sequences.

For $\sigma = (\sigma_m)_{m \geq 1}$ in $h_{\mathbb{C}}^{-1}$ and $n \geq 1$ define an entire function $\phi_n(\sigma, \lambda)$ by

$$\phi_n(\sigma, \lambda) = \prod_{m \neq n} \frac{\overline{\sigma}_m - \lambda}{m^2 \pi^2} = \prod_{m \neq n} \Big(1 + \frac{\sigma_m - \lambda}{m^2 \pi^2} \Big),$$

where $\overline{\sigma}_m \overset{\text{Def}}{=} m^2 \pi^2 + \sigma_m$ throughout this appendix. Note that the infinite product above converges absolutely and locally uniformly in $h_{\mathbb{C}}^{-1} \times \mathbb{C}$ as the sum $\sum_{m \geq 1} \frac{|\sigma_m| + |\lambda|}{\pi^2 m^2}$ converges locally uniformly in $h_{\mathbb{C}}^{-1} \times \mathbb{C}$. Indeed, for any $N \in \mathbb{N}$ and any integer $p \geq 0$ one can write

$$\sum_{m=N}^{N+p} \frac{|\sigma_m|}{m^2} = \sum_{m=N}^{N+p} \frac{|\sigma_m|}{m^{1+1/4}} \frac{1}{m^{3/4}} \leq C \sqrt{\sum_{m=N}^{N+p} \frac{|\sigma_m|^2}{m^2} \frac{1}{m^{1/2}}}, \tag{9.1}$$

where $C \overset{\text{Def}}{=} \sqrt{\sum_{m \geq 1} \frac{1}{m^{3/2}}} < \infty$. As the sum $\sum_{m \geq 1} \frac{|\sigma_m|^2}{m^2} = \|\sigma\|_{h^{-1}}^2$ is locally bounded in h^{-1} and as $1/m^{1/2}$ converges to zero monotonely as $m \to \infty$ the sum $\sum_{m \geq 1} \frac{|\sigma_m|^2}{m^2} \frac{1}{m^{1/2}}$ converges locally uniformly for σ in h^{-1} by the Dirichlet criterium of uniform convergence. Hence by (9.1), the sum $\sum_{m \geq 1} \frac{|\sigma_m| + |\lambda|}{\pi^2 m^2}$

converges locally uniformly in $h^{-1}_{\mathbb{C}} \times \mathbb{C}$. In particular, ϕ_n is an analytic function on $h^{-1}_{\mathbb{C}} \times \mathbb{C}$.
For q in $H_0^{-1}(\mathbb{T}^1)$ and $m \geq 1$, define a linear functional $A_m(q)$ on the space of entire functions by

$$A_m(q)\phi = \frac{1}{2\pi} \int_{\Gamma_m} \frac{\phi(\lambda)}{\sqrt[c]{\Delta^2(\lambda, q) - 4}} \, d\lambda.$$

One can choose the contours Γ_m to be locally independent of q, and arbitrarily close to the real intervals

$$G_m(q) = [\lambda_{2m-1}(q), \lambda_{2m}(q)] \tag{9.2}$$

so that A_m are actually well defined on the space of real analytic functions on the real line.
For each $n \geq 1$ we then consider on the real Hilbert space $h^{-1} \times H_0^{-1}$ the functional equation

$$F^n(\sigma, q) = 0,$$

where $F^n = (F^n_m)_{m \geq 1}$ with

$$F^n_m(\sigma, q) = \begin{cases} A^n_m(q)\phi_n(\sigma), & m \neq n, \\ \overline{\sigma}_n - \tau_n(q), & m = n, \end{cases} \tag{9.3}$$

and, for $m \neq n$,

$$A^n_m = w^n_m A_m, \quad w^n_m = 2\pi m \frac{n^2 - m^2}{n^2}.$$

In fact, each function F^n_m is well defined and real analytic on some complex neighborhood U of $h^{-1} \times H_0^{-1}$, independent of n and m.
We show that under some mild provisions there exists a unique solution $\sigma^n(q)$ of $F^n(\sigma, q) = 0$, which is real analytic in q and extends to some complex neighborhood of $H_0^{-1}(\mathbb{T}^1)$ independently of n. We then verify that $\overline{\sigma}^n_m = \tau_m + O(\gamma_m^2/m)$. By [13] this solution satisfies for $q \in L^2_0$

$$A_n(q)\phi_n(\sigma^n(q)) = \frac{\pi n}{2}.$$

By analyticity and density, this equation will hold on some complex neighborhood $\mathcal{W}$ of $H_0^{-1}(\mathbb{T}^1)$ as well, so the functions

$$\psi_n \overset{\mathtt{Def}}{=} \frac{2}{\pi n} \phi_n(\sigma^n)$$

will have the required properties.

9.1 Real Solutions

Before constructing real solutions we first establish the proper setting of the functionals F^n.

Lemma 9.2. *For each $n \geq 1$, equation (9.3) defines a map*

$$F^n : h^{-1} \times H_0^{-1} \to h^{-1}$$
$$(\sigma, q) \mapsto F^n(\sigma, q),$$

which is real analytic and extends analytically to a complex neighborhood U of $h^{-1} \times H_0^{-1}$. Moreover, this neighborhood U can be chosen independently of n and so that all F^n are locally uniformly bounded on it.

Proof. Fix $n \geq 1$ and consider F_m^n for $m \neq n$. As in [13], by the definition of ϕ_n and the fact that $\sqrt[c]{\Delta^2 - 4}$ can be written as

$$\frac{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}}{m^2\pi^2} \, 2(-1)^{m+1} \sqrt[+]{\lambda - \lambda_0} \prod_{l \neq m} \frac{\sqrt[+]{(\lambda_{2l} - \lambda)(\lambda_{2l-1} - \lambda)}}{l^2\pi^2}$$

for λ close to Γ_m, we have

$$F_m^n(\sigma, q) = \frac{1}{2\pi} \int_{\Gamma_m} \frac{\overline{\sigma}_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} \, w_m^n \zeta_m^n(\lambda) \, d\lambda,$$

where

$$\zeta_m^n(\lambda) = \frac{(-1)^{m+1}}{2\sqrt[+]{\lambda - \lambda_0}} \frac{n^2\pi^2}{\overline{\sigma}_n - \lambda} \prod_{l \neq m} \frac{\overline{\sigma}_l - \lambda}{\sqrt[+]{(\lambda_{2l} - \lambda)(\lambda_{2l-1} - \lambda)}}. \tag{9.4}$$

Using Lemma 9.4 and [13, Lemma M.1] we get, locally uniformly on a complex neighborhood U of $h^{-1} \times H_0^{-1}$ and uniformly in $m \neq n$,

$$|F_m^n(\sigma, q)| = O(\max_{\lambda \in \Gamma_m} |\overline{\sigma}_m - \lambda|) = O(\rho_m),$$

where the sequence $(\rho_m)_{m \geq 1}$, given by

$$\rho_m \overset{\texttt{Def}}{=} |\overline{\sigma}_m - \tau_m| + |\gamma_m| + 1/m \tag{9.5}$$

is bounded in h^{-1} locally uniformly on U (cf. Proposition 10.1).
We already noticed that each function F_m^n is real analytic on U. Analyticity of the entire map F^n then follows with [25, Theorem 3, Appendix A]. □

Lemma 9.3. *For each $\varepsilon > 0$ and each point (σ^0, q) in $h^{-1} \times H_0^{-1}$ there exists a number $m_0 \in \mathbb{N}$ such that, on a complex neighborhood of (σ^0, q),*

$$w_m^n \zeta_m^n(\lambda) = 1 + \mathtt{le}(\varepsilon)$$

for any $n \geq 1$, $m \geq m_0$ with $m \neq n$, and $\lambda \in \Gamma_m$, where the contours Γ_m are chosen at a distance of $|\gamma_m| + 1/m$ around τ_m.

Proof. For λ close to G_m, the term $w_m^n \zeta_m^n(\lambda)$ can be written in the form

$$w_m^n \zeta_m^n(\lambda) = \frac{\pi m}{\sqrt[+]{\lambda - \lambda_0}} \cdot \frac{n^2\pi^2 - m^2\pi^2}{\overline{\sigma}_n - \lambda} \cdot (-1)^{m+1} \prod_{l \neq m} \frac{\overline{\sigma}_l - \lambda}{\sqrt[+]{(\lambda_{2l} - \lambda)(\lambda_{2l-1} - \lambda)}}. \tag{9.6}$$

Note that, on a complex neighborhood around any given point in $h^{-1} \times H_0^{-1}$, we may choose $\delta > 0$ and contours Γ_m around G_m of order $|\gamma_m| + 1/m$ such that

$$\inf_{n \neq m} \min_{\lambda \in \Gamma_m} |\overline{\sigma}_n - \lambda| \geq \delta.$$

Then, uniformly on a suitably chosen complex neighborhood of any point in $h^{-1} \times H_0^{-1}$, and uniformly for $m \neq n$ and $\lambda \in \Gamma_m$, the first term on the right hand side in (9.6) is of the form

$$\frac{\pi m}{\sqrt[+]{\lambda - \lambda_0}} = 1 + O\Big(\frac{1}{m}\Big),$$

whereas the second term can be estimated by

$$\begin{aligned} \frac{n^2\pi^2 - m^2\pi^2}{\overline{\sigma}_n - \lambda} &= 1 + O\Big(\frac{|\sigma_n| + |\lambda - m^2\pi^2|}{|n^2 - m^2|}\Big) \\ &= 1 + O\Big(\sum_{l \neq m} \frac{|\sigma_l|}{|l^2 - m^2|} + \frac{|\lambda - m^2\pi^2|}{m}\Big) \\ &= 1 + O\Big(\frac{\|\sigma\|_{h^{-1}}}{\sqrt{m}} + \|\sigma\|_{h^{-1}}^{[m]} + \frac{|\gamma_m| + 1/m + |\tau_m - m^2\pi^2|}{m}\Big), \end{aligned}$$

where for the last inequality we have used Lemma 8.1 and the fact that the contours Γ_m are at a distance of order $|\gamma_m| + 1/m$ around τ_m. The desired estimate then follows from the estimate

$$||\sigma||_{h^{-1}}^{[m]} \leq ||\sigma^0||_{h^{-1}}^{[m]} + ||\sigma - \sigma^0||_{h^{-1}}$$

together with Proposition 10.1. The third term on the right hand side of (9.6) is estimated by Lemma 8.6. □

Using similar arguments as in the proof of Lemma 9.3 one proves

Lemma 9.4. *For each point* $(\sigma, q) \in h^{-1} \times H_0^{-1}(\mathbb{T}^1)$ *there exists a complex neighborhood* U *of* (σ, q) *such that*

$$w_m^n \zeta_m^n(\lambda) = O(1)$$

uniformly for $n \neq m$ *and* $\lambda \in U_m$ *with* $(U_k)_{k \geq 1}$ *chosen as in Corollary 10.2.*

Next we consider the Jacobian of F^n with respect to σ. At any given point in $h^{-1} \times H_0^{-1}$ this Jacobian is a bounded linear operator $Q^n : h^{-1} \to h^{-1}$, which is represented by an infinite matrix (Q_{mr}^n) with elements

$$Q_{mr}^n = \frac{\partial F_m^n}{\partial \sigma_r} = \frac{\partial}{\partial \sigma_r} A_m^n \phi_n = A_m^n \frac{\partial \phi_n}{\partial \sigma_r}, \quad m, r \neq n, \tag{9.7}$$

with $\frac{\partial \phi_n}{\partial \sigma_r} = \frac{1}{r^2\pi^2} \prod_{l \neq n,r} \frac{\overline{\sigma}_l - \lambda}{l^2\pi^2} = \frac{\phi_n}{\overline{\sigma}_r - \lambda}$ while

$$Q_{mn}^n = Q_{nm}^n = \delta_{mn}.$$

The following simple observation, which is used several times below, is proven exactly as in [13].

Lemma 9.5. *If* ϕ *is real analytic on the real line, and* $A_m \phi = 0$ *for some* $m \geq 1$, *then* ϕ *has a root in* $[\lambda_{2m-1}(q), \lambda_{2m}(q)]$.

This lemma shows that we have to look for the zero σ_m^n in the interval $G_m(q) = [\lambda_{2m-1}(q), \lambda_{2m}(q)]$. It therefore makes sense to restrict ourselves to the open domain $V \subset h^{-1} \times H_0^{-1}$ characterized by

$$\frac{\lambda_{2k-2} + \lambda_{2k-1}}{2} < \overline{\sigma}_k < \frac{\lambda_{2k} + \lambda_{2k+1}}{2}, \quad k \geq 1.$$

As a consequence, any solution (σ, q) in V leads to a monotone sequence $\overline{\sigma}_m^n$, which in turn makes σ unique.

Lemma 9.6.

(a) On $V \subset h^{-1} \times H_0^{-1}$, the diagonal elements Q^n_{mm} never vanish, and for any point $(\sigma, q) \in V$ and any $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that, locally uniformly on a sufficiently small neighborhood of (σ, q) in V,

$$Q^n_{mm} = 1 + \mathtt{le}(\varepsilon) \quad \text{for any } m \geq m_0 \text{ and } n \geq 1 \text{ with } n \neq m.$$

(b) Locally uniformly on V, and uniformly in $n \geq 1$,

$$Q^n_{mr} = O\Big(\frac{\rho_m}{|m^2 - r^2|}\Big)$$

for $m \neq r$ and $m, r \neq n$, with ρ_m defined as in (9.5).

Proof. As in [13], one easily shows that, for $m, r \neq n$,

$$Q^n_{mr} = \frac{1}{2\pi} \int_{\Gamma_m} \left(\frac{\overline{\sigma}_m - \lambda}{\overline{\sigma}_r - \lambda} \right) \frac{w^n_m \zeta^n_m(\lambda)}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} \, d\lambda. \tag{9.8}$$

The claimed estimates are then proven as in [13], using [13, Lemma M.1] as well as Lemma 9.4 for (a) and, for (b), in addition

$$\frac{\overline{\sigma}_m - \lambda}{\overline{\sigma}_r - \lambda} = O\Big(\frac{\rho_m}{|m^2 - r^2|}\Big)$$

with ρ_m defined as in (9.5). By (9.8), Q^n_{mm} does not vanish as ζ^n_m has no root in $[\lambda_{2m-1}, \lambda_{2m}]$. □

Lemma 9.7. *At any point in V the Jacobian Q^n of F^n with respect to σ is of the form*

$$Q^n = D^n + K^n$$

where $D^n : h^{-1} \to h^{-1}$ is an isomorphism in diagonal form, and $K^n : h^{-1} \to h^{-1}$ is compact.

Proof. Set $D^n \overset{\mathtt{Def}}{=} \mathtt{diag}(Q^n_{mm})$. By the preceding lemma, $D^n : h^{-1} \to h^{-1}$ has a bounded inverse. Moreover, $K^n = Q^n - D^n$ is a bounded linear operator on h^{-1} with vanishing diagonal, and off-diagonal elements

$$K^n_{mr} = Q^n_{mr} = O\left(\frac{\rho_m}{|m^2 - r^2|}\right), \quad m \neq r$$

again by the preceding lemma.
By Lemma 8.2 and as $(\rho_m)_{m\geq 1} \in h^{-1}$

$$\sum_{\substack{m,r\\ m\neq r}} \Big(\frac{r}{|m^2-r^2|}\,\frac{\rho_m}{m}\Big)^2 < \infty$$

so K^n is Hilbert-Schmidt, hence compact. □

Lemma 9.8. *At any given point in $V \subset h^{-1} \times H_0^{-1}$, each Jacobian Q^n for $n \geq 1$ is one-to-one and hence a linear isomorphism $h^{-1} \to h^{-1}$.*

Proof. Fix $n \geq 1$. To show that Q^n is one-to-one, suppose that $Q^n h = 0$ for some $h \in h^{-1}$. Then clearly $h_n = 0$, since $Q^n_{nr} = \delta_{nr}$ for $r \geq 1$ by definition. Moreover, as in [13], one infers from $Q^n h = 0$ and (9.7) that

$$A_m \psi = 0 \quad \text{for any } m \neq n,$$

where ψ is the entire function

$$\psi(\lambda) \overset{\texttt{Def}}{=} \phi_n \cdot \sum_{r\neq n} \frac{h_r}{\overline{\sigma}_r - \lambda}.$$

To see that $\psi(\lambda)$ is entire note that the sum $\sum_{r\neq n} \frac{h_r}{\overline{\sigma}_r-\lambda}$ converges uniformly on every bounded compact set B of $\lambda's$ in $\mathbb{C}$ such that $B \cap \{\overline{\sigma}_r\}_{r\neq n} = \emptyset$. In particular, $\sum_{r\neq n} \frac{h_r}{\overline{\sigma}_r-\lambda}$ is a meromorphic function of λ with simple poles at $\{\overline{\sigma}_r\}_{r\neq n}$. As $\phi_n(\lambda)$ is an entire function of λ with zeroes at $\{\overline{\sigma}_r\}_{r\neq n}$ the function $\psi(\lambda)$ is entire as well.
Hence, by Lemma 9.5, ψ has for any $m \neq n$ a root ξ_m in the interval $[\lambda_{2m-1}, \lambda_{2m}]$. Consequently, letting $\overline{\sigma}_n = \xi_n = \tau_n$,

$$\psi_*(\lambda) \overset{\texttt{Def}}{=} \frac{\overline{\sigma}_n - \lambda}{n^2\pi^2}\psi = \sum_{r\neq n} \frac{h_r}{\overline{\sigma}_r - \lambda}\phi_*, \quad \text{with} \quad \phi_* = \prod_{l\geq 1} \frac{\overline{\sigma}_l - \lambda}{l^2\pi^2},$$

is also an entire function with roots ξ_m for any $m \geq 1$. Using Liouville's theorem, we now show that, in fact, $\psi_* = 0$.
Note that, on the circles $|\lambda| = R_k \overset{\texttt{Def}}{=} (k+\frac{1}{2})^2\pi^2$,

$$\sum_{r\neq n} \Big|\frac{h_r}{\overline{\sigma}_r - \lambda}\Big| = O\Big(\frac{|h_k|}{k} + \sum_{r\neq k} \frac{|h_r|}{|r^2-k^2|}\Big) = O(k^{-1/2}\|h\|_{h^{-1}} + \|h\|^{[k]}_{h^{-1}})$$

by Lemma 8.1. Moreover, by Lemma 8.4, there exists $k_0 \in \mathbb{N}$ such that, on the circles $|\lambda| = R_k$ for $k \geq k_0$,

$$\phi_*(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Big(1 + \mathtt{le}(1/2)\Big),$$

and

$$\chi_*(\lambda) \overset{\mathtt{Def}}{=} \prod_{l \geq 1} \frac{\xi_l - \lambda}{l^2\pi^2} = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Big(1 + \mathtt{le}(1/2)\Big).$$

Combining the previous three estimates and using the definition of ψ_*, the quotient ψ_*/χ_* is an entire function which tends to 0 uniformly on the circles $|\lambda| = R_k$ as $k \to \infty$. By Liouville's theorem this is only possible for $\psi_* = 0$, hence $h = 0$ by evaluating ψ_* at the points $\overline{\sigma}_m, m \neq n$.
This shows that Q^n is one-to-one. By the preceding lemma and the Fredholm Alternative, Q^n is thus an isomorphism. □

Lemmas 9.2 and 9.8 allow us to apply the implicit function theorem to any particular solution of $F^n(\sigma, q) = 0$ in the domain V. In fact, arguing as in [13] we get the following result:

Proposition 9.9. *For any $n \geq 1$ there exists a unique real analytic map*

$$\sigma^n : H_0^{-1} \to h^{-1}$$

with graph in V such that

$$F^n\left(\sigma^n(q), q\right) = 0$$

everywhere. Indeed, $\overline{\sigma}_m^n(q) \in G_m(q)$ for each $m \geq 1$ at every point $q \in H_0^{-1}(\mathbb{T}^1)$.

9.2 Complex Extension

Proposition 9.10. *Each real analytic map $\sigma^n : H_0^{-1} \to h^{-1}$ of Proposition 9.9 extends to a common complex neighborhood of H_0^{-1}.*

Proof. To verify that the solutions σ^n of Proposition 9.9 all extend to a common complex neighborhood of $h^{-1} \times H_0^{-1}$ independent of n, we first show that, at every real point $q \in H_0^{-1}$, the inverses of the Jacobians

$$Q^n\left(\sigma^n(q), q\right) = \frac{\partial F^n}{\partial \sigma}\left(\sigma^n(q), q\right)$$

are bounded uniformly in n.

Recall that, by Proposition 9.9, the solutions $\sigma^n(q)$ are in the compact subset $\Pi(q)$ of h^{-1},

$$\Pi(q) = \prod_{m\geq 1} \left(G_m(q) - m^2\pi^2\right).$$

Fix $q \in H_0^{-1}$. Consider the Jacobian $Q^n = (Q^n_{mr})$ at a point in $\Pi(q) \times \{q\} \subseteq V$. By (9.7),

$$Q^n_{mr} = \frac{w^n_m}{2\pi} \int_{\Gamma_m} \frac{\partial \phi_n}{\partial \sigma_r} \frac{d\lambda}{\sqrt[c]{\Delta^2(\lambda) - 4}}.$$

In this identity for Q^n_{mr} we can pass to the limit $n \to \infty$ to obtain

$$Q^n_{mr} \to Q^*_{mr} \overset{\text{Def}}{=} m \int_{\Gamma_m} \frac{\partial \phi_*}{\partial \sigma_r} \frac{d\lambda}{\sqrt[c]{\Delta^2(\lambda) - 4}}, \quad \phi_* \overset{\text{Def}}{=} \prod_{l\geq 1} \frac{\overline{\sigma}_l - \lambda}{l^2\pi^2},$$

where ϕ_* is the limit of $\phi_n \overset{\text{Def}}{=} \prod_{l\neq n} \frac{\overline{\sigma}_l - \lambda}{l^2\pi^2} = \frac{n^2\pi^2}{\overline{\sigma}_l - \lambda}\phi_*$. Inspecting the proof of Lemma 9.3, one finds that the estimate of $w^n_m \zeta^n_m$ from Lemma 9.3 holds uniformly for $\sigma \in \Pi(q)$. It follows that the asymptotic estimates from Lemma 9.6 for Q^n_{mm} and Q^n_{mr} $(m \neq r)$ hold uniformly for $\sigma \in \Pi(q)$ as well. Moreover, it is clear that for $m, r \neq n$

$$Q^n_{mr} \to Q^*_{mr} \tag{9.9}$$

uniformly for $\sigma \in \Pi(q)$. Consequently, the Q^*_{mr} satisfy the corresponding asymptotic estimates from Lemma 9.6 uniformly for $\sigma \in \Pi(q)$, and define a bounded operator Q^* on h^{-1}.

To show the convergence $Q^n \to Q^*$ (uniform in $\sigma \in \Pi(q)$), write $Q^n = D^n + K^n$ and $Q^* = D^* + K^*$ as a sum of a diagonal and an off-diagonal part, similarly as in Lemma 9.7. Then, for any $m_0 \geq 1$,

$$\begin{aligned} \|D^* - D^n\|_{\mathcal{L}(h^{-1})} &\leq \sup_{m\neq n} |Q^*_{mm} - Q^n_{mm}| \\ &\leq \max_{1\leq m\leq m_0} |Q^*_{mm} - Q^n_{mm}| + \sup_{m>m_0} \Big(|Q^*_{mm} - 1| + |Q^n_{mm} - 1|\Big) \end{aligned}$$

which by (9.9) and Lemma 9.6 (a) is uniformly small for $\sigma \in \Pi(q)$. The norm $||K^* - K^n||_{\mathcal{L}(h^{-1})}$ is estimated by the Hilbert-Schmidt norm of $K^* - K^n$

$$\|K^* - K^n\|_{\mathcal{L}(h^{-1})} \leq \Big(\sum_{\substack{m,r\neq n \\ m\neq r}} \frac{|r|^2}{|m|^2} \, |Q^*_{mr} - Q^n_{mr}|^2 \Big)^{1/2}.$$

To estimate the latter sum, decompose, for $m_0 \in \mathbb{N}$ arbitrary, the range $A = \{(m,r) \in \mathbb{N}^2 \mid m, r \neq n,\ m \neq r\}$ of the sum into the subsets

$$\begin{aligned} A_1 &= A \cap \{1 \leq m \leq m_0,\ 1 \leq r \leq 2m_0\}, \\ A_2 &= A \cap \{m > m_0,\ r \geq 1\}, \\ A_3 &= A \cap \{1 \leq m \leq m_0,\ r > 2m_0\}, \end{aligned}$$

and prove that each of the corresponding sums converges to 0 separately. As A_1 is finite the corresponding sum converges to zero by (9.9). On A_2 use the estimate from Lemma 9.6 (b) for $|Q^n_{mr}|$ and the corresponding estimate for $|Q^*_{mr}|$ to get, uniformly for $\sigma \in \Pi(q)$,

$$\begin{aligned} \Big(\sum_{(m,r)\in A_2} \frac{|r|^2}{|m|^2} \, |Q^*_{mr} - Q^n_{mr}|^2 \Big)^{1/2} &\leq C \,\Big(\sum_{m > m_0} |m|^{-2} |\rho_m|^2 \sum_{r \neq m} \frac{|r|^2}{|m^2 - r^2|^2} \Big)^{1/2} \\ &= O\Big(\|\overline{\rho}\|^{[m_0]}_{h^{-1}} \Big), \end{aligned}$$

where the sequence $\overline{\rho} = (\overline{\rho}_m)_m$ in h^{-1} is given by $\overline{\rho}_m = \sup_{\sigma \in \Pi(q)} \rho_m$.
On the set A_3, using estimate (8.2), one gets uniformly for $\sigma \in \Pi(q)$,

$$\begin{aligned} \Big(\sum_{(m,r)\in A_3} \frac{|r|^2}{|m|^2} \, |Q^*_{mr} - Q^n_{mr}|^2 \Big)^{1/2} &\leq C \,\Big(\sum_{m \leq m_0} |m|^{-2} |\rho_m|^2 \sum_{r > 2m_0} \frac{|r|^2}{|m^2 - r^2|^2} \Big)^{1/2} \\ &= O\Big(\|\overline{\rho}\|_{h^{-1}} \, m_0^{-1/2} \Big). \end{aligned}$$

Combining the previous estimates we conclude that $Q^n \to Q^*$ in the operator norm on h^{-1} uniformly for $\sigma \in \Pi(q)$. In particular, Q^* is real analytic in $\sigma \in \Pi(q)$.
It follows from Lemma 9.5 that the diagonal elements Q^*_{mm} don't vanish since $\partial\phi_*/\partial\sigma_m$ has no root in G_m. Hence, by the same arguments used in the proofs of Lemmas 9.7 and 9.8, Q^* is boundedly invertible on h^{-1} at every point in $\Pi(q)$.
As the set $\Pi(q)$ is compact in h^{-1}, Q^* is indeed uniformly boundedly invertible for $\sigma \in \Pi(q)$. Then also $Q^n(\sigma, q)$ is boundedly invertible uniformly for $\sigma \in \Pi(q)$, for all large n. Hence Q^n is boundedly invertible for any $n \in \mathbb{N}$ and $\sigma \in \Pi(q)$.
Using that, by Lemma 9.2, the maps F^n are analytic and locally uniformly bounded on a common complex neighborhood of V independent of n, one then argues as in [13] to show that there exists a complex neighborhood of

$\Pi(q) \times \{q\}$ on which the Q^n ($n \geq 1$) are uniformly boundedly invertible. The result then follows from the implicit function theorem. $\square$

9.3 Asymptotics

Proposition 9.11. *The components of $\sigma^n = (\sigma_m^n)_{m\geq 1}$ satisfy, for any $m \geq 1$,*

$$|\overline{\sigma}_m^n - \tau_m| \leq C\ \frac{|\gamma_m^2|}{m}$$

locally uniformly in a complex neighborhood of H_0^{-1} and uniformly in n.

Proof. There is nothing to prove for $\overline{\sigma}_n^n = \tau_n$, so consider $\overline{\sigma}_m^n$ with $m \neq n$. In view of the construction of ψ_n by the implicit function theorem, we have a first crude estimate

$$\overline{\sigma}_m^n = \tau_m + h_{\mathbb{C}}^{-1}(m) = \tau_m + O(m), \tag{9.10}$$

locally uniformly in a complex neighborhood of $H_0^{-1}(\mathbb{T}^1)$.
As in [13] we find that

$$(\overline{\sigma}_m^n - \tau_m) w_m^n \zeta_m^n(\tau_m) = \frac{1}{2\pi} \int_{\Gamma_m} \frac{(\lambda - \overline{\sigma}_m^n)\,(w_m^n \zeta_m^n(\lambda) - w_m^n \zeta_m^n(\tau_m))}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}}\ d\lambda. \tag{9.11}$$

If $\gamma_m = 0$, then the right hand side of (9.11) vanishes, and there is nothing to prove as $\zeta_m^n(\tau_m) \neq 0$ and $\bar{\sigma}_m^n = \tau_m$. So assume that $\gamma_m \neq 0$. Choose the contour Γ_m at a distance of order γ_m around τ_m. By [13, Lemma M.1],

$$\begin{aligned} |(\overline{\sigma}_m^n - \tau_m) w_m^n \zeta_m^n(\tau_m)| \quad &\leq \quad \max_{\lambda\in\Gamma_m} |\lambda - \overline{\sigma}_m^n| \max_{\lambda\in\Gamma_m} |w_m^n \zeta_m^n(\lambda) - w_m^n \zeta_m^n(\tau_m)| \\ \leq \max_{\lambda\in\Gamma_m} |\lambda - \overline{\sigma}_m^n| M_m \gamma_m & \end{aligned} \tag{9.12}$$

where M_m is the maximum of $|\frac{d}{d\lambda}(w_m^n \zeta_m^n(\lambda))|$ over the convex hull of Γ_m. Using Lemma 9.4 and Cauchy's estimate one gets as in the proof of Proposition D.9 in [13], $|\frac{d}{d\lambda}(w_m^n \zeta_m^n(\lambda))| = O(1/m)$ on the convex hull of Γ_m. To show that the term $w_m^n \zeta_m^n(\tau_m)$ on the left hand side of (9.12) is uniformly bounded away from zero, one argues as in [13]: choose a complex neighborhood in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ of a potential $q \in H_0^{-1}(\mathbb{T}^1)$ for which isolating neighborhoods exist, and use Lemma 9.3 for large m.

Then, using the crude estimate $\overline{\sigma}_m^n = \tau_m + O(m)$, we get $|\lambda - \overline{\sigma}_m^n| = O(|\gamma_m| + m)$ for λ in Γ_m and hence from (9.12)

$$|\overline{\sigma}_m^n - \tau_m| = \max_{\lambda \in \Gamma_m} |\lambda - \overline{\sigma}_m^n| \, O\left(\frac{|\gamma_m|}{m}\right) = O(|\gamma_m|).$$

But this in turn implies

$$\max_{\lambda \in \Gamma_m} |\lambda - \overline{\sigma}_m^n| \leq |\overline{\sigma}_m^n - \tau_m| + \max_{\lambda \in \Gamma_m} |\lambda - \tau_m| = O(|\gamma_m|)$$

which finally gives the claimed estimate. This estimate holds locally uniformly on some complex neighborhood of H_0^{-1} and uniformly in n. □

10 Appendix C: Auxiliary results

10.1 Isolating neighborhoods

By definition two complex numbers $a, b \in \mathbb{C}$ are *lexicographically ordered* $a \prec b$ iff

$$\mathtt{Re}(a) < \mathtt{Re}(b), \text{ or } \mathtt{Re}(a) = \mathtt{Re}(b) \text{ and } \mathtt{Im}(a) \leq \mathtt{Im}(b).$$

The periodic and the Dirichlet spectrum of the Hill operator $L_q \overset{\mathtt{Def}}{=} -\frac{d^2}{dx^2} + q$ with potential q from the Sobolev space $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ were investigated in [14] where the problem was reduced to the properties of the spectrum of the impedance operator (cf. [7, 8, 18, 19]). In particular, it was proved that the periodic spectrum of the Hill operator L_q is discrete and can be ordered lexicographically $\lambda_0(q) \prec \lambda_1(q) \prec \lambda_2(q) \prec ...$ with $\mathtt{Re}(\lambda_k) \to \infty$ as $k \to \infty$ (cf. Theorem 11 in [14]). Consider the spectral quantities

$$\tau_k(q) \overset{\mathtt{Def}}{=} \frac{\lambda_{2k}(q) + \lambda_{2k-1}(q)}{2}$$

$$\gamma_k(q) \overset{\mathtt{Def}}{=} \lambda_{2k}(q) - \lambda_{2k-1}(q)$$

where $\lambda_0(q) \prec \lambda_1(q) \prec \lambda_2(q) \prec ...$ are listed with multiplicities. The results in [14] (cf. Theorem 2) together with the results for the impedance operator in [7] (cf. Lemma 4.8) show that the Dirichlet spectrum of L_q is discrete and can be ordered lexicographically $\mu_1(q) \prec \mu_2(q) \prec ...$ for q from a sufficiently narrow complex neighborhood of $H_0^{-1}(\mathbb{T}^1)$.

Proposition 10.1. *There exists a complex neighborhood $\mathcal{V}$ of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ such that the maps*

$$q \mapsto (k^2\pi^2 - \tau_k(q))_{k\geq 1} \ \textit{and} \ q \mapsto (k^2\pi^2 - \mu_k(q))_{k\geq 1}$$

are analytic from $\mathcal{V}$ into $h_{\mathbb{C}}^{-1}$, and the map

$$\mathcal{V} \to h_{\mathbb{C}}^0, \ q \mapsto (\gamma_k^2(q)/k^2)_{k\geq 1}$$

is analytic as well.

Proof. It follows from [14, Theorem 2, Theorem 10] and the results established in [18, 19] for the periodic spectrum of the impedance operator that there exists a complex neighborhood $\mathcal{V}$ of $H_0^{-1}(\mathbb{T}^1)$ such that the sequences $(k^2\pi^2 - \tau_k(q))_{k\geq 1}$ and $(|\gamma_k(q)|)_{k\geq 1}$ belong to the sequence space $h_{\mathbb{C}}^{-1}$ and considered as maps from $\mathcal{V}$ to $h_{\mathbb{C}}^{-1}$ are locally bounded. Indeed, by [14, Theorem 2, Theorem 10] it is sufficient to prove the above statements for the spectral quantities $\tilde{\tau}_k(r) \overset{\texttt{Def}}{=} (\tilde{\lambda}_{2k-1} + \tilde{\lambda}_{2k})/2$ and $\tilde{\gamma}_k(r) \overset{\texttt{Def}}{=} \tilde{\lambda}_{2k} - \tilde{\lambda}_{2k-1}$ where $\tilde{\lambda}_0 \prec \tilde{\lambda}_1 \prec \tilde{\lambda}_2 \prec ...$ is the lexicographically ordered periodic spectrum (listed with multiplicities) of the impedance operator $T_r \overset{\texttt{Def}}{=} -\frac{d^2}{dx^2} - 2r\frac{d}{dx}$ with r from a sufficiently narrow open complex neighborhood $\mathcal{U}$ of $L_0^2(\mathbb{T}^1)$ in $L_0^2(\mathbb{T}^1, \mathbb{C})$. It follows from [14, Theorem 2] and [14, Lemma 7] that for any fixed $r \in L_0^2(\mathbb{T}^1)$ there exist a complex neighborhood $U(r) \subset L_0^2(\mathbb{T}^1, \mathbb{C})$ of r and a constant $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and any $v \in U(r)$ the periodic spectrum of the impedance operator T_v lies in the union of the sets $\{\lambda \in \mathbb{C} \mid \texttt{Re}(\lambda) < (n_0^2 - n_0)\pi\}$ and $\{\lambda \in \mathbb{C} \mid |\lambda - n^2\pi^2| \leq n\}$ $(n \geq n_0)$. Using the asymptotics $\tilde{\lambda}_{2k}(r), \tilde{\lambda}_{2k-1}(r) = k^2\pi^2 + h^{-1}(k)$, $(h^{-1}(k))_k \in h^{-1}$, from [18, Theorem 1.1 ii)] and taking n_0 bigger if necessary we obtain that for any $n \geq n_0$, $\tilde{\lambda}_{2n-1}(r)$ and $\tilde{\lambda}_{2n}(r)$ are the unique eigenvalues of T_r inside the complex disk in $\mathbb{C}$ of radius n around $n^2\pi^2$. Using the continuity (with respect to $v \in U(r)$) of the Riesz projectors of T_v corresponding to the contours $\Gamma_n \overset{\texttt{Def}}{=} \{|\lambda - n^2\pi^2| = n\}$ (cf. [14, Lemma 7]) and applying Lemma 4.10 in [16] (stating that two projectors whose difference has small norm have isomorphic ranges) we obtain that for any $v \in U(r)$ and any $n \geq n_0$, $\tilde{\lambda}_{2n-1}(v)$ and $\tilde{\lambda}_{2n}(v)$ are the only two eigenvalues (counted with multiplicities) of T_v in the complex disk in $\mathbb{C}$ of radius n around $n^2\pi^2$. Denote by $\tilde{\Delta}(\lambda, v)$ the discriminant of the impedance operator T_v (cf. [14, Section 5.1]). By the Floquet theory $\tilde{\lambda}_{2n-1}(v)$ and $\tilde{\lambda}_{2n-1}(v)$ are roots of the equation

$$\tilde{\Delta}^2(\lambda, v) - 4 = 0 \tag{10.1}$$

and vice versa. Following the proof of (1.10) and (1.11) in [18, Theorem 1.1 ii)] and using that the eigenvalues of T_v are the roots of equation (10.1) together with the asymptotic formulas for the discriminant $\tilde\Delta$ in [19, Lemma 2.4 iii)] one proves that the sequences $(k^2\pi^2 - \tilde\tau_k(v))_{k\geq 1}$ and $(|\tilde\gamma_k(v)|)_{k\geq 1}$ belong to the space $h^{-1}_{\mathbb{C}}$ and considered as maps from $U(r)$ to $h^{-1}_{\mathbb{C}}$ are locally uniformly bounded. Taking $\mathcal{U} \overset{\text{Def}}{=} \bigcup_{r\in L^2_0(\mathbb{T}^1)} U(r)$ we define $\mathcal{V} \overset{\text{Def}}{=} R(\mathcal{U})$ where R denotes the Riccati map.
Shrinking the neighborhood $\mathcal{V}$ of $H^{-1}_0(\mathbb{T}^1)$ if necessary we can ensure that, for any fixed $k\geq 1$, the functions $\gamma_k^2(q)$ and $\tau_k(q)$ are analytic. In particular, the map $q\mapsto (k^2\pi^2-\tau_k(q))_{k\geq 1}\in h^{-1}_{\mathbb{C}}$ is analytic on $\mathcal{V}$ by Theorem 3 from [25, Appendix A]. On the other side one has

$$\begin{aligned} ||(\gamma_k^2(q)/k^2)_{k\geq 1}||^2_{h^0} &\overset{\text{Def}}{=} \sum_{k\geq 1} |\gamma_k(q)|^4/k^4 \\ &\leq \Big(\sum_{k\geq 1} |\gamma_k(q)|^2/k^2\Big)^2 = ||\gamma(q)||^4_{h^{-1}} \end{aligned} \qquad (10.2)$$

which implies that the map $q\mapsto(\gamma_k^2(q)/k^2)_{k\geq 1}$ is locally bounded on $\mathcal{V}$ and hence analytic.
Shrinking the neighborhood $\mathcal{V}$ of $H^{-1}_0(\mathbb{T}^1)$ if necessary once more and applying the results from [7, Section 4, Theorem 4.9 and 4.11] together with [14, Theorem 2 and §4.5] we conclude the analyticity of the map $q\mapsto (k^2\pi^2-\mu_k)_{k\geq 1}$. □

The following corollary ensures the existence of a special system of mutually disjoint open sets $\{U_n\}_{n\geq 1}$ in $\mathbb{C}$ needed to apply the product estimates of Section 8.

Corollary 10.2. *There exists a complex neighborhood $\mathcal{W}$ of $H^{-1}_0(\mathbb{T}^1)$ in $H^{-1}_0(\mathbb{T}^1,\mathbb{C})$ such that for any $q_0\in\mathcal{W}$ there exist a neighborhood $V(q_0)\subset\mathcal{W}$ of q_0 and a system of mutually disjoint, convex, open neighborhoods $\{U_n\}_{n\geq 1}$ in $\mathbb{C}$ such that for any $n\geq 1$ the interval $G_n(q)\overset{\text{Def}}{=}\{t\lambda_{2n}(q)+(1-t)\lambda_{2n-1}(q)\mid 0\leq t\leq 1\}$ as well as $\mu_n(q)$ lie in U_n for any $q\in V(q_0)$. Moreover, for any $0<\beta\leq\pi/2$ there exists a number $n_0\geq 1$ such that for any $n\geq n_0$ the neighborhood U_n can be chosen of the form $U_n\overset{\text{Def}}{=}\{\lambda\in\mathbb{C}\mid |\sqrt{\lambda}-n\pi|<\beta\}$.*

Proof. Denote by $\mathcal{V}$ the complex neighborhood given by Proposition 10.1. In order to prove the corollary it suffices to show that for any $q_0\in H^{-1}_0(\mathbb{T}^1)$

there exist an open, complex neighborhood $V(q_0) \subset \mathcal{V}$ and a system of mutually disjoint open neighborhoods $\{U_n\}_{n\geq 1}$ in $\mathbb{C}$ satisfying the properties stated in Corollary 10.2. The neighborhood $\mathcal{W} \subset \mathcal{V}$ is then defined by $\mathcal{W} \stackrel{\text{Def}}{=} \bigcup_{q_0 \in H_0^{-1}(\mathbb{T}^1)} V(q_0)$.
Let us fix $q_0 \in H_0^{-1}(\mathbb{T}^1)$ and $0 < \beta \leq \pi/2$. It follows from Proposition 10.1 that for arbitrary $\delta > 0$ there exists $n_0 > 0$ so that, for any $n \geq n_0$

$$||(\gamma_k^2(q)/k^2)_{k\geq 1}||_{h^0}^{[n]} \leq ||(\gamma_k^2(q_0)/k^2)_{k\geq 1}||_{h^0}^{[n]} + ||((\gamma_k^2(q) - \gamma_k^2(q_0))/k^2)_{k\geq 1})||_{h^0} < \delta \tag{10.3}$$

uniformly in a neighborhood $U(q_0) \subset \mathcal{W}$, where $||\cdot||_{h^0}^{[n]}$ is given by formula (8.1). In particular, for $n \geq n_0$ and $q \in U(q_0)$

$$|\gamma_n(q)| < \sqrt{\delta} n.$$

Again by Proposition 10.1, taking n_0 bigger and shrinking the neighborhood $U(q_0)$ if necessary, one gets by the same argument that for any $q \in U(q_0)$ and $n \geq n_0$

$$|\tau_n(q) - n^2\pi^2| < \sqrt{\delta} n, \quad |\mu_n(q) - n^2\pi^2| < \sqrt{\delta} n$$

as well. Therefore, taking $\delta = \delta(\beta) > 0$ sufficiently small, we find $n_0 > 0$ and a complex neighborhood $U(q_0)$ of $q_0 \in H_0^{-1}(\mathbb{T}^1)$ such that for any $n \geq n_0$ the interval $G_n(q)$ and $\mu_n(q)$ lie in $U_n \stackrel{\text{Def}}{=} \{\lambda \in \mathbb{C} \mid |\sqrt{\lambda} - n\pi| < \beta\}$ for any $q \in U(q_0)$.
By Theorem 3 in [14] and the continuity of the Dirichlet eigenvalues $\mu_k(q)$ as functions of the potential $q \in H_0^{-1}(\mathbb{T}^1)$

$$\lambda_1(q) \leq \mu_1(q) \leq \lambda_2(q) < ... < \lambda_{2n_0-1}(q) \leq \mu_{n_0}(q) \leq \lambda_{2n_0}(q) < ...$$

Hence, as q_0 is real there exist convex mutually disjoint open neighborhoods $U_1, ..., U_{n_0-1}$ with $U_{n_0-1} \cap U_{n_0} = \emptyset$, such that for any $1 \leq n \leq n_0 - 1$ the interval $G_n(q_0)$ and $\mu_n(q_0)$ lie in U_n. As $\mu_n(q)$, $\tau_n(q)$ and $\gamma_n^2(q)$ are continuous in $q \in \mathcal{V}$ we conclude that there exists a complex neighborhood $U'(q_0)$ of q_0 such that for any $1 \leq n \leq n_0 - 1$ and $q \in U'(q_0)$ the interval $G_n(q)$ and $\mu_n(q)$ lie in U_n. Finally, taking $V(q_0) \stackrel{\text{Def}}{=} U(q_0) \cap U'(q_0)$ we complete the proof. □

Definition 10.3. *A system of mutually disjoint neighborhoods as in Corollary 10.2 is called a* system of isolating neighborhoods.

It follows from Corollary 10.2 that for any $n \geq 1$ we can choose a contour with counterclockwise orientation Γ_n (independent of $q \in V(q_0)$) in U_n such that for any $n \geq 1$ and $q \in V(q_0)$ the interval $G_n(q)$ and $\mu_n(q)$ lie in the interior of Γ_n.
The following simple lemma is used several times in the proof of the product estimates in Section 8.

Lemma 10.4. *Let $\{U_n\}_{n\geq 1}$ be a system of isolating neighborhoods and $n_0 > 0$ be as in Corollary 10.2. Then there exist constants $0 < \rho_1 < \rho_2$ such that for any $n \geq n_0$, $k \neq n$ and $\lambda \in U_n$ one has*

$$\rho_1 |k^2 - n^2| \leq |k^2\pi^2 - \lambda| \leq \rho_2 |k^2 - n^2|.$$

10.2 Discriminant and its properties

Following [14, Section 4.3] we define the *discriminant* $\Delta(\lambda, q)$ of the Hill operator L_q for q in a sufficiently small complex neighborhood $\mathcal{W}$ of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ by

$$\Delta(\lambda, q) \overset{\text{Def}}{=} \tilde{\Delta}(\lambda + ||r||^2, r) \tag{10.4}$$

where $r = R^{-1}(q) \in L_0^2(\mathbb{T}^1, \mathbb{C})$ and $R^{-1} : \mathcal{W} \to L_0^2(\mathbb{T}^1, \mathbb{C})$ is the inverse of the Riccati map $R : r \mapsto r' + r^2 - ||r||^2$ (cf. [14, Theorem 2]), $||r||^2 \overset{\text{Def}}{=} \int_0^1 r(x)^2 dx \in \mathbb{C}$, and $\tilde{\Delta}(\lambda, r)$ denotes the discriminant of the impedance operator $T_r(u) \overset{\text{Def}}{=} -u'' - 2ru'$.

Proposition 10.5. *There exists a complex neighborhood $\mathcal{W}$ of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$ such that the discriminant $\Delta(\lambda, q)$ is an analytic function on $\mathbb{C} \times \mathcal{W}$. Moreover, if $q \in \mathcal{W}$ then*

$$\Delta^2(\lambda, q) - 4 = 4(\lambda_0 - \lambda) \prod_{k \geq 1} \frac{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}{k^4 \pi^4} \tag{10.5}$$

where $\lambda_0 \prec \lambda_1 \prec \lambda_2 \prec \ldots$ is the periodic spectrum of L_q.

Proof. By definition, the discriminant of the impedance operator T_r is given by $\tilde{\Delta}(\tilde{\lambda}, r) \overset{\text{Def}}{=} \tilde{y}_1(1, \tilde{\lambda}, r) + \tilde{y}_2'(1, \tilde{\lambda}, r)$ where $\tilde{y}_1(x, \tilde{\lambda}, r)$ and $\tilde{y}_2(x, \tilde{\lambda}, r)$ are the fundamental solutions of the equation $T_r(u) = \tilde{\lambda} u$. By Theorem 1.3 and 1.4 in [7] $\tilde{\Delta}(\tilde{\lambda}, r)$ is analytic on $\mathbb{C} \times L_0^2(\mathbb{T}^1, \mathbb{C})$ and together with Theorem 2 in [14] the first statement of the proposition then follows.

Theorem 4.7 in [7] shows that, for any $r \in L^2_0(\mathbb{T}^1, \mathbb{C})$, the order of the entire function $\tilde{\lambda} \mapsto \tilde{\Delta}(\tilde{\lambda}, r)$ is less than or equal to $\frac{1}{2}$.
Therefore, by Hadamard's Theorem [26] and Lemma 10.6 below, there exists a complex neighborhood $\mathcal{U}$ of $L^2_0(\mathbb{T}^1)$ in $L^2_0(\mathbb{T}^1, \mathbb{C})$ such that for any $r \in \mathcal{U}$,

$$\tilde{\Delta}^2(\tilde{\lambda}, r) - 4 = c(r)\tilde{\lambda} \prod_{k \geq 1} \left(1 - \frac{\tilde{\lambda}}{\tilde{\lambda}_{2k}}\right)\left(1 - \frac{\tilde{\lambda}}{\tilde{\lambda}_{2k-1}}\right)$$

where $0 = \tilde{\lambda}_0 \prec \tilde{\lambda}_1 \prec \tilde{\lambda}_2 \prec \ldots$ is the periodic spectrum of the impedance operator T_r and $c(r)$ possibly depends on r. Using the asymptotic estimates $\tilde{\lambda}_{2k}, \tilde{\lambda}_{2k-1} = \pi^2 k^2 + O(k)$ (cf. Corollary 10.2), we obtain

$$\begin{aligned} \tilde{\Delta}^2(\tilde{\lambda}, r) - 4 &= c(r)\tilde{\lambda} \prod_{k \geq 1} \frac{\pi^4 k^4}{\tilde{\lambda}_{2k}\tilde{\lambda}_{2k-1}} \frac{(\tilde{\lambda}_{2k} - \tilde{\lambda})(\tilde{\lambda}_{2k-1} - \tilde{\lambda})}{\pi^4 k^4} \\ &= a(r)\tilde{\lambda} \prod_{k \geq 1} \frac{(\tilde{\lambda}_{2k} - \tilde{\lambda})(\tilde{\lambda}_{2k-1} - \tilde{\lambda})}{\pi^4 k^4} \end{aligned}$$

where $a(r) \stackrel{\mathtt{Def}}{=} c(r) / \prod_{k \geq 1} \frac{\tilde{\lambda}_{2k}\tilde{\lambda}_{2k-1}}{\pi^4 k^4}$. Taking $\kappa_n \stackrel{\mathtt{Def}}{=} (n + 1/2)^2 \pi^2$ and applying Lemma 8.4 we obtain

$$\tilde{\Delta}^2(\kappa_n, r) - 4 = a(r)(\sin^2 \sqrt{\kappa_n})(1 + o(1)) = a(r) + o(1).$$

On the other side, for a real potential, $r \in L^2_0(\mathbb{T}^1)$, Theorem 1.8 in [7] shows that $\tilde{\Delta}(\kappa_n, r) = 2\cos\sqrt{\kappa_n} + o(1) = o(1)$, hence $a(r) = -4$ and thus

$$\tilde{\Delta}^2(\tilde{\lambda}, r) - 4 = -4\tilde{\lambda} \prod_{k \geq 1} \frac{(\tilde{\lambda}_{2k} - \tilde{\lambda})(\tilde{\lambda}_{2k-1} - \tilde{\lambda})}{\pi^4 k^4}. \tag{10.6}$$

As $a(r)$ is an analytic function on $\mathcal{U}$ and $a(r) = -4$ for $r \in L^2_0(\mathbb{T}^1)$ we obtain that $a(r) \equiv -4$ on $\mathcal{U}$. Using formula (10.4) and the identities $||r||^2 = -\lambda_0(q)$ and $\tilde{\lambda}_m = \lambda_m + ||r||^2$ (cf. [14, Theorem 10]) the second statement of Proposition 10.5 follows. $\square$

Lemma 10.6. *There exists a complex neighborhood $\mathcal{U}$ of $L^2_0(\mathbb{T}^1)$ in $L^2_0(\mathbb{T}^1, \mathbb{C})$ such that for any $r \in \mathcal{U}$ the periodic spectrum of the impedance operator T_r (counted with multiplicities) coincides with the roots of the equation $\tilde{\Delta}^2(\tilde{\lambda}, r) - 4 = 0$ (counted with their algebraic multiplicities).*

Proof. The standard Floquet theory arguments show that the eigenvalues of T_r are roots of $\tilde{\Delta}^2(\tilde{\lambda}, r) - 4 = 0$ and vice versa. In order to prove that they have the same multiplicities remark that for real potentials $r \in L_0^2(\mathbb{T}^1)$ the lemma follows from [22, Theorem 3.4]. Indeed as in [18], performing a change of the variables $y = y(x) \overset{\texttt{Def}}{=} \int_0^x (1/\rho^2(x))dx$, $\rho(x) \overset{\texttt{Def}}{=} \exp(\int_0^x r(v)dv) > 0$, one transforms the impedance operator T_r to the operator $\tilde{T}_r \overset{\texttt{Def}}{=} -\rho^{-4}\frac{d^2}{dy^2}$ on the torus $\mathbb{T}_l^1 \overset{\texttt{Def}}{=} \mathbb{R}/l\mathbb{Z}$ with period $l \overset{\texttt{Def}}{=} y(1)$, and applies [22, Theorem 3.4] to the latter operator. To prove the result for potentials r from a complex neighborhood of $L_0^2(\mathbb{T}^1)$ we use the same perturbation arguments as in the proof of Proposition 10.1. □

As in [13], for real potentials, we define the *canonical root* $\sqrt[c]{\Delta(\lambda, q) - 4}$ on the set $\mathbb{C} \setminus \bigcup_{k\geq 0}[\lambda_{2k-1}(q), \lambda_{2k}(q)]$, $\lambda_{-1} \overset{\texttt{Def}}{=} -\infty$, as the w-coordinate of the point $(\lambda, w) \in \Sigma_q = \{(\lambda, w) \in \mathbb{C}^2 |\ w^2 = \Delta(\lambda, q) - 4\}$ lying on the branch of the Riemannian surface Σ_q uniquely defined by the requirement that $\sqrt[c]{\Delta(\lambda, q) - 4}$ tends to a positive number as $\lambda \to \tau_1(q)$ with $\texttt{Im}(\lambda) < 0$. For complex potentials this root is defined by continuous extension.
Analogously, for real $a < b$, the *standard root* $\sqrt[s]{(b-\lambda)(\lambda - a)}$ is defined on $\mathbb{C} \setminus [a, b]$ by the condition that $\sqrt[s]{(b-\lambda)(\lambda - a)}$ tends to a positive number as $\lambda \to (a + b)/2$ with $\texttt{Im}(\lambda) < 0$. For $a, b \in \mathbb{C}$ with $a \prec b$ the standard root is defined in a similar fashion – see [13].
As usual $\sqrt[+]{\lambda}$ denotes the branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ defined by the requirement that $\sqrt[+]{\lambda} > 0$ for $\lambda > 0$.

Proposition 10.7. *If a sequence $(q_n)_{n\geq 1} \subset H_0^{-1}(\mathbb{T}^1)$ converges weakly in $H_0^{-1}(\mathbb{T}^1)$ then there exist an element $q \in H_0^{-1}(\mathbb{T}^1)$ and a subsequence $(q_{n_j})_{j\geq 1}$ such that for any $k \geq 1$, $I_k(q_{n_j}) \to I_k(q)$ as $j \to \infty$.*

Proof. By the definition of the k'th action variable

$$I_k(q_n) \overset{\texttt{Def}}{=} \frac{1}{\pi}\int_{\Gamma_k} \lambda \frac{\partial \Delta(\lambda, q_n)/\partial\lambda}{\sqrt[c]{\Delta^2(\lambda, q_n) - 4}}\, d\lambda$$
$$= \frac{2}{\pi}\int_{\tilde{\lambda}_{2k-1}(r_n)}^{\tilde{\lambda}_{2k}(r_n)} \tilde{\lambda} \frac{\partial \tilde{\Delta}(\tilde{\lambda}, r_n)/\partial\tilde{\lambda}}{\sqrt[c]{\tilde{\Delta}^2(\tilde{\lambda}, r_n) - 4}}\, d\tilde{\lambda} \qquad (10.7)$$

where $r_n = R^{-1}(q_n)$ and $\tilde{\lambda}_m = \lambda_m + ||r||^2$ $(m \geq 0)$ are the periodic eigenvalues of the impedance operator. It follows from Lemma 10.8 below that the sequence $r_n \overset{\text{Def}}{=} R^{-1}(q_n)$ is bounded in $L^2_0(\mathbb{T}^1)$. Therefore there exists a subsequence $(r_{n_j})_{j\geq 1}$ which converges weakly to some element $r \in L^2_0(\mathbb{T}^1)$. Take $q \overset{\text{Def}}{=} R(r)$. It follows from Lemma 3.1 and 3.2 in [7] that $\tilde{\Delta}(\tilde{\lambda}, r_n) \to \tilde{\Delta}(\tilde{\lambda}, r)$ uniformly on bounded sets in $\mathbb{C}$. Taking the limit in formula (10.7) we obtain that $I_k(q_{n_j}) \to I_k(q)$ as $j \to \infty$ for any $k \geq 1$. $\square$

Lemma 10.8. *The inverse of the Riccati map $R^{-1} : H^{-1}_0(\mathbb{T}^1) \to L^2_0(\mathbb{T}^1)$ is bounded and*

$$||R^{-1}(q)|| \leq \max\{1, C||q||^2_{H^{-1}_0}\}$$

where C is a universal constant.

Proof. As in the proof of Lemma 1.1 in [12] (with notations as in [12]) we see that for $M \geq 1$ and

$$\lambda \in \mathtt{Ext}_M \overset{\text{Def}}{=} \{\lambda \in \mathbb{C} |\ \mathtt{Re}(\lambda) \leq |\,\mathtt{Im}(\lambda)| - M)\}$$

the operator $\lambda - L_q$, when expressed in Fourier space, is of the form

$$D^{1/2}_\lambda (I_\lambda - S_\lambda) D^{1/2}_\lambda,$$

where $||I_\lambda||_{\mathcal{L}(h^0_\mathbb{C})} = 1$, $D^{1/2}_\lambda$ is bounded as a map from $h^1_\mathbb{C}(\mathbb{Z}\setminus\{0\})$ to $h^0_\mathbb{C}(\mathbb{Z}\setminus\{0\})$ as well as from $h^0_\mathbb{C}(\mathbb{Z}\setminus\{0\})$ to $h^{-1}_\mathbb{C}(\mathbb{Z}\setminus\{0\})$, and

$$\begin{aligned} ||S_\lambda||_{\mathcal{L}(h^0_\mathbb{C})} &\leq 8\Big(\sup_{k\in\mathbb{Z}} \frac{\langle k\rangle^2}{M + k^2\pi^2}\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}} \frac{1}{M + n^2\pi^2}\Big)^{1/2} ||q||_{H^{-1}_0} \\ &\leq 8\Big(\frac{\coth\sqrt{M}}{\sqrt{M}}\Big)^{1/2} ||q||_{H^{-1}_0} \leq 16||q||_{H^{-1}_0}/M^{1/4} \end{aligned} \tag{10.8}$$

with $\langle k\rangle \overset{\text{Def}}{=} 1 + |k|$. Taking $M = \max\{1, C_1||q||^4_{H^{-1}_0}\}$ with $C_1 > 0$ sufficiently big we ensure that $||S_\lambda||_{\mathcal{L}(h^0_\mathbb{C})} < 1$ and therefore $\mathtt{Ext}_M$ is contained in the resolvent set of L_q (cf. [12, Lemma 1.1]). On the other side, for $q \in H^{-1}_0(\mathbb{T}^1)$, $\lambda_0(q) = -||r||^2 \leq 0$ (cf. [14, Lemma 1]). Therefore $||r||^2 = |\lambda_0(q)| \leq M = \max\{1, C_1||q||^4_{H^{-1}_0}\}$. $\square$

It follows from the definition of $\Delta(\lambda, q)$, defined on a complex neighborhood $\mathcal{W}$ of $H^{-1}_0(\mathbb{T}^1)$ in $H^{-1}_0(\mathbb{T}^1, \mathbb{C})$, and the properties of the discriminant $\tilde{\Delta}(\tilde{\lambda}, r)$

of the impedance operator (see [19, Lemma 3.1 i)]) that the zeroes of the entire function

$$\dot{\Delta}(\lambda, q) \overset{\text{Def}}{=} \frac{\partial \Delta(\lambda, q)}{\partial \lambda},$$

$\dot{\lambda}_j = \dot{\lambda}_j(q)$, can be ordered lexicographically $\dot{\lambda}_1 \prec \dot{\lambda}_2 \prec \dot{\lambda}_3 \prec \dots$ and $\lim_{k\to\infty} \mathrm{Re}(\dot{\lambda}_k) = \infty$. For real $q \in H_0^{-1}(\mathbb{T}^1)$, $\dot{\lambda}_k(q)$ is the unique root of $\dot{\Delta}(\lambda, q)$ in the real interval $[\lambda_{2k-1}, \lambda_{2k}]$ (cf. [22, §8]). Moreover, it follows from Lemma 3.1 *ii)* in [19] that the map $q \mapsto (\dot{\lambda}_k(q) - k^2\pi^2)_{k\geq 1}$ is locally bounded when considered as a map $\mathcal{W} \to h_{\mathbb{C}}^{-1}$. As each of its components $\dot{\lambda}_k(q) - k^2\pi^2$ is analytic in q we conclude that this map is analytic as well. Arguing as in the proof of Corollary 10.2 one proves that for any $q_0 \in H_0^{-1}(\mathbb{T}^1)$ there exist a complex neighborhood $V(q_0) \subset \mathcal{W}$ of q_0 and a system of isolating neighborhoods $\{U_k\}_{k\geq 1}$ chosen as in Corollary 10.2 so that in addition to the properties stated in Corollary 10.2, $\dot{\lambda}_k(q)$ is the unique root of $\dot{\Delta}(\lambda, q)$ in U_k for any $q \in V(q_0)$ and $k \geq 1$ (cf. [19, Lemma 3.1]). Arguing as in the proof of Proposition 10.5 one sees that $\dot{\Delta}(\lambda, q)$ has a product representation (see also [13], Proposition B.13)

$$\dot{\Delta}(\lambda, q) = -\prod_{j\geq 1} \frac{\dot{\lambda}_j - \lambda}{j^2\pi^2}. \tag{10.9}$$

Proposition 10.9. *Let $q_0 \in H_0^{-1}(\mathbb{T}^1)$ and $0 < \epsilon \leq 1/2$. Then there exist a complex neighborhood $U(q_0) \subset \mathcal{W}$ of q_0 (with $\mathcal{W}$ chosen as above) and $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $q \in U(q_0)$*

$$\dot{\lambda}_n(q) = \tau_n(q) + \mathtt{le}(\epsilon)|\gamma_n(q)|.$$

Proof. Taking $q \in \mathcal{W}$ and applying Proposition 10.5 we obtain for any $n \in \mathbb{N}$

$$\Delta^2(\lambda, q) - 4 = \frac{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}{n^2\pi^2} \chi_n(\lambda, q), \tag{10.10}$$

where

$$\chi_n(\lambda, q) \overset{\text{Def}}{=} 4\frac{\lambda - \lambda_0(q)}{n^2\pi^2} \prod_{k\in\mathbb{N}\setminus\{n\}} \frac{(\lambda_{2k}(q) - \lambda)(\lambda_{2k-1}(q) - \lambda)}{k^4\pi^4}. \tag{10.11}$$

Lemma 8.8 shows that for any $q_0 \in H_0^{-1}(\mathbb{T}^1)$ and $\epsilon > 0$ there exist a neighborhood $V(q_0) \subset \mathcal{W}$ of q_0, $m_0 \in \mathbb{N}$ and a constant $0 < \beta < \pi/2$ such that for $n \geq m_0$, $q \in V(q_0)$ and $\sqrt{\lambda} = n\pi + \beta$

$$\chi_n(\lambda, q) = 1 + \mathtt{le}(\epsilon/4). \tag{10.12}$$

For β as above we can choose by Corollary 10.2 $n_0 \in \mathbb{N}$, a neighborhood $U(q_0)$ of q_0 in $\mathcal{W}$, and a system of isolating neighborhoods $\{U_k\}_{k\geq 1}$ so that the disk of radius $|\gamma_n(q)|$ around any $\lambda \in U_n$ is contained in the neighborhood $\{\lambda \in \mathbb{C} | |\sqrt{\lambda} - n\pi| < \beta)\}$ for any $q \in U(q_0)$ and any $n \geq n_0$. By Cauchy's estimate, for $\lambda \in U_n$

$$|\gamma_n(q)||\dot{\chi}_n(\lambda, q)| \leq \epsilon/4. \tag{10.13}$$

Recall that by shrinking the neighborhood $U(q_0)$ if necessary, we can ensure that $\dot{\lambda}_k(q)$ is the unique root of $\dot{\Delta}(\lambda, q)$ in U_k for any $k \geq 1$. As $\dot{\lambda}_n(q) = \tau_n(q)$ if $\gamma_n(q) = 0$, it suffices to consider $q \in U(q_0) \setminus D_n$ (with $n \geq n_0$) By (10.10) the equation $\frac{d}{d\lambda}|_{\lambda=\dot{\lambda}_n(q)}(\Delta^2(\lambda, q) - 4) = 0$ can be written as

$$2(\dot{\lambda}_n - \tau_n(q))\chi_n(\dot{\lambda}_n, q) + (\lambda_{2n}(q) - \dot{\lambda}_n)(\lambda_{2n-1}(q) - \dot{\lambda}_n)\dot{\chi}_n(\dot{\lambda}_n, q) = 0. \tag{10.14}$$

Hence, with $\dot{\chi}_n = \dot{\chi}_n(\dot{\lambda}_n)$ and $\chi_n = \chi_n(\dot{\lambda}_n)$ and in view of (10.12) and (10.13),

$$\begin{aligned} |\dot{\lambda}_n - \tau_n| &= |\lambda_{2n} - \dot{\lambda}_n||\dot{\lambda}_n - \lambda_{2n-1}||\dot{\chi}_n/2\chi_n| \\ &\leq \frac{|\lambda_{2n} - \dot{\lambda}_n||\dot{\lambda}_n - \lambda_{2n-1}|}{|\gamma_n|} \; \frac{\epsilon/4}{2(1-\epsilon/4)} \\ &\leq \frac{\epsilon}{7} \; \frac{|\lambda_{2n} - \dot{\lambda}_n||\dot{\lambda}_n - \lambda_{2n-1}|}{|\gamma_n|} \end{aligned} \tag{10.15}$$

where we used that $0 < \epsilon \leq 1/2$. In the case q is real valued the claimed estimate then follows immediately as $\lambda_{2n-1}(q) \leq \dot{\lambda}_n(q) \leq \lambda_{2n}(q)$. In the general case we estimate $|\lambda_j - \dot{\lambda}_n|$ for $j = 2n, 2n-1$ as follows. For any $q \in U(q_0)$ and $\mu \in \mathbb{C}$ with $|\mu - \tau_n(q)| \leq |\gamma_n(q)/2|$ $(n \geq n_0)$ one has by Cauchy's formula

$$2\Delta(\mu)\dot{\Delta}(\mu) = \frac{d}{d\mu}(\Delta(\mu)^2 - 4) = \frac{1}{2\pi i} \int_{\{|\lambda - \tau_n| = |\gamma_n|\}} \frac{\Delta(\lambda)^2 - 4}{(\lambda - \mu)^2} \, d\lambda. \tag{10.16}$$

Recall that $\dot{\Delta}(\lambda)$ has a product representation

$$\dot{\Delta}(\mu, q) = -\prod_{j\geq 1} \frac{\dot{\lambda}_j - \mu}{j^2\pi^2}. \tag{10.17}$$

Shrinking the neighborhood $U(q_0)$ as well as choosing n_0 larger and $0 < \beta < \pi/2$ smaller if necessary it then follows from Lemma 8.5 that for $|\mu - \tau_n| \leq$

$|\gamma_n/2|$, with $n \geq n_0$ and $q \in U(q_0)$

$$\dot{\Delta}(\mu) = -\frac{\dot{\lambda}_n - \mu}{n^2\pi^2} \; \frac{(-1)^{n+1}}{2} \; (1 + \mathtt{le}(\epsilon/2)).$$

Further, by (10.10) and (10.12)

$$\Delta(\mu)^2 - 4 = \frac{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}{n^2\pi^2}(1 + \mathtt{le}(\epsilon/4)).$$

As $\Delta(\lambda_j) = (-1)^n\, 2$ for $j = 2n, 2n-1$ it then follows from (10.16) that

$$2\frac{\dot{\lambda}_n - \lambda_j}{n^2\pi^2}(1 + \mathtt{le}(\epsilon/2)) = \frac{1}{2\pi i} \int\limits_{\{|\lambda - \tau_n| = |\gamma_n|\}} \frac{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}{n^2\pi^2}(1 + \mathtt{le}(\epsilon/4))\frac{d\lambda}{(\lambda - \lambda_j)^2}\,.$$

By the trivial inequality $|\gamma_n|/2 \leq |\lambda_j - \lambda| \leq 3|\gamma_n|/2$ valid on the circles $\{|\lambda - \tau_n| = |\gamma_n|\}$ we obtain that $\frac{|\lambda_{2n} - \lambda|}{|\lambda_{2n-1} - \lambda|} \leq 3$. Hence, as $0 < \epsilon \leq 1/2$, the identity above leads to

$$|\dot{\lambda}_n - \lambda_j| \leq \frac{3}{2} \; \frac{1 + \epsilon/4}{1 - \epsilon/2} \; |\gamma_n| \leq \frac{9}{4}|\gamma_n|$$

and by (10.15) this then leads to the claimed estimate

$$|\dot{\lambda}_n - \tau_n| \leq \epsilon|\gamma_n|$$

for any $q \in U(q_0)$ and $n \geq n_0$. □

Recall that $D_n \overset{\mathtt{Def}}{=} \{q \in \mathcal{W} \mid \gamma_n(q) = 0\}$.

Proposition 10.10. *Let $q_0 \in D_n$ with $n \geq 1$ arbitrary. Then there exists an open neighborhood $V(q_0)$ of q_0 in $\mathcal{W}$ such that*

$$\dot{\lambda}_n(q) = \tau_n(q) + O(|\gamma_n|^2).$$

Proof. Given $q_0 \in D_n$ we find a neighborhood $V(q_0)$ of q_0 and a system of isolating neighborhoods $\{U_k\}_{k \geq 1}$ so that $\dot{\lambda}_k(q) \in U_k$ $(k \in \mathbb{N})$. As in the proof of Proposition 10.9 one sees that $\dot{\lambda}_n$ satisfies equation (10.14). Note that $\dot{\lambda}_n(q) = \tau_n(q)$ for any q in $V(q_0)$ with $\gamma_n(q) = 0$. As $\inf_{\lambda \in U_n} |\chi_n(\lambda)| > 0$ uniformly for $q \in V(q_0)$ equation (10.14) can be rewritten (for q in $V(q_0)$ and $V(q_0)$ sufficiently small) as

$$\dot{\lambda}_n - \tau_n = \left(\frac{\gamma_n}{2}\right)^2 \frac{\dot{\chi}_n/\chi_n}{2 + (\dot{\lambda}_n - \tau_n)\dot{\chi}_n/\chi_n}$$

where $\dot{\chi}_n \stackrel{\text{Def}}{=} \dot{\chi}_n(\dot{\lambda}_n(q), q)$ and $\chi_n \stackrel{\text{Def}}{=} \chi_n(\dot{\lambda}_n(q), q)$. For $q \to q_0$ one has $\tau_n - \dot{\lambda}_n \to 0$ and hence $\chi_n(\dot{\lambda}_n, q) \to \chi_n(\tau_n(q_0), q_0) \neq 0$ as well as $\dot{\chi}_n(\dot{\lambda}_n, q) \to \dot{\chi}_n(\tau_n(q_0), q_0)$. In particular $\dot{\chi}_n(\dot{\lambda}_n(q), q)/\chi_n(\dot{\lambda}_n(q), q)$ is bounded in a neighborhood of q_0 and as $\tau_n - \dot{\lambda}_n \to 0$ for $q \to q_0$ the statement of the proposition then follows. □

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