LOW-CODIMENSIONAL ASSOCIATED PRIMES OF GRADED COMPONENTS OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $R = \oplus_{n\geq 0} R_n$ be a homogeneous noetherian ring and let M be a finitely generated graded R-module. Let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to the irrelevant ideal $R_+ := \oplus_{n>0} R_n$ of R. We show that if R_0 is a domain, there is some $s \in R_0\backslash\{0\}$ such that the $(R_0)_s$-modules $H^i_{R_+}(M)_s$ are torsion-free (or vanishing) for all i.

On use of this, we can deduce the following results on the asymptotic behaviour of the n-th graded component $H^i_{R_+}(M)_n$ of $H^i_{R_+}(M)$ for $n \to -\infty$:

If R_0 is a domain or essentially of finite type over a field, the set

$$\{\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big) \mid \mathrm{height}(\mathfrak{p}_0) \leq 1\}$$

is asymptotically stable for $n \to -\infty$.

If R_0 is semilocal and of dimension 2, the modules $H^i_{R_+}(M)$ are tame. If R_0 is in addition a domain or essentially of finite type over a field, the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)$ is asymptotically stable for $n \to -\infty$.

1. INTRODUCTION

Let $R = \oplus_{n\geq 0} R_n$ be a homogeneous noetherian ring, so that R is $\mathbb{N}_0$-graded, the base ring R_0 is noetherian and R is generated over R_0 by finitely many elements $\ell_0, \cdots, \ell_r \in R_1$. Let $R_+ := \oplus_{n>0} R_n \subseteq R$ denote the irrelevant ideal of R and let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module. For each $i \in \mathbb{N}_0$, let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to R_+, furnished with its natural grading (cf [4, Chap. 12]). For each $n \in \mathbb{Z}$, let $H^i_{R_+}(M)_n$ denote the n-th graded component of $H^i_{R_+}(M)$.

In this paper we are interested in the asymptotic behaviour of the set $\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M)_n\big)$ of associated primes of the R_0-modules $H^i_{R_+}(M)_n$ for $n \to -\infty$. The obvious question in this context is, whether the above set is asymptotically stable for $n \to -\infty$,

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thus whether there is some $n_0 \in \mathbb{Z}$ such that $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_{n_0}\right)$ for all $n \leq n_0$. We shall express this for short by saying that we have "asymptotic stability (of associated primes) at level i". On use of examples constructed by Singh [13] or by Katzman [6] one can in fact see, that the mentioned asymptotic stability need not hold, even in the following surprisingly simple cases (cf [1]):

(1.1) $R_0 = \mathbb{Z}[\mathbf{x},\mathbf{y},\mathbf{z}]_{(p,\mathbf{x},\mathbf{y},\mathbf{z})}$; $R = M = R_0[\mathbf{u},\mathbf{v},\mathbf{w}]/(\mathbf{xu}+\mathbf{yv}+\mathbf{wz})$;

(1.2) $R_0 = K[\mathbf{x},\mathbf{y},\mathbf{s},\mathbf{t}]_{(\mathbf{x},\mathbf{y},\mathbf{s},\mathbf{t})}$; $R = M = R_0[\mathbf{u},\mathbf{v}]/(\mathbf{s}\mathbf{x}^2\mathbf{v}^2 - (\mathbf{t}+\mathbf{s})\mathbf{xyuv} + \mathbf{t}\mathbf{y}^2\mathbf{u}^2)$;

where $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v},\mathbf{w}$ are indeterminates, $p \in \mathbb{N}$ is a prime, K is a field and R is given the standard grading. It is important to notice that in the above two examples asymptotic stability is hurt in different ways: namely in (1.1) we always have $\{\mathfrak{p}_0\} \subseteq \mathrm{Ass}_{R_0}\left(H^3_{R_+}(R)_n\right) \subseteq \{\mathfrak{p}_0, \mathfrak{m}_0\}$ with equality on either side for infinitely many n, where $\mathfrak{p}_0 := (\mathbf{x},\mathbf{y},\mathbf{z})R_0$ and $\mathfrak{m}_0 := (\mathbf{x},\mathbf{y},\mathbf{z},p)R_0$. On the other hand in (1.2) the set $\bigcup_{n<0} \mathrm{Ass}_{R_0}\left(H^2_{R_+}(R)_n\right)$ is infinite. Clearly, there are also positive results concerning the above asymptotic stability question. Let us mention a few of them:

(1.3) *If the R-modules $H^j_{R_+}(M)$ are finitely generated for all $j < i$, we have asymptotic stability at level i* (cf [2]). *Moreover (under mild conditions on R_0) the "asymptotic set of prime divisors of $H^i_{R_+}(M)_n$ for $n \to -\infty$" is determined in local terms of M* (cf [3]).

(1.4) *If R_0 is (semi-) local and of dimension ≤ 1, we have asymptotic stability at any level i.*

One of the principal aims of this paper is to extend (1.4) to the case where R_0 is still of dimension 1 but no longer semilocal. In view of the local result (1.4) it is equivalent to ask whether the set $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ of R-associated primes of $H^i_{R_+}(M)$ is finite for all $i \in \mathbb{N}_0$. This obviously leads to the question whether for any noetherian ring A, any finitely generated A-module N and any ideal $\mathfrak{a} \subseteq A$ with $\dim(N/\mathfrak{a}N) = 1$ the set $\mathrm{Ass}_A\left(H^i_{\mathfrak{a}}(N)\right)$ is finite for each $i \in \mathbb{N}_0$. In the local case much can be said in such a situation (cf [12] and also [11]), contrary to the non-local case.

In particular cases (1.4) has found an extension to the non-local case, namely:

(1.5) *If R is a Cohen-Macaulay (CM) ring with $\dim(R_0) = 1$ and if M is a CM-module, we have asymptotic stability at any level i* (cf [10]).

(1.6) *If $\dim(R_0) = 1$ and if $R = M = R_0[\mathbf{x}_1, \cdots, \mathbf{x}_r]/(f_1, \cdots, f_n)$ with a regular sequence of homogeneous polynomials $f_1, \cdots, f_n$, we have asymptotic stability at any level i* (cf [9]).

To extend (1.4) to the non-local case, we prove that under mild conditions on R_0 but without any restriction on $\dim(R_0)$ we have "asymptotic stability of associated primes in codimension 1" at any level i, more precisely (cf 3.4, 3.7):

(1.7) *Assume that R_0 is a finite integral extension of a domain A_0 such that $\mathfrak{q}_0 \cap A_0 = 0$ for each minimal prime $\mathfrak{q}_0$ of R_0, or assume that R_0 is essentially of finite type over a field. Then, for each $i \in \mathbb{N}$:*

a) *The set $\mathcal{T}^i(M) := \{\mathfrak{p} \in \mathrm{Ass}_R \left(H^i_{R_+}(M)\right) \mid \mathrm{height}(\mathfrak{p} \cap R_0) \leq 1\}$ is finite;*

b) *the set $\mathcal{T}^i(M)_n := \{\mathfrak{p}_0 \in \mathrm{Ass}_{R_0} \left(H^i_{R_+}(M)_n\right) \mid \mathrm{height}(\mathfrak{p}_0) \leq 1\}$ is asymptotically stable for $n \to -\infty$.*

Specializing to the case $\dim(R_0) = 1$ we now easily get the requested extension of (1.4) under the additional condition that R_0 is a finite integral extension of a domain or essentially of finite type over a field (cf 3.10).

Our second aim is to extend (1.4) to the case in which R_0 is semilocal and of dimension 2. Without any further restriction on R_0 we get the following weaker result (cf 4.7 a)):

(1.8) *If R_0 is semilocal and of dimension 2, the R-module $H^i_{R_+}(M)$ is tame for each $i \in \mathbb{N}_0$,*

(which means that either $H^i_{R_+}(M)_n = 0$ for all $n \ll 0$ or $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$). In the special case where R and M are both CM this is shown in [10]. Under additional mild restrictions on R_0 we get the requested "local" extension of (1.4) to the case $\dim(R_0) = 2$, namely (cf 4.8):

(1.9) *Assume that R_0 is semilocal of dimension 2 and either a finite integral extension of a domain or essentially of finite type over a field. Then, we have asymptotic stability of associated primes at any level i.*

The crucial result of our paper is in fact the above statement (1.7). It is rather obvious that the proof of this statement has to use that the R_0-modules $H^i_{R_+}(M)$ behave well along a dense open set of $\mathrm{Spec}(R_0)$. And this is what we prove first, namely (cf 2.5):

(1.10) *If R_0 is a domain, there is some $s \in R_0 \setminus \{0\}$ such that the $(R_0)_s$-module $\left(H^i_{R_+}(M)\right)_s$ is torsion free (or vanishes) for each $i \in \mathbb{N}_0$.*

Our paper leaves open some questions which arise naturally. So in view of (1.1), (1.2) and (1.9) it seems obvious to ask:

(1.11) *Do we have asymptotic stability of associated primes at any level i if R_0 is a local (regular) domain of dimension 3?*

In view of the unexpected behaviour of graded components of local cohomology modules (cf [3], [7]) one should be careful in giving conjectures on this subject. Nevertheless, let us ask one more question - the "stability analogue" of the corresponding "finiteness problem" posed in [7]:

(1.12) *Is the set* $\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *asymptotically stable for* $n \to -\infty$ *for each* $i \in \mathbb{N}_0$*?*

Clearly, an affirmative answer to this problem of "asymptotic stability of supports" would imply tameness of all the modules $H^i_{R_+}(M)$ and hence answer the "tameness-problem" (cf [2]) affirmatively.

As for the unexplained terminology we refer to [4] and [5].

2. Torsion-Freeness of Local Cohomology

As in the introduction, let $R = \oplus_{n\geq 0} R_n$ denote a homogeneous noetherian ring and let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module. Also, let $R_+ := \oplus_{n>0} R_n \subseteq R$ denote the irrelevant ideal of R.

In this section we assume that the base ring R_0 is a domain and show that there is an element $s \in R_0\backslash\{0\}$ such that the localized local cohomology modules $H^i_{R_+}(M)_s$ are torsion-free (or 0) over $(R_0)_s$ for all $i \in \mathbb{N}_0$.

We first give a few preliminaries.

2.1. **Remark.** A) (cf [2]) For each $i \in \mathbb{N}_0$ we have

$$\mathrm{Ass}_R\left(H^i_{R_+}(M)\right) = \left\{\mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\right\}.$$

B) Assume now that R_0 is an integral domain. Let $s \in R_0\backslash\{0\}$ and let $i \in \mathbb{N}_0$. Then $R_s = (R_0)_s \otimes_{R_0} R$ is a homogeneous noetherian ring with irrelevant ideal $(R_s)_+ = R_+R_s = (R_+)_s$ and the (graded) flat base change property of local cohomology modules gives rise to an isomorphism of graded R_s-modules $H^i_{R_+}(M)_s \cong H^i_{(R_+)_s}(M_s)$. Therefore and in view of part A) the following statements are equivalent:

(i) $H^i_{R_+}(M)_s$ is a torsion-free $(R_0)_s$-module;

(ii) $H^i_{(R_s)_+}(M_s)$ is a torsion-free $(R_0)_s$-module;

(iii) If $\mathfrak{p} \in \mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$, then $s \in \mathfrak{p}$ or $\mathfrak{p} \cap R_0 = 0$.

(We convene that the zero module over a domain is free and torsion-free of rank 0).

C) Keep the notations and hypotheses of part B). The aim of this section is to show that we can choose $s \in R_0\backslash\{0\}$ such that the equivalent statements B) (i) - (iii) hold for all values of i. Clearly, to show this we may replace R and M by R_t and M_t respectively, where $t \in R_0\backslash\{0\}$ is arbitrary. •

If T is a module over the noetherian commutative ring A we use $\dim_A(T)$ and $\mathrm{pdim}_A(T)$ to denote the Krull- and the projective dimension of T respectively. We convene that $\dim_A(0) = \mathrm{pdim}_A(0) = -\infty$.

2.2. **Lemma.** *Let R_0 be an infinite domain with quotient field K. Moreover, let $d := \dim_{K\otimes_{R_0}R}(K \otimes_{R_0} M) \geq 0$ and let $\mathbf{x}_1, \cdots, \mathbf{x}_d$ be indeterminates. Then, there is an element $t \in R_0\backslash\{0\}$ and a homomorphism of homogeneous $(R_0)_t$-algebras*

$$(R_0)_t[\underline{\mathbf{x}}] = (R_0)_t[\mathbf{x}_1, \cdots, \mathbf{x}_d] \xrightarrow{\varphi} R_t$$

such that $\dim_{K[\underline{\mathbf{x}}]}(K \otimes_{(R_0)_t} M_t) = d$ and such that there are isomorphisms of graded $(R_0)_t[\underline{\mathbf{x}}]$-modules

$$H^i_{(R_0)_t[\underline{\mathbf{x}}]_+}(M_t) \cong H^i_{(R_t)_+}(M_t), \quad (\forall i \in \mathbb{N}_0).$$

Proof: Let $\mathfrak{a} := 0 \underset{R}{:} M$. Then $K \otimes_{R_0} \mathfrak{a} = 0 \underset{K\otimes_{R_0}R}{:} (K \otimes_{R_0} M)$ shows that $K \otimes_{R_0} (R/\mathfrak{a}) \cong (K \otimes_{R_0} R)/(K \otimes_{R_0} \mathfrak{a})$ is a homogeneous K-algebra of finite type and has dimension d. So, by the homogeneous normalization lemma there is a finite injective homomorphism of homogeneous K-algebras

$$K[\underline{\mathbf{x}}] = K[\mathbf{x}_1, \cdots, \mathbf{x}_d] \to K \underset{R_0}{\otimes} (R/\mathfrak{a}).$$

In particular we have $\sqrt{(\mathbf{x}_1, \cdots, \mathbf{x}_d)(K \otimes_{R_0} R/\mathfrak{a})} = (K \otimes_{R_0} R/\mathfrak{a})_+$. As $R/\mathfrak{a}$ is of finite type over R_0, we thus find an element $t \in R_0\backslash\{0\}$ and a finite injective homomorphism of homogeneous $(R_0)_t$-algebras

$$(R_0)_t[\underline{\mathbf{x}}] \to R_t/\mathfrak{a}_t$$

such that $\sqrt{(\mathbf{x}_1, \cdots, \mathbf{x}_d)(R_t/\mathfrak{a}_t)} = (R_t/\mathfrak{a}_t)_+$. Therefore, there is a homomorphism of homogeneous $(R_0)_t$-algebras

$$(R_0)_t[\underline{\mathbf{x}}] \xrightarrow{\varphi} R_t$$

such that $\sqrt{(\underline{\mathbf{x}})R_t + \mathfrak{a}_t} = \sqrt{(\mathbf{x}_1, \cdots, \mathbf{x}_d)R_t + \mathfrak{a}_t} = (R_t)_+$ and $\varphi^{-1}(\mathfrak{a}_t) = 0$. The latter equality gives $(0 \underset{K[\underline{\mathbf{x}}]}{:} K \otimes_{(R_0)_t} M_t) = 0$ and hence $\dim_{K[\underline{\mathbf{x}}]}(K \otimes_{(R_0)_t} M_t) = d$. As $\mathfrak{a}_t M_t = 0$, the graded base ring independence of local cohomology gives rise to isomorphisms of graded $(R_0)_t[\underline{\mathbf{x}}]$-modules

$$\begin{aligned} H^i_{(R_0)_t[\underline{\mathbf{x}}]_+}(M_t) &\cong H^i_{(\underline{\mathbf{x}})R_t}(M_t) \cong H^i_{(\underline{\mathbf{x}})R_t+\mathfrak{a}_t}(M_t) \\ &= H^i_{\sqrt{(\underline{\mathbf{x}})R_t+\mathfrak{a}_t}}(M_t) = H^i_{(R_t)_+}(M_t). \end{aligned}$$

■

2.3. **Lemma.** *Let R_0 be an infinite domain with quotient field K and let $d := \dim_{K\otimes_{R_0}R}(K \otimes_{R_0} M)$. Then, there is some $t \in R_0\backslash\{0\}$ such that*

$$H^i_{(R_t)_+}(M_t) = 0 \text{ for all } i > d.$$

Proof: If $d < 0$ we have $K \otimes_{R_0} M = 0$ and so find some $t \in R_0 \backslash \{0\}$ with $M_t = 0$ and hence with $H^i_{(R_t)_+}(M_t) = 0$ for all $i \geq 0$. If $d \geq 0$, apply 2.2 and observe that $H^i_{(R_0)_t[\underline{\mathbf{x}}]_+}(M_t) = H^i_{(\mathbf{x}_1, \cdots, \mathbf{x}_d)(R_0)_t[\underline{\mathbf{x}}]}(M_t) = 0$ for all $i > d$. ■

2.4. **Lemma.** *Let $R = R_0[\underline{\mathbf{x}}] = R_0[\mathbf{x}_1, \cdots, \mathbf{x}_d]$ be a polynomial ring over the noetherian domain R_0 with quotient field K and let M be a finitely generated graded R-module. Assume that $K \otimes_{R_0} M$ is a free $K \otimes_{R_0} R = K[\underline{\mathbf{x}}]$-module.*

Then, there is an element $s \in R_0 \backslash \{0\}$ such that $H^i_{(R_s)_+}(M_s)$ is free over $(R_0)_s$ if $i = d$ and vanishes if $i \neq d$.

Proof: As M is finitely generated, there is some $s \in R_0 \backslash \{0\}$ such that M_s is a graded free module of finite rank over $R_s = (R_0)_s[\underline{\mathbf{x}}]$. As $H^i_{(R_s)_+}(R_s)$ is free over $(R_0)_s$ if $i = d$ and vanishes otherwise, our claim is clear. ■

Now, we are ready to prove the announced main result

2.5. **Theorem.** *Let R_0 be a domain. Then, there is an element $s \in R_0 \backslash \{0\}$ such that $H^i_{R_+}(M)_s$ is a torsion-free $(R_0)_s$-module for all $i \in \mathbb{N}_0$.*

Proof: If $\dim(R_0) = 0, R_0$ is a field and we can choose $s = 1$. So, let $\dim(R_0) > 0$. Then in particular R_0 is infinite. Let K denote the quotient field of R_0. As M is finitely generated over R, there is some $t \in R_0 \backslash \{0\}$ such that M_t is torsion-free over $(R_0)_t$. Replacing R by R_t and M by M_t we thus may assume that M is torsion-free over R_0. This implies that $H^0_{R_+}(M) \subseteq M$ is torsion-free over R_0. So, it suffices to find some $s \in R_0 \backslash \{0\}$ such that the $(R_0)_s$-modules $H^i_{R_+}(M)_s$ are torsion-free for all $i > 0$. This, we do by induction on $d := \dim_{K \otimes_{R_0} R}(K \otimes_{R_0} M)$.

If $d \leq 0$, we have $(K \otimes_{R_0} M)_n = 0$ for all $n \gg 0$. As M is finitely generated over R we thus find some $s \in R_0 \backslash \{0\}$ such that $(M_s)_n = 0$ for all $n \gg 0$ so that $H^i_{(R_s)_+}(M_s) = 0$ for all $i > 0$. In view of 2.1 B) we get $H^i_{R_+}(M)_s = 0$ for all $i > 0$.

So, let $d > 0$. As $\dim_{K \otimes_{R_0} R}\big(K \otimes_{R_0} \big(M/\Gamma_{R_+}(M)\big)\big) \leq d$ and $H^i_{R_+}(M) \cong H^i_{R_+}\big(M/\Gamma_{R_+}(M)\big)$ for all $i > 0$ we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. By 2.2 and 2.1 C) we may assume that $R = R_0[\underline{\mathbf{x}}] = R_0[\mathbf{x}_1, \cdots, \mathbf{x}_d]$ is a polynomial ring.

So, let $d > 1$. Next, we show that there is some $t \in R_0 \backslash \{0\}$ such that $H^d_{R_+}(M)_t$ is torsion-free over $(R_0)_t$. To do this, we first assume that $K \otimes_{R_0} M$ is torsion-free of rank r over $K \otimes_{R_0} R = K[\underline{\mathbf{x}}]$. Then, there is an exact sequence of graded $K[\underline{\mathbf{x}}]$-modules

$$0 \to K \underset{R_0}{\otimes} M \to \overset{r}{\underset{i=1}{\oplus}} K[\underline{\mathbf{x}}](a_i) \to T \to 0$$

with $\dim_{K[\underline{\mathbf{x}}]}(T) < d$. So, there is some $v \in R_0\backslash\{0\}$ such that there is an exact sequence of finitely generated graded $(R_0)_v[\underline{\mathbf{x}}] = R_v$-modules

$$0 \to M_v \to \bigoplus_{i=1}^{r} R_v(a_i) \to U \to 0$$

with $\dim_{K[\underline{\mathbf{x}}]}(K \otimes_{(R_0)_v} U) < d$. Now, we can and do replace R by R_v and M by M_v. Then, the above sequence takes the form

$$0 \to M \to \bigoplus_{i=1}^{r} R(a_i) \to U \to 0$$

with $\dim_{K\otimes_{R_0}R}(K \otimes_{R_0} U) < d$. So, by induction there is some $t \in R_0\backslash\{0\}$ such that $H^{d-1}_{R_+}(U)_t$ is torsion-free over $(R_0)_t$. If we apply cohomology to the above sequence and then localize at t, we get an exact sequence of graded R_t-modules

$$0 \to H^{d-1}_{R_+}(U)_t \to H^d_{R_+}(M)_t \to \bigoplus_{i=1}^{r} H^d_{R_+}(R)_t(a_i).$$

As the last module in this sequence is free over $(R_0)_t$, we see that $H^d_{R_+}(M)_t$ is torsion-free over $(R_0)_t$.

Assume now, that the $K[\underline{\mathbf{x}}] = K \otimes_{R_0} R$-module $K \otimes_{R_0} M$ is not torsion-free. Let $Q := \{\mathfrak{q} \in \mathrm{Ass}_R(M)\backslash\{0\} \mid \mathfrak{q} \cap R_0 = 0\}$. As

$$\{\mathfrak{q}K[\underline{\mathbf{x}}] \mid \mathfrak{q} \in Q\} = \mathrm{Ass}_{K[\underline{\mathbf{x}}]}(K \underset{R_0}{\otimes} M)\backslash\{0\} \neq \emptyset,$$

we find some homogeneous element $a \in \bigcap_{\mathfrak{q}\in Q} \mathfrak{q}\backslash\{0\}$. Let $N := \Gamma_{aR}(M)$. As $K \otimes_{R_0} N \cong \Gamma_{aK[\underline{\mathbf{x}}]}(K \otimes_{R_0} N)$ and $0 \in \mathrm{Ass}_{K[\underline{\mathbf{x}}]}(K \otimes_{R_0} M)$ we have

$$\mathrm{Ass}_{K[\underline{\mathbf{x}}]}(K \otimes_{R_0} N) = \{\mathfrak{q}K[\underline{\mathbf{x}}] \mid \mathfrak{q} \in Q\}$$

and

$$\mathrm{Ass}_{K[\underline{\mathbf{x}}]}(K \otimes_{R_0} M/N) = \mathrm{Ass}_{K[\underline{\mathbf{x}}]}\big((K \otimes_{R_0} M)/\Gamma_{aK[\underline{\mathbf{x}}]}(K \otimes_{R_0} M)\big) = \{0\}.$$

Therefore $\overline{d} := \dim_{K[\underline{\mathbf{x}}]}(K \otimes_{R_0} N) < d$ and $K \otimes_{R_0} M/N$ is torsion-free over $K[\underline{\mathbf{x}}]$. By 2.3 there is some $v \in R_0\backslash\{0\}$ such that $H^i_{(R_v)_+}(N_v) = 0$ for all $i > \overline{d}$. As usual we can replace R by R_v and M by M_v and hence assume that $H^i_{R_+}(N) = 0$ for all $i > \overline{d}$. Now, as $K \otimes_{R_0} M/N$ is torsion-free over $K[\underline{\mathbf{x}}]$, there is some $t \in R_0\backslash\{0\}$ such that $H^d_{R_+}(M/N)_t$ is torsion-free over $(R_0)_t$. Applying cohomology to the exact sequence $0 \to N \to M \to M/N \to 0$, localizing at t and keeping in mind that $H^d_{R_+}(N) = 0$ we get a monomorphism of graded R-modules $H^d_{R_+}(M) \rightarrowtail H^d_{R_+}(M/N)$. So $H^d_{R_+}(M)_t$ is torsion-free over $(R_0)_t$ and we have found the requested element t in general.

We now replace R by $R_t = (R_0)_t[\underline{\mathbf{x}}]$ and M by M_t and hence assume that $H^d_{R_+}(M)$ is torsion-free over R_0. As $R_+ = (\mathbf{x}_1, \cdots, \mathbf{x}_d)R$ we have $H^i_{R_+}(M) = 0$ for all $i > d$. It thus remains to find an element $s \in R_0\backslash\{0\}$ such that $H^i_{R_+}(M)_s$ is torsion-free over $(R_0)_s$ for all $i \in \{1, \cdots, d-1\}$. We do this by induction on the projective dimension

$p = \mathrm{pdim}_{K[\underline{\mathbf{x}}]}(K \otimes_{R_0} M)$ of the $K[\underline{\mathbf{x}}]$-module $K \otimes_{R_0} M$. If $p \leq 0$, we are done by 2.4. So, let $p > 0$. There is an exact sequence of graded R-modules

$$0 \to L \xrightarrow{\varepsilon} \bigoplus_{i=1}^{r} R(a_i) \xrightarrow{\lambda} M$$

which induces a minimal graded presentation

$$0 \longrightarrow K \otimes_{R_0} L \xrightarrow{K \otimes \varepsilon} \bigoplus_{i=1}^{r} (K \otimes_{R_0} R)(a_i) \xrightarrow{K \otimes \lambda} K \otimes_{R_0} M \longrightarrow 0$$

of the graded $K[\underline{\mathbf{x}}] = K \otimes_{R_0} R$-module $K \otimes_{R_0} M$. In particular $\mathrm{pdim}_{K[\underline{\mathbf{x}}]}(K \otimes_{R_0} L) < p$. Moreover, there is some $w \in R_0 \setminus \{0\}$ such that the induced homomorphism $\lambda_w : \oplus_{i=1}^{r} R_w(a_i) \to M_w$ becomes surjective. Thus, we may replace R, M and L by R_w, M_w and L_w respectively and hence assume that we have an exact sequence of graded R-modules

$$0 \to L \xrightarrow{\varepsilon} \bigoplus_{i=1}^{r} R(a_i) \xrightarrow{\lambda} M \to 0$$

with $\mathrm{pdim}_{K[\underline{\mathbf{x}}]}(K \otimes_{R_0} L) < p$.

Now, by induction, there is some $s \in R_0 \setminus \{0\}$ such that $H^i_{R_+}(L)_s$ is torsion-free over $(R_0)_s$ for all $i \in \mathbb{N}_0$. If we apply cohomology to the above exact sequence we get monomorphisms $H^i_{R_+}(M) \rightarrowtail H^{i+1}_{R_+}(L)$ for $i = 1, \cdots, d-1$. It follows that $H^i_{R_+}(M)_s$ is torsion-free over $(R_0)_s$ for $i = 1, \cdots, d-1$. ■

3. Associated Primes of Local Cohomology in Codimension One

Again, let $R = \oplus_{n \geq 0} R_n$ and $M = \oplus_{n \in \mathbb{Z}} M_n$ be as in the introduction. We now study the one-codimensional associated primes of the graded components of the local cohomology modules $H^i_{R_+}(M)$.

3.1. **Notation and Remark.** A) Let $i \in \mathbb{N}_0$. We set:

$$\mathcal{T}^i = \mathcal{T}^i(M) := \{\mathfrak{p} \in \mathrm{Ass}_R \left(H^i_{R_+}(M)\right) \mid \text{ height}(\mathfrak{p} \cap R_0) \leq 1\}.$$

Moreover, for each $n \in \mathbb{Z}$ we set:

$$\mathcal{T}^i_n = \mathcal{T}^i_n(M) := \{\mathfrak{p}_0 \in \mathrm{Ass}_{R_0} \left(H^i_{R_+}(M)_n\right) \mid \text{ height}(\mathfrak{p}_0) \leq 1\}.$$

B) Clearly $\mathcal{T}^i_n = \emptyset$ for all $n \gg 0$ (cf [4, 15.1.5]). Moreover by 2.1 A)

$$\mathcal{T}^i = \{\mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \bigcup_{n \in \mathbb{Z}} \mathcal{T}^i_n\}.$$

•

3.2. **Convention.** Let $(\mathcal{S}_n)_{n\in\mathbb{Z}}$ be a family of sets. We say that the set $\mathcal{S}_n$ is *asymptotically stable* for $n \to -\infty$ if there is some $n_0 \in \mathbb{Z}$ such that $\mathcal{S}_n = \mathcal{S}_{n_0}$ for all $n \leq n_0$.

3.3. **Lemma.** *Let $i \in \mathbb{N}_0$ and let $\mathcal{S} \subseteq \mathcal{T}^i$. For each $n \in \mathbb{Z}$ let $\mathcal{S}_n := \{\mathfrak{p}_0 \in \mathcal{T}^i_n \mid \mathfrak{p}_0 + R_+ \in \mathcal{S}\}$. Then*

a) $\mathcal{S} = \{\mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \bigcup_{n\in\mathbb{Z}} \mathcal{S}_n\}$;

b) *$\mathcal{S}$ is finite if and only if $\mathcal{S}_n$ is asymptotically stable for $n \to -\infty$.*

Proof: a) is obvious by 3.1 B).

b) Assume first that $\mathcal{S}$ is finite. Then, by statement a) the set $\tilde{\mathcal{S}} := \bigcup_{n\in\mathbb{Z}} \mathcal{S}_n$ is finite, too. Now, let $\mathfrak{p}_0 \in \tilde{\mathcal{S}}$. Then $(R_0)_{\mathfrak{p}_0}$ is a local ring of dimension ≤ 1 and hence by [1, 3.5 e)] the set $\mathrm{Ass}_{R_{0\mathfrak{p}_0}} \left(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n\right)$ is asymptotically stable for $n \to -\infty$. In view of the natural isomorphisms of $(R_0)_{\mathfrak{p}_0}$-modules $\left(H^i_{R_+}(M)_n\right)_{\mathfrak{p}_0} \cong H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n$ we thus see that either $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ for all $n \ll 0$ or $\mathfrak{p}_0 \notin \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ for all $n \ll 0$. In the first case $\mathfrak{p}_0 \in \mathcal{S}_n$ for all $n \ll 0$, in the second $\mathfrak{p}_0 \notin \mathcal{S}_n$ for all $n \ll 0$. As $\tilde{\mathcal{S}}$ is finite, this shows that $\mathcal{S}_n$ is asymptotically stable for $n \to -\infty$.

Assume now, that $\mathcal{S}_n$ is asymptotically stable for $n \to -\infty$. As $\mathcal{S}_n$ is finite for each $n \in \mathbb{Z}$ and $\mathcal{S}_n = \emptyset$ for all $n \gg 0$ (cf 3.1 B)) the set $\bigcup_{n\in\mathbb{Z}} \mathcal{S}_n$ is finite. By a) it follows that $\mathcal{S}$ is finite. ∎

3.4. **Proposition.** *Assume that R_0 is a finite integral extension of a domain A_0 such that $\mathfrak{q}_0 \cap A_0 = 0$ for each minimal prime $\mathfrak{q}_0$ of R_0. Then, for each $i \in \mathbb{N}_0$*

a) *$\mathcal{T}^i(M)$ is finite;*

b) *$\mathcal{T}^i_n(M)$ is asymptotically stable for $n \to -\infty$.*

Proof: In view of (3.3) it suffices to prove statement a). Let $\ell_0, \cdots, \ell_r \in R_1$ be such that $R = R_0[\ell_0, \cdots, \ell_r]$ and let $A := A_0[\ell_0, \cdots, \ell_r]$. Then, A is a homogeneous noetherian subring of R such that $A_+R = R_+$ and such that R is a finite integral extension of A. In particular, M is a finitely generated graded A-module.

Now, by the graded base-ring independence of local cohomology there is an isomorphism of graded A-modules $H^i_{R_+}(M) \cong H^i_{A_+}(M)$. If we apply 2.5 and 2.1 B) to the finitely generated A-module M, we find some $s \in A_0\backslash\{0\}$ such that $\mathfrak{r} \cap A_0 = 0$ or $s \in \mathfrak{r} \cap A_0$ for each $\mathfrak{r} \in \mathrm{Ass}_A\left(H^i_{R_+}(M)\right)$.

As $\mathfrak{q}_0 \cap A_0 = 0$ for each minimal prime $\mathfrak{q}_0$ of R_0 we have $\mathrm{height}(sR_0) \geq 1$.

Let $\mathfrak{p} \in \mathcal{T}^i(M)$. Then $\mathfrak{p} \cap A \in \mathrm{Ass}_A\left(H^i_{R_+}(M)\right)$ and hence $\mathfrak{p} \cap A_0 = 0$ or $s \in \mathfrak{p} \cap A_0$. In the first case $\mathfrak{p} \cap R_0$ must be one of the finitely many minimal primes of R_0. If $s \in \mathfrak{p} \cap A_0$ we conclude from $\mathrm{height}(\mathfrak{p} \cap R_0) \leq 1 \leq \mathrm{height}(sR_0)$ that $\mathfrak{p} \cap R_0$ is one of the finitely many minimal primes of sR_0. So, the set $\{\mathfrak{p} \cap R_0 \mid \mathfrak{p} \in \mathcal{T}^i(M)\}$ is finite and hence so is $\mathcal{T}^i(M)$ (cf 2.1 A)). ∎

3.5. **Corollary.** *Assume that R_0 is an integral domain. Then, for each $i \in \mathbb{N}_0$ the conclusions of 3.4 hold.*

3.6. **Lemma.** *Let $i \in \mathbb{N}_0$ and assume that $\mathcal{T}^i(\Gamma_{\mathfrak{q}_0 R}(M))$ and $\mathcal{T}^i(M/\Gamma_{\mathfrak{q}_0 R}(M))$ are finite for some minimal prime $\mathfrak{q}_0$ of R_0 with $\mathfrak{q}_0 \supseteq 0 \underset{R_0}{:} M$. Then $\mathcal{T}^i(M)$ is finite.*

Proof: Let $\mathfrak{q}_0^{(1)}, \mathfrak{q}_0^{(2)}, \cdots, \mathfrak{q}_0^{(t)} = \mathfrak{q}_0$ be the different minimal primes of R_0 containing $0 \underset{R_0}{:} M$, $(t \in \mathbb{N}_0)$. Observe that $\mathfrak{p} \cap R_0$ is of height ≤ 1 and contains $0 \underset{R_0}{:} M$ for each $\mathfrak{p} \in \mathcal{T}^i(M)$.

Now, let $\overline{M} := M/\Gamma_{\mathfrak{q}_0 R}(M)$. Observe that $\mathfrak{q}_0 \not\supseteq 0 \underset{R_0}{:} \overline{M}$. By our hypotheses $\mathcal{T}^i(\overline{M})$ is finite. It thus suffices to show that $\mathcal{T}^i(M) \setminus \mathcal{T}^i(\overline{M})$ is finite. So, let $\mathfrak{p} \in \mathcal{T}^i(M) \setminus \mathcal{T}^i(\overline{M})$ and let $\mathfrak{p}_0 = \mathfrak{p} \cap R_0$. Keep in mind that height$(\mathfrak{p}_0) \leq 1$. The exact sequence of graded R-modules

$$H^{i-1}_{R_+}(\overline{M}) \xrightarrow{\delta} H^i_{R_+}(\Gamma_{\mathfrak{q}_0 R}(M)) \to H^i_{R_+}(M) \to H^i_{R_+}(\overline{M})$$

yields $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{coker}(\delta))$. In particular we have $\mathfrak{p} \in \mathrm{Supp}(\Gamma_{\mathfrak{q}_0 R}(M)) \subseteq \mathrm{Var}(\mathfrak{q}_0 R)$ and hence $\mathfrak{q}_0 \subseteq \mathfrak{p}_0$. Assume first that $\bigcap_{j=1}^{t-1} \mathfrak{q}_0^{(j)} \subseteq \mathfrak{p}_0$. Then $\mathfrak{p}_0$ must be a minimal prime of the ideal $\bigcap_{j=1}^{t-1} \mathfrak{q}_0^{(j)} + \mathfrak{q}_0$ and this leaves us with only finitely many possible choices for $\mathfrak{p}_0$ and hence for $\mathfrak{p} = \mathfrak{p}_0 + R_+$ (cf 2.1 A)).

Assume now that $\bigcap_{j=1}^{t-1} \mathfrak{q}_0^{(j)} \not\subseteq \mathfrak{p}_0$. If $\mathfrak{p}_0 \supseteq 0 \underset{R_0}{:} \overline{M}$, then $\mathfrak{p}_0$ contains a minimal prime $\mathfrak{r}_0$ of $0 \underset{R_0}{:} \overline{M}$. As $\mathfrak{r}_0 \neq \mathfrak{q}_0^{(j)}$ for $j = 1, \cdots, t-1$ and $\mathfrak{r}_0 \neq \mathfrak{q}_0$ it follows that $\mathfrak{r}_0$ is not minimal in R_0, thus $\mathfrak{r}_0 = \mathfrak{p}_0$. Again, in this case we are left with only finitely many possibilities for $\mathfrak{p} = \mathfrak{p}_0 + R_+$. If $\mathfrak{p}_0 \not\supseteq 0 \underset{R_0}{:} \overline{M}$ then $\overline{M}_{\mathfrak{p}_0} = 0$. On use of the graded flat base change property of local cohomology it follows that $H^{i-1}_{R_+}(\overline{M})_{\mathfrak{p}_0} = H^i_{R_+}(\overline{M})_{\mathfrak{p}_0} = 0$ and hence $H^i_{R_+}(\Gamma_{\mathfrak{q}_0 R}(M))_{\mathfrak{p}_0} \cong H^i_{R_+}(M)_{\mathfrak{p}_0}$. But this implies $\mathfrak{p} \in \mathrm{Ass}_R\left(H^i_{R_+}(\Gamma_{\mathfrak{q}_0 R}(M))\right)$ and so $\mathfrak{p}$ belongs to the finite set $\mathcal{T}^i(\Gamma_{\mathfrak{q}_0 R}(M))$. ∎

3.7. **Theorem.** *Assume that R_0 is essentially of finite type over a field. Then, for each $i \in \mathbb{N}$*

a) $\mathcal{T}^i(M)$ *is finite;*

b) $\mathcal{T}^i_n(M)$ *is asymptotically stable for $n \to -\infty$.*

Proof: By 3.3 it suffices to prove statement a). There is a subring $A_0 \subseteq R_0$ and a multiplicative set $S_0 \subseteq A_0$ such that A_0 is of finite type over a field and $R_0 = S_0^{-1}A_0$. Let $\ell_0, \cdots, \ell_r \in R_1$ be such that $R = R_0[\ell_0, \cdots, \ell_r]$ and let $A := A_0[\ell_0, \cdots, \ell_r]$. Then A is a noetherian homogeneous subring of R such that $R = S_0^{-1}A$. Let $m_1, \cdots, m_s \in M$ be homogeneous elements with $M = \sum_{j=1}^{s} Rm_j$ and let $N := \sum_{j=1}^{s} Am_j$. Then, N is a finitely generated graded A-module with $S_0^{-1}N = M$. So by the graded flat base change property of local cohomology there is an isomorphism of graded R-modules $H^i_{R_+}(M) = H^i_{(S_0^{-1}A)_+}(S_0^{-1}N) \cong S_0^{-1}H^i_{A_+}(N)$ which shows that $\mathcal{T}^i(M) \subseteq \{S_0^{-1}\mathfrak{q} \mid \mathfrak{q} \in \mathcal{T}^i(N)\}$. It thus suffices to show that $\mathcal{T}^i(N)$ is finite. This allows to replace R and M by A and N respectively and hence to assume that R_0 is of finite type over a field.

Let $\mathfrak{q}_0^{(1)}, \mathfrak{q}_0^{(2)}, \cdots, \mathfrak{q}_0^{(t)}$ be the different minimal primes of R_0 containing $0 \underset{R_0}{:} M$, ($t \in \mathbb{N}_0$). We proceed by induction on t. If $t = 0$, we have height$(0 \underset{R_0}{:} M) > 0$. As height$(\mathfrak{p} \cap R_0) \leq 1$ and $\mathfrak{p} \cap R_0 \supseteq 0 \underset{R_0}{:} M$ for each $\mathfrak{p} \in \mathcal{T}^i(M)$ our claim follows from 2.1 A). So, let $t > 0$ and set $\mathfrak{q}_0 := \mathfrak{q}_0^{(t)}$, $\overline{M} := M/\Gamma_{\mathfrak{q}_0 R}(M)$. Then $\mathrm{Ass}_R(\overline{M}) = \mathrm{Ass}_R(M)\setminus\mathrm{Var}(\mathfrak{q}_0 R)$ shows that $\mathfrak{q}_0^{(1)}, \cdots, \mathfrak{q}_0^{(t-1)}$ are the different minimal primes of R_0 containing $0 \underset{R_0}{:} \overline{M}$. So, by induction $\mathcal{T}^i(\overline{M})$ is finite.

By 3.6 it remains to show that $\mathcal{T}^i\left(\Gamma_{\mathfrak{q}_0 R}(M)\right)$ is finite. Therefore we may replace M by $\Gamma_{\mathfrak{q}_0 R}(M)$ and hence assume that $\mathfrak{q}_0^n M = 0$ for some $n \in \mathbb{N}$. So M becomes a graded finitely generated module over $R/\mathfrak{q}_0^n R$ and by the graded base ring independence property of local cohomology there is an isomorphism of graded $R/\mathfrak{q}_0^n R$-modules $H^i_{R_+}(M) \cong H^i_{(R/\mathfrak{q}_0^n R)_+}(M)$. Thus, we may replace in addition R by $R/\mathfrak{q}_0^n R$ and hence assume that $\mathfrak{q}_0$ is the unique minimal prime ideal of R_0. Now, by Noether's normalization lemma, R_0 is a finite integral extension of a noetherian domain A_0. As $\mathfrak{q}_0$ is the unique minimal prime of R_0 we have $\mathfrak{q}_0 \cap A_0 = 0$. So, by 3.4 the set $\mathcal{T}^i(M)$ is finite. ∎

3.8**. Notation and Remark.** A) Assume that R_0 is of finite dimension d. Let $i \in \mathbb{N}_0$. We set

$$\mathcal{S}^i = \mathcal{S}^i(M) := \{\mathfrak{p} \in \mathrm{Ass}_R\left(H^i_{R_+}(M)\right) \mid \dim(R_0/\mathfrak{p} \cap R_0) \geq d-1\}.$$

Moreover for each $n \in \mathbb{Z}$ we set

$$\mathcal{S}^i_n = \mathcal{S}^i_n(M) := \{\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) \mid \dim(R_0/\mathfrak{p}_0) \geq d-1\}.$$

B) In the notations of 3.1 we clearly have

$$\mathcal{S}^i \subseteq \mathcal{T}^i \text{ and } \mathcal{S}^i_n = \{\mathfrak{p}_0 \in \mathcal{T}^i_n \mid \mathfrak{p}_0 + R_+ \in \mathcal{S}^i\} \text{ for all } n \in \mathbb{Z}.$$

•

3.9. **Proposition.** *Assume that* $\dim(R_0) < \infty$ *and that* R_0 *is a finite integral extension of a domain* A_0. *Then, for each* $i \in \mathbb{N}_0$

a) $\mathcal{S}^i(M)$ *is finite;*

b) $\mathcal{S}^i_n(M)$ *is asymptotically stable for* $n \to -\infty$.

Proof: In view of 3.8 B) and 3.3 it is enough to prove statement a). This is done similarly as in the proof of 3.4 just on use on the inequality $\dim(R_0/sR_0) \leq \dim(R_0) - 1$ instead of the inequality height $(sR_0) \geq 1$. ∎

3.10. **Corollary.** *Let* $\dim(R_0) \leq 1$. *Assume that* R_0 *is either semilocal or a finite integral extension of a domain or essentially of finite type over a field. Then, for each* $i \in \mathbb{N}_0$

a) $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ *is finite;*

b) $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for* $n \to -\infty$.

Proof: As $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right) = \mathcal{S}^i(M) = \mathcal{T}^i(M)$ and $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \mathcal{S}^i_n(M) = \mathcal{T}^i_n(M)$ in this case, we conclude by 3.3 if R_0 is semilocal and by 3.9 resp. 3.7 otherwise. ∎

4. Semilocal Base Rings of Dimension 2

We keep our previous notations and hypotheses. In this section our interest is focussed on the asymptotic behaviour of the graded components of the local cohomology modules $H^i_{R_+}(M)$ in the case where R_0 is a semilocal ring of dimension ≤ 2.

4.1. **Lemma.** *Assume that* $(R_0, \mathfrak{m}_0)$ *is local and of dimension* ≤ 2. *Let* $x_0 \in \mathfrak{m}_0$ *be* $M/\Gamma_{\mathfrak{m}_0 R}(M)$*-regular and such that* $\dim(R_0/x_0R_0) \leq 1$. *Then, for each* $i \in \mathbb{N}_0$, *the graded* R*-module* $\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)/x_0 H^i_{R_+}(M)\right)$ *is artinian.*

Proof: Let $\overline{M} := M/\Gamma_{\mathfrak{m}_0 R}(M)$. If we apply cohomology to the short exact sequence $0 \to \overline{M} \xrightarrow{x_0} \overline{M} \to \overline{M}/x_0\overline{M} \to 0$ and keep in mind that the functor $\Gamma_{\mathfrak{m}_0 R}$ is left-exact, we get a monomorphism of graded R-modules

$$0 \to \Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(\overline{M})/x_0 H^i_{R_+}(\overline{M})\right) \to \Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(\overline{M}/x_0\overline{M})\right).$$

Graded base ring independence of local cohomology yields a natural isomorphism $H^i_{R_+}(\overline{M}/x_0\overline{M}) \cong H^i_{(R/x_0R)_+}(\overline{M}/x_0\overline{M})$. As $(R/x_0R)_0 \cong R_0/x_0R_0$ is of dimension ≤ 1 it follows that $\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(\overline{M}/x_0\overline{M})\right)$ is an artinian R-module (cf [1, (2.5)]). So the R-module $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(\overline{M})\ /x_0 H^i_{R_+}(\overline{M})\big)$ is artinian, too. As $H^j_{R_+}\big(\Gamma_{\mathfrak{m}_0 R}(M)\big)$ is artinian for each $j \in \mathbb{N}_0$ (cf [1, 2.3]) we get two exact sequences of graded R-modules

$0 \to A \to H^i_{R_+}(M) \to U \to 0$; $0 \to U \to H^i_{R_+}(\overline{M}) \to B \to 0$ in which A and B are artinian. Therefore, we get two exact sequences of graded R-modules

$$0 \to \overline{A} \to H^i_{R_+}(M)/x_0 H^i_{R_+}(M) \to U/x_0U$$
$$0 \to \overline{B} \to U/x_0U \to H^i_{R_+}(\overline{M})/x_0 H^i_{R_+}(\overline{M})$$

in which $\overline{A}$ is a homomorphic image of A and $\overline{B}$ is a homomorphic image of the artinian R-module $\mathrm{Tor}^R_1(B, R/x_0R)$. So $\overline{A}$ and $\overline{B}$ are both artinian.

If we apply the left-exact functor $\Gamma_{\mathfrak{m}_0 R}$ to the above sequences, we get our claim. ■

4.2. **Lemma.** *Let $(R_0, \mathfrak{m}_0)$ be local of dimension $d > 0$. Let $x_0 \in \mathfrak{m}_0$ be a parameter for R_0 and let $i \in \mathbb{N}_0$. Assume that for infinitely many $n \in \mathbb{Z}$ we have $\dim_{R_0}\left(H^i_{R_+}(M)_n/x_0H^i_{R_+}(M)_n\right) \geq d-1$. Then, there is some $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$ with $\dim(R_0/\mathfrak{p}_0) \geq d-1$ and such that $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ for all $n \ll 0$.*

Proof: By our hypotheses there is a minimal prime $\mathfrak{q}$ of x_0R_0 such that $\dim(R_0/\mathfrak{q}) = d-1$ and $\mathfrak{q} \in \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n/x_0H^i_{R_+}(M)_n\right)$ for infinitely many $n \in \mathbb{Z}$. So, on use of the graded flat base change property of local cohomology and as in addition $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$ it follows that $H^i_{(R_\mathfrak{q})_+}(M_\mathfrak{q})_n \cong \left(H^i_{R_+}(M)_n\right)_\mathfrak{q} \neq 0$ for infinitely many $n < 0$. As $(R_\mathfrak{q})_0 = (R_0)_\mathfrak{q}$ is local and of dimension ≤ 1, there is some $\mathfrak{s} \in \mathrm{Spec}\left((R_\mathfrak{q})_0\right)$ such that $\mathfrak{s} \in \mathrm{Ass}_{(R_\mathfrak{q})_0}\left(H^i_{(R_\mathfrak{q})_+}(M_\mathfrak{q})_n\right)$ for all $n \ll 0$ (cf 3.10). With $\mathfrak{p}_0 := \mathfrak{s} \cap R_0$, another use of the graded flat base change property of local cohomology gives our claim. ■

4.3. **Lemma.** *Let $(R_0, \mathfrak{m}_0)$ be local and of dimension ≤ 2. Let $i \in \mathbb{N}$ and assume that $H^i_{R_+}(M)_n = 0$ for infinitely many $n < 0$. Then $H^i_{R_+}(M)_n = 0$ for all $n \ll 0$.*

Proof: As $H^i_{R_+}(M)_n = 0$ is equivalent to $\mathrm{Ass}_{R_0}\left(H^i_{R_0}(M_n)\right) = \emptyset$ we may conclude by 3.10 b) whenever $\dim(R_0) \leq 1$. So, let $\dim(R_0) = 2$. Choose an element $x_0 \in \mathfrak{m}_0$ which avoids all minimal primes of R_0 and all members of $\mathrm{Ass}_R(M)\backslash\mathrm{Var}(\mathfrak{m}_0R)$. Then $\dim(R_0/x_0R_0) = 1$ and x_0 is $M/\Gamma_{\mathfrak{m}_0R}(M)$-regular, so that $\Gamma_{\mathfrak{m}_0R}\left(H^i_{R_+}(M)/x_0H^i_{R_+}(M)\right)$ is artinian (cf 4.1). Assume that $H^i_{R_+}(M)_n \neq 0$ for infinitely many $n < 0$. Then, by Nakayama, $H^i_{R_+}(M)_n/x_0H^i_{R_+}(M)_n \neq 0$ for infinitely many $n < 0$.

If $\dim_{R_0}\left(H^i_{R_+}(M)_n/\ x_0H^i_{R_+}(M)_n\right) \geq 1$ for infinitely many $n < 0$ we conclude by 4.2 that $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$, a contradiction. Therefore

$$\dim_{R_0}\left(H^i_{R_+}(M)_n/x_0H^i_{R_+}(M)_n\right) = 0$$

and hence

$$\Gamma_{\mathfrak{m}_0R}\left(H^i_{R_+}(M)/x_0H^i_{R_+}(M)\right)_n = \Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n/x_0H^i_{R_+}(M)_n\right) \neq 0$$

for infinitely many $n < 0$. As $\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)/x_0 H^i_{R_+}(M)\right)$ is artinian it follows hat $\Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n/x_0 H^i_{R_+}(M)_n\right) \neq 0$ for all $n \ll 0$. This leads to the contradiction that $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$. ■

4.4. **Lemma.** *Let* $(R_0, \mathfrak{m}_0)$ *be local, let* $i \in \mathbb{N}_0, n \in \mathbb{Z}$ *and let* $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$. *Let* $x_0 \in \mathfrak{m}_0$ *be* $H^i_{R_+}(M)_n/\Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n\right)$*-regular. Then*

$$\Gamma_{\mathfrak{m}_0 R}\left(H^i_R(M)\right)_n = \Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n\right) \nsubseteq x_0 H^i_{R_+}(M)_n.$$

Proof: Let $H := H^i_{R_+}(M)_n$. Then $\mathfrak{m}_0 \in \mathrm{Ass}_{R_+}(H)$ implies that $\Gamma_{\mathfrak{m}_0}(H) \neq 0$. As x_0 is $H/\Gamma_{\mathfrak{m}_0}(H)$-regular we have $\Gamma_{\mathfrak{m}_0}(H) = \Gamma_{x_0 R_0}(H)$ and hence $\Gamma_{\mathfrak{m}_0}(H) \cap x_0 H = \Gamma_{x_0 R_0}(H) \cap x_0 H = x_0 \Gamma_{x_0 R_0}(H) = x_0 \Gamma_{\mathfrak{m}_0}(H)$. By Nakayama we have $x_0 \Gamma_{\mathfrak{m}_0}(H) \subsetneqq \Gamma_{\mathfrak{m}_0}(H)$ and hence $\Gamma_{\mathfrak{m}_0}(H) \nsubseteq x_0 H$. ■

4.5. **Lemma.** *Let* $(R_0, \mathfrak{m}_0)$ *be local and of dimension* ≤ 2. *Let* $i \in \mathbb{N}_0$ *and assume that the set* $\mathcal{S}^i(M)$ *of 3.8 is finite and that* $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *for infinitely many* $n < 0$. *Then* $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *for all* $n \ll 0$.

Proof: If $\dim(R_0) \leq 1$ we may conclude by 3.10 b). So, let $\dim(R_0) = 2$. As $\mathcal{S}^i(M)$ is finite, the set $\tilde{\mathcal{S}} := \bigcup_{n\in\mathbb{Z}} \mathcal{S}^i_n(M) = \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) \backslash \{\mathfrak{m}_0\}$ finite, too (cf 2.1 A)). So, there is some $x_0 \in \mathfrak{m}_0$ which avoids at the same time all members of $\tilde{\mathcal{S}}$, all members of $\mathrm{Ass}_R(M) \backslash \mathrm{Var}(\mathfrak{m}_0 R)$ and all minimal primes of R_0. Therefore $\dim(R_0/x_0 R_0) = 1$, x_0 is $M/\Gamma_{\mathfrak{m}_0 R}(M)$-regular and moreover x_0 is $H^i_{R_+}(M)_n/\Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n\right)$-regular for all $n \in \mathbb{Z}$.

In particular $\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)/x_0 H^i_{R_+}(M)\right)$ is an artinian R-module (cf 4.1). As

$$U := \left(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) + x_0 H^i_{R_+}(M)\right)/x_0 H^i_{R_+}(M)$$

is a submodule of $H^i_{R_+}(M)/x_0 H^i_{R_+}(M)$ and is $\mathfrak{m}_0 R$-torsion, U is a submodule of $\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)/x_0 H^i_{R_+}(M)\right)$. So, U is artinian. Now, by our hypotheses and by 4.4 we have $\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)\right)_n \nsubseteq x_0 H^i_{R_+}(M)_n$ for infinitely many $n < 0$, so that the n-th graded component U_n of U is non-vanishing for infinitely many $n < 0$. As U is artinian, it follows that $U_n \neq 0$ for all $n \ll 0$. But this implies that $\Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n\right) = \Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)\right)_n \neq 0$ for all $n \ll 0$ and hence gives our claim. ■

4.6. **Reminder and Remark.** A) According to [2] we say that $H^i_{R_+}(M)$ is *asymptotically gap free* or *tame* if either $H^i_{R_+}(M)_n = 0$ for all $n \ll 0$ or $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$.

B) If the set $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ is asymptotically stable for $n \to -\infty$, then $H^i_{R_+}(M)$ is tame. •

4.7. **Theorem.** *Let* R_0 *be semilocal and of dimension* ≤ 2. *Let* $i \in \mathbb{N}_0$. *Then*

a) $H^i_{R_+}(M)$ *is tame;*

b) *if the set* $\mathcal{S}^i(M)$ *of 3.8 is finite, the set* $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for* $n \to -\infty$.

Proof: Let $\mathfrak{m}_0^{(1)}, \cdots, \mathfrak{m}_0^{(r)}$ be the different maximal ideals of R_0. In view of the natural isomorphisms of $(R_0)_{\mathfrak{m}_0^{(j)}}$-modules $\left(H^i_{R_+}(M)_n\right)_{\mathfrak{m}_0^{(j)}} \cong H^i_{(R_{\mathfrak{m}_0^{(j)}})_+}(M_{\mathfrak{m}_0^{(j)}})_n$ for all $j \in \{1, \cdots, r\}$ and $n \in \mathbb{Z}$, one may immediately pass to the case where $(R_0, \mathfrak{m}_0)$ is local.

Now, statement a) is clear by 4.3. In order to prove statement b), set $\mathcal{W}^i_n(M) = \{\mathfrak{m}_0\} \cap \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ and observe that $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \mathcal{S}^i_n(M) \cup \mathcal{W}^i_n(M)$ for each $n \in \mathbb{Z}$.

As $\mathcal{S}^i(M)$ is finite, $\mathcal{S}^i_n(M)$ is asymptotically stable for $n \to -\infty$ (cf 3.8 B), 3.3 b)). By 4.5 the set $\mathcal{W}^i_n(M)$ is asymptotically stable for $n \to -\infty$. This proves our claim. ■

4.8. **Corollary.** *Let* R_0 *be semilocal and of dimension* ≤ 2. *Assume that* R_0 *is either a finite integral extension of a domain or essentially of finite type over a field. Then, for each* $i \in \mathbb{N}_0$, *the set* $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ *is asymptotically stable for* $n \to -\infty$.

Proof: Clear from 4.7 b), 3.9, 3.7 and 3.8 B). ■

References

[1] M. BRODMANN, S. FUMASOLI and R. TAJAROD: *Local cohomology over homogeneous rings with one-dimensional local base ring*, to appear in Proceedings of the AMS.

[2] M. BRODMANN and M. HELLUS: *Cohomological patterns of coherent sheaves over projective schemes*, Journal of Pure and Applied Algebra 172 (2002), 165 - 182.

[3] M. BRODMANN, M. KATZMAN and R.Y. SHARP: *Associated primes of graded components of local cohomology modules*, Transactions of the AMS 354, No 11 (2002), 4261 - 4283.

[4] M. BRODMANN and R.Y. SHARP: *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics 60, Cambridge University Press (1998).

[5] D. EISENBUD: *Commutative algebra with a view towards algebraic geometry*, Springer, New York (1996).

[6] M. KATZMAN: *An example of an infinite set of associated primes of a local cohomology module*, Journal of Algebra 252 (2002), 161 - 166.

[7] M. KATZMAN and R.Y. SHARP: *Some properties of top graded local cohomology modules*, preprint.

[8] C.S. LIM: *Graded local cohomology modules and their associated primes: The Cohen-Macaulay case*, preprint.

[9] C.S. LIM: *Graded local cohomology modules and their associated primes*, to appear in Communications in Algebra.

[10] C.S. LIM: *Graded local cohomology and its associated primes*, PhD Thesis, Michigan State University (2002).

[11] T. MARLEY: *The associated primes of local cohomology modules of small dimension*, Manuscripta Mathematica 104 (2001), 519 - 525.

[12] T. MARLEY and J.C. VASSILEV: *Cofiniteness and associated primes of local cohomology modules*, preprint.

[13] A.K. SINGH: *p-torsion elements in local cohomology modules*, Mathematics Research Letters 7 (2000), 165 - 176.

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