An Elementary Proof of Poncelet's Theorem

(on the occasion to its bicentennial)

Lorenz Halbeisen (University of Zürich)
Norbert Hungerbühler (ETH Zürich)

Abstract

We present an elementary proof of Poncelet's Theorem which relies only on Pascal's Theorem in the projective plane.

1 Introduction

In 1813, while Poncelet was in captivity as war prisoner in the Russian city of Saratov, he discovered the following theorem:

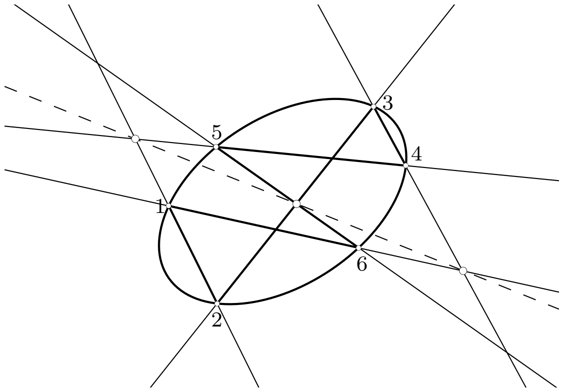
PONCELET'S THEOREM. *Let K and C be smooth conics in general position which neither meet nor intersect. Suppose there is an n-sided polygon inscribed in K and circumscribed about C. Then for any point P of K, there exists an n-sided polygon, also inscribed in K and circumscribed about C, which has P as one of its vertices.*

After his return to France, Poncelet published a proof in his book [9], which appeared in 1822. Until now, several proofs and generalizations have been found involving the theory of elliptic functions or measure theory (see, e.g., Griffiths and Harris [6, 7], and Flatto [5] or Dragović and Radnović [4] for a general overview). However, according to Flatto [5, p. 2], none of these proofs is elementary.

The aim of this paper is to give an elementary proof of PONCELET'S THEOREM. More precisely, we will show that PONCELET'S THEOREM is a purely combinatorial consequence of PASCAL'S THEOREM. Before we give several forms of the latter, let us introduce some notation: For two points a and b, let $a-b$ denote the line through a and b, and for two lines ℓ_1 and ℓ_2, let $\ell_1 \wedge \ell_2$ denote the intersection point of these lines in the projective plane. In abuse of notation, we often write $a - b - c$ in order to emphasize that the points a, b, c are collinear. In the sequel, points are often labeled with numbers, and lines with encircled numbers like ③.

In this terminology, PASCAL'S THEOREM and its equivalent forms read as follows:

Pascal's Theorem (*cf.* [8])



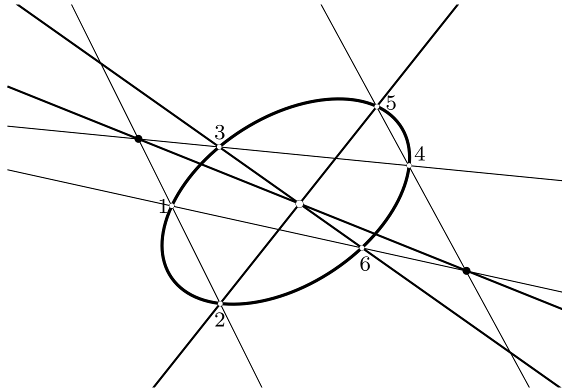
Any six points 1, …, 6 lie on a conic

if and only if

$$(1 - 2) \wedge (4 - 5)$$
$$(2 - 3) \wedge (5 - 6)$$
$$(3 - 4) \wedge (6 - 1)$$

are collinear.

Carnot's Theorem (*cf.* [2, no. 396])



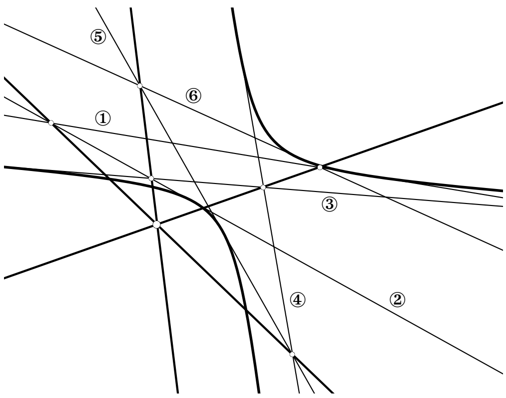
Any six points 1, …, 6 lie on a conic

if and only if

$$[(1 - 2) \wedge (3 - 4)] - [(4 - 5) \wedge (6 - 1)]$$
$$(2 - 5)$$
$$(3 - 6)$$

are concurrent.

Brianchon's Theorem (*cf.* [1])



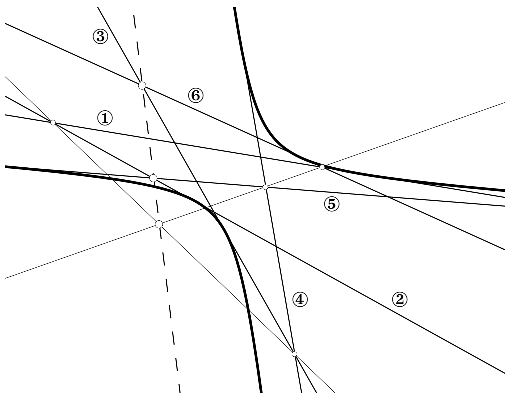
Any six lines ①, …, ⑥ are tangent to a conic

if and only if

$$(\textcircled{1} \wedge \textcircled{2}) - (\textcircled{4} \wedge \textcircled{5})$$
$$(\textcircled{2} \wedge \textcircled{3}) - (\textcircled{5} \wedge \textcircled{6})$$
$$(\textcircled{3} \wedge \textcircled{4}) - (\textcircled{6} \wedge \textcircled{1})$$

are concurrent.

Carnot's Theorem*



Any six lines ①, …, ⑥ are tangent to a conic

if and only if

$$[(\textcircled{1} \wedge \textcircled{2}) - (\textcircled{3} \wedge \textcircled{4})] \wedge [(\textcircled{4} \wedge \textcircled{5}) - (\textcircled{6} \wedge \textcircled{1})]$$
$$(\textcircled{2} \wedge \textcircled{5})$$
$$(\textcircled{3} \wedge \textcircled{6})$$

are collinear.

As the real projective plane is self-dual, PASCAL'S THEOREM and BRIANCHON'S THEOREM are equivalent. Moroever CARNOT'S THEOREM and ts dual CARNOT'S THEOREM* are just reformulations of PASCAL'S THEOREM and BRIANCHON'S THEOREM by exchanging the points 3 and 5, and the lines ③ and ⑤, respectively. Recall that if two adjacent points, say 1 and 2, coincide, then the corresponding line $1-2$ becomes a tangent with 1 as contact point. Similarly, if two lines, say ① and ②, coincide, then ①$\wedge$② becomes the contact point of the tangent ①. As a last remark we would like to mention that a conic is in general determined by five points, by five tangents, or by a combination like three tangents and two contact points of these tangents.

The paper is organized as follows: In Section 2, we prove PONCELET'S THEOREM for the special case of triangles and at the same time we develop the kind of combinatorial arguments we shall use later. Section 3 contains the crucial tool which allows to show that PONCELET'S THEOREM holds for an arbitrary number of edges. Finally, in Section 4, we use the developed combinatorial technics in order to prove some additional symmetry properties of Poncelet-polygons.

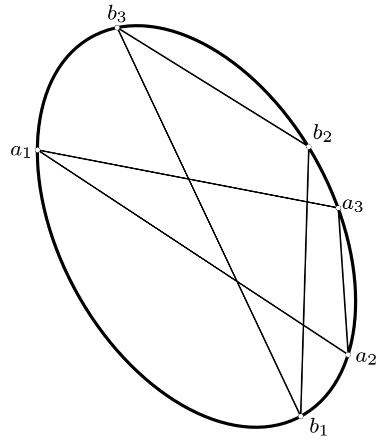
2 Poncelet's Theorem for Triangles

For triangles, PONCELET'S THEOREM states as follows: *If there is a triangle, which is inscribed in some conic K and circumscribed about another conic C, then any point of K is the vertex of some triangle which is also inscribed in K and circumscribed about C.*

In order to prove PONCELET'S THEOREM for triangles, we will show that if the six vertices of two triangles lie on a conic K, then the six sides of the triangles are tangents to some conic C.

The crucial point in the proof of the following theorem (as well as in the proofs of the other theorems of this paper) is to find the suitable numbering of points and edges, and to apply some form of PASCAL'S THEOREM in order to find collinear points or concurrent lines.

THEOREM 2.1. *If two triangles are inscribed in a conic and the two triangles do not have a common vertex, then the six sides of the triangles are tangent to a conic.*

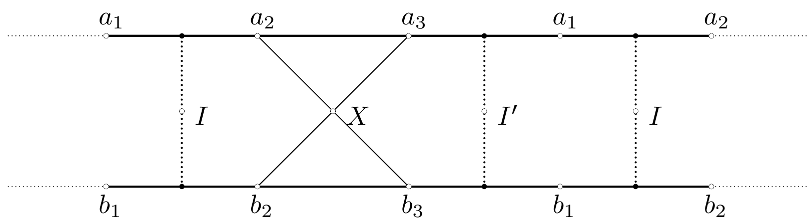


Proof. Let K be a conic in which two triangles $\triangle a_1 a_2 a_3$ and $\triangle b_1 b_2 b_3$ are inscribed where the two triangles do not have a common vertex.

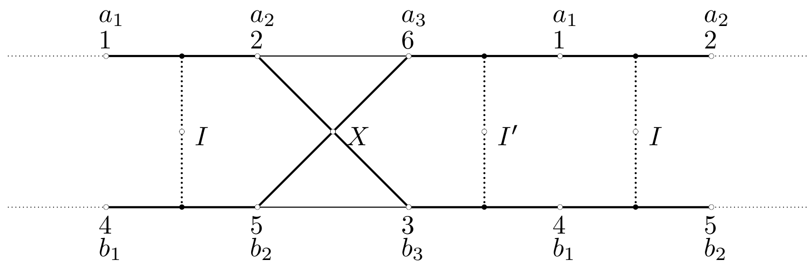
Firstly, we introduce the following three intersection points:

$$I := (a_1 - a_2) \wedge (b_1 - b_2)$$
$$X := (a_2 - b_3) \wedge (b_2 - a_3)$$
$$I' := (a_3 - a_1) \wedge (b_3 - b_1)$$

In order to visualize the intersection points I, X, and I', we break up the conic K and draw it as two straight lines, one for each triangle:



Now, we number the six points $a_1, a_2, a_3, b_1, b_2, b_3$ on the conic K as shown in the following figure:

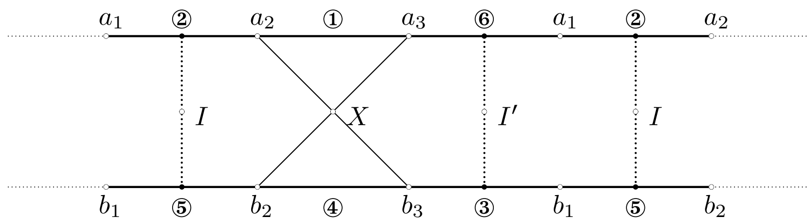


By PASCAL'S THEOREM we get that the three intersection points

$$(1-2)\wedge(4-5)\qquad(2-3)\wedge(5-6)\qquad(3-4)\wedge(6-1)$$

are collinear, which is the same as saying that the points $I-X-I'$ are collinear.

In the next step, we label the sides of the triangles as shown in the following figure:



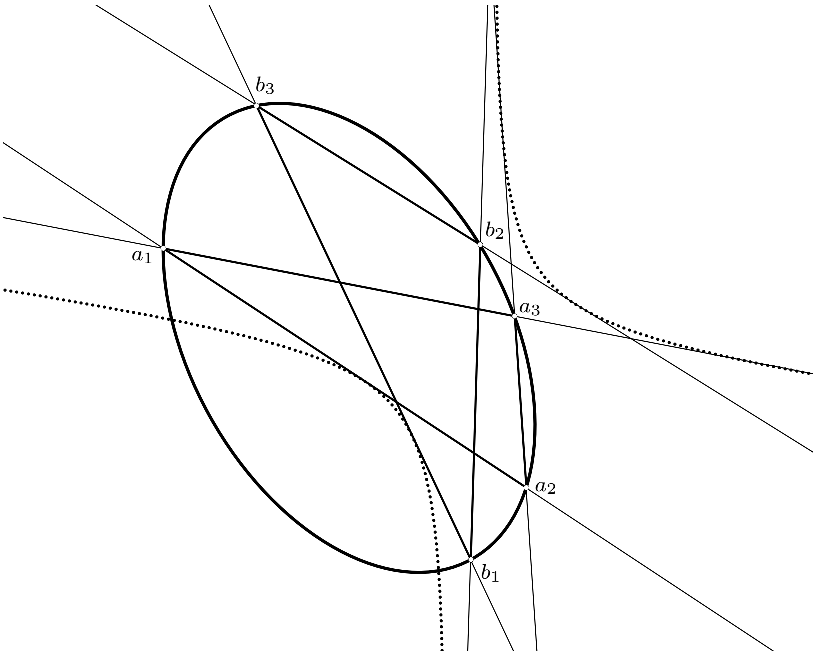
By CARNOT'S THEOREM* we get that the six sides ①, ..., ⑥ of the two triangles are tangents to a conic if and only if

$$[(① \wedge ②)-(③ \wedge ④)]\wedge[(④ \wedge ⑤)-(⑥ \wedge ①)]$$

$$(② \wedge ⑤)$$

$$(③ \wedge ⑥)$$

are collinear. Now, this is the same as saying that the points $X-I-I'$ are collinear, which, as we have seen above, is equivalent to a_1,a_2,a_3,b_1,b_2,b_3 lying on a conic.



q.e.d.

As an immediate consequence we get PONCELET'S THEOREM for triangles:

COROLLARY 2.2 (Poncelet's Theorem for Triangles). *Let K and C be smooth conics which neither meet nor intersect. Suppose there is a triangle $\triangle a_1a_2a_3$ inscribed in K and circumscribed about C. Then for any point b_1 of K, there exists a triangle $\triangle b_1b_2b_3$ which is also inscribed in K and circumscribed about C.*

Proof. Given two conics K and C. Let the triangle $\triangle a_1a_2a_3$ be inscribed in K and circumscribed about C. Furthermore, let b_1 be an arbitrary point on K which is distinct from a_1, a_2, a_3, and let b_2 and b_3 be distinct points on K such that $b_1 - b_2$ and $b_1 - b_3$ are two tangents to C. By construction, we get that five sides of the triangles $\triangle a_1a_2a_3$ and $\triangle b_1b_2b_3$ are tangents to the conic C. On the other hand, by THEOREM 2.1, we know that all six sides of these triangles are tangents to some conic C'. Now, since a conic is determined by five tangents, C' and C coincide, which implies that the triangle $\triangle b_1b_2b_3$ is circumscribed about C. *q.e.d.*

The following fact about *Poncelet-triangles* has a near relative for even-sided polygons (see Section 4):

FACT 2.3. *Let K and C be conics and let the triangle $\triangle a_1a_2a_3$ be inscribed in K and circumscribed about C. Furthermore, let t_1, t_2, t_3 be the contact points of the three tangents $a_2 - a_3, a_3 - a_1, a_1 - a2$. Then the three lines $a_1 - t_1, a_2 - t_2, a_3 - t_3$ meet in a point.*

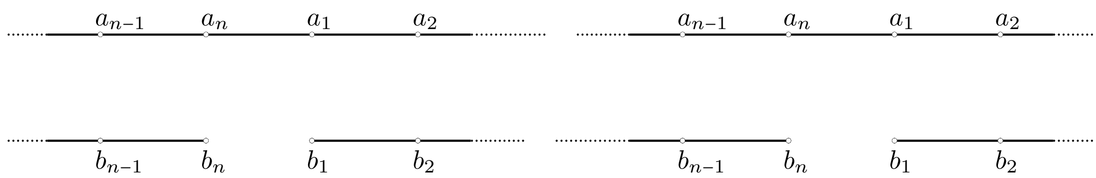
Proof. Label the three sides of the triangles as follows:

$$① = a_2 - a_3 = ② \qquad ③ = a_3 - a_1 = ④ \qquad ⑤ = a_1 - a_2 = ⑥$$

Then $① \wedge ② = t_1$, $② \wedge ③ = t_2$, $④ \wedge ⑤ = t_3$, and by BRIANCHON'S THEOREM we get that $a_1 - t_1, a_2 - t_2, a_3 - t_3$ meet in a point. *q.e.d.*

3 The General Case

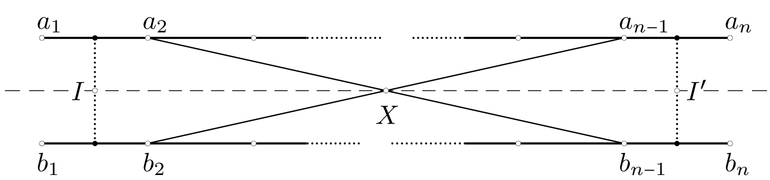
Let K and C be smooth conics in general position which neither meet nor intersect. We assume that there is an n-sided polygon $a_1, \dots, a_n$ which is inscribed in K such that all its n sides $a_1 - a_2, a_2 - a_3, \dots, a_n - a_1$ are tangent to C. Choose an arbitrary point b_1 on K which is distinct from the points $a_1, \dots, a_n$. From b_1 we draw a tangent to C which meets K in a second point b_2. Notice that since K and C neither meet nor intersect, b_1 is distinct from b_2, and also from the points a_i. From b_2 we draw the other tangent to C and get a third point b_3. Proceeding this way, we get an $(n-1)$-sided polygonal chain $b_1, \dots, b_n$ where all $n-1$ sides $b_1 - b_2, b_2 - b_3, \dots, b_{n-1} - b_n$ are tangent to C, but we do not know whether $b_n - b_1$ is tangent to C too. If we break up the conic K and draw it as two straight lines, one for each polygon, we get the following situation:



In order to prove PONCELET'S THEOREM, we have to show that $b_n - b_1$ is also tangent to C, which will follow quite easily from the following result:

LEMMA 3.1. *For $n \geq 4$, the three intersection points*

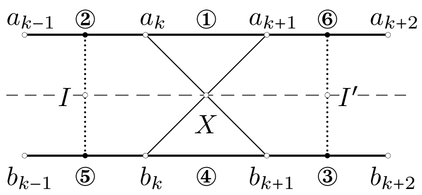
$$\begin{aligned} I &:= (a_1 - a_2) \wedge (b_1 - b_2) \\ X &:= (a_2 - b_{n-1}) \wedge (b_2 - a_{n-1}) \\ I' &:= (a_{n-1} - a_n) \wedge (b_{n-1} - b_n) \end{aligned}$$



are collinear, which is visualized by the dashed line.

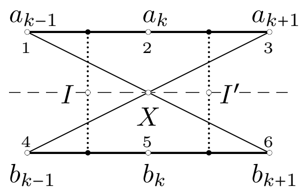
Proof. Depending on the parity of n, we have one of the following anchorings, from which we will work step by step outwards.

n even, with $k = \frac{n}{2}$:



By Carnot's Theorem* we have that $I - X - I'$ are collinear.

n odd, with $k = \frac{n+1}{2}$:



By Pascal's Theorem we have that $I - X - I'$ are collinear.

The lemma will now follow by the following two claims:

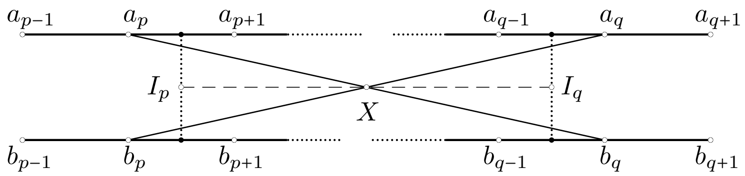
Claim 1. *Let p and q be integers with $2 \leq p < q \leq a_{n-1}$. Further, let*

$$I_{p-1} := (a_{p-1} - a_p) \wedge (b_{p-1} - b_p), \qquad I_p := (a_p - a_{p+1}) \wedge (b_p - b_{p+1}),$$

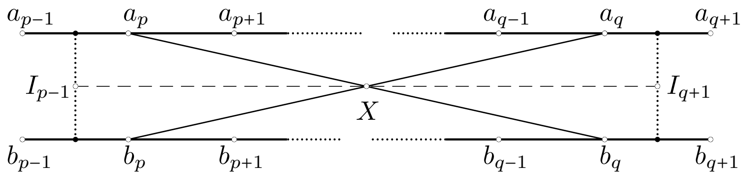
$$I_q := (a_{q-1} - a_q) \wedge (b_{q-1} - b_q), \qquad I_{q+1} := (a_q - a_{q+1}) \wedge (b_q - b_{q+1}),$$

$$X := (a_p - b_q) \wedge (b_p - a_q).$$

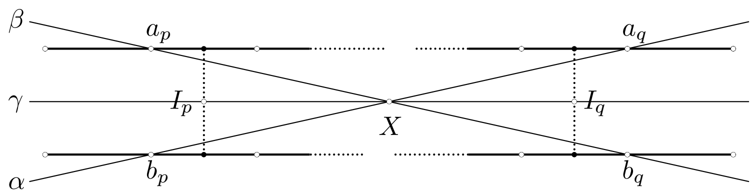
If $I_p - X - I_q$ are collinear, then $I_{p-1} - X - I_{q+1}$ are collinear too. This implication is visualized in the following figure:



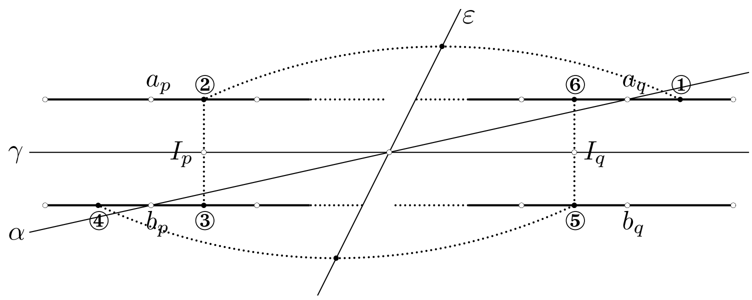
$$\Downarrow$$



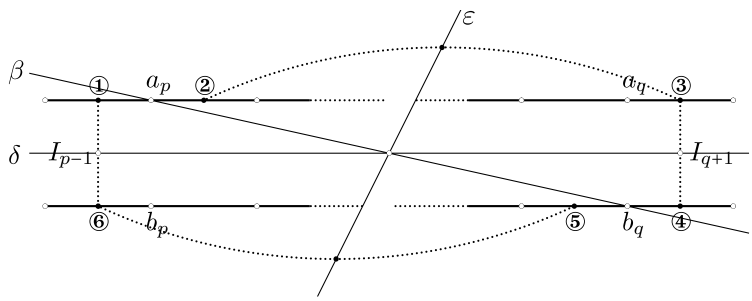
Proof of Claim 1.



(a) By assumption, the lines α, β, γ meet in X.



(b) By Brianchon's Theorem, the lines α, γ, ε are concurrent.



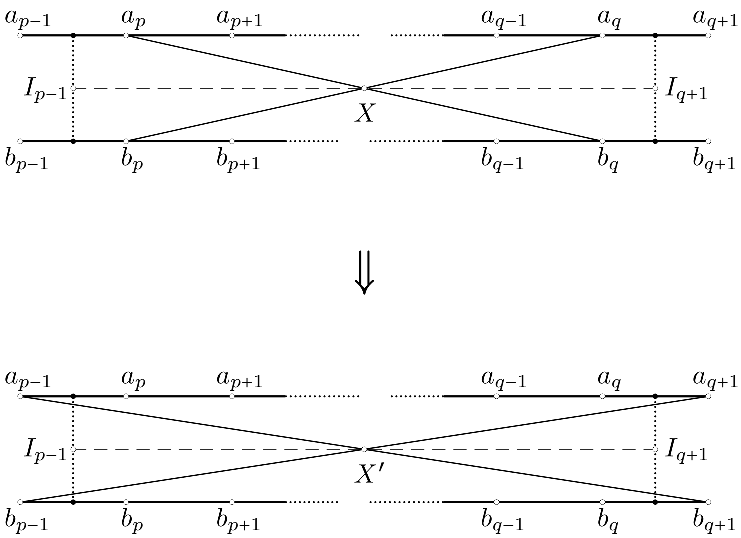
(c) By Brianchon's Theorem, the lines β, ε, δ are concurrent.

By (a) & (b) we get that α, β, and ε meet in X, and by (c) we get that also α, β, and δ meet in X, which implies that $I_{p-1} - X - I_{q+1}$ are collinear. *q.e.d.*

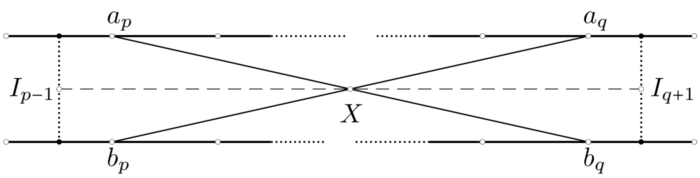
Claim 2. *Let I_{p-1}, I_{q+1}, and X be as above, and let*

$$X' := (a_{p-1} - b_{q+1}) \wedge (b_{p-1} - a_{q+1})\,.$$

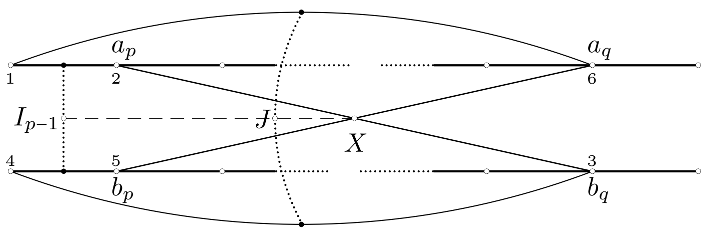
If $I_{p-1} - X - I_{q+1}$ are collinear, then $I_{p-1} - X' - I_{q+1}$ are collinear too. This implication is visualized in the following figure:



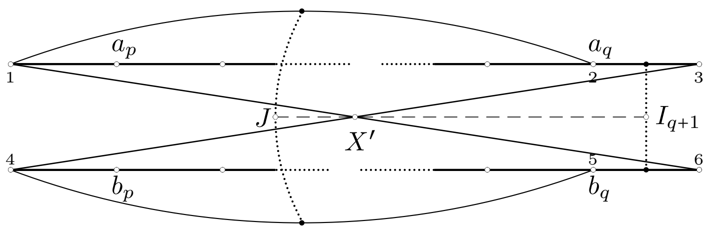
Proof of Claim 2.



(a) By assumption, the points $I_{p-1} - X - I_{q+1}$ are collinear.



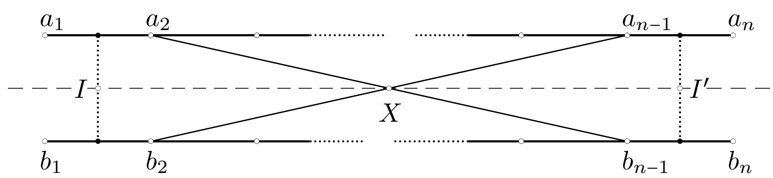
(b) By Pascal's Theorem, the points $I_{p-1} - X - J$ are collinear.



(c) By Pascal's Theorem, the points $X' - J - I_{q+1}$ are collinear.

By (a) & (b) we get that $I_{p-1} - J - I_{q+1}$ are collinear, and by (c) we get that X' lies on $J - I_{q+1}$, hence, $I_{p-1} - X' - I_{q+1}$ are collinear. *q.e.d.*

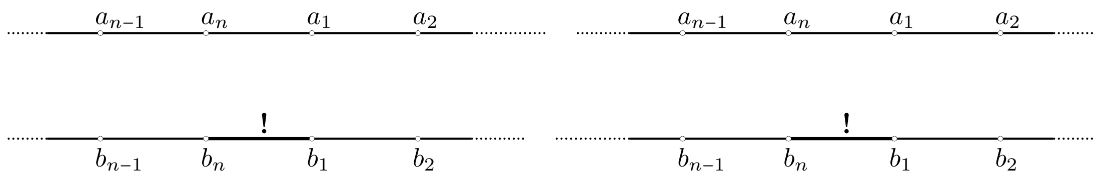
By an iterative application of Claim 1 & 2, we finally get the situation



in which $I - X - I'$ are collinear. *q.e.d.*

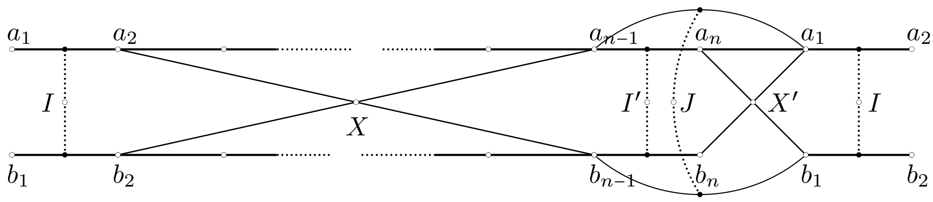
With similar arguments as in the proof of Poncelet's Theorem for triangles (Corollary 2.2), we can now proof the general case:

Theorem 3.2 (Poncelet's Theorem). *Let K and C be smooth conics in general position which neither meet nor intersect. We assume that there is an n-sided polygon $a_1, \dots, a_n$ which is inscribed in K such that all its n sides $a_1 - a_2, a_2 - a_3, \dots, a_n - a_1$ are tangent to C. Further we assume that there is an $(n-1)$-sided polygonal chain $b_1, \dots, b_n$ whose $n-1$ sides are tangent to C. Then also $b_n - b_1$ is tangent to C.*

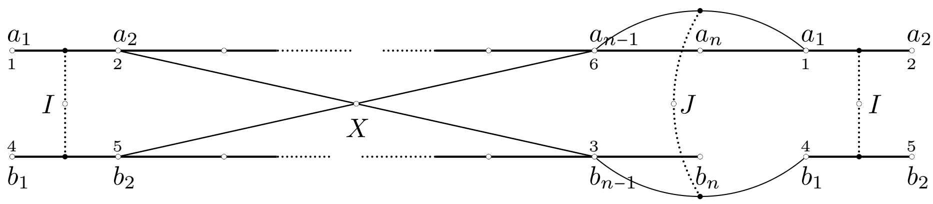


Proof. By LEMMA 3.1 we know that $I - X - I'$ are collinear, where $I = (a_1 - a_2) \wedge (b_1 - b_2)$, $I' = (a_{n-1} - a_n) \wedge (b_{n-1} - b_n)$, and $X = (a_2 - b_{n-1}) \wedge (b_2 - a_{n-1})$. In order to show that $b_n - b_1$ is tangent to C, we have to introduce two more intersection points:

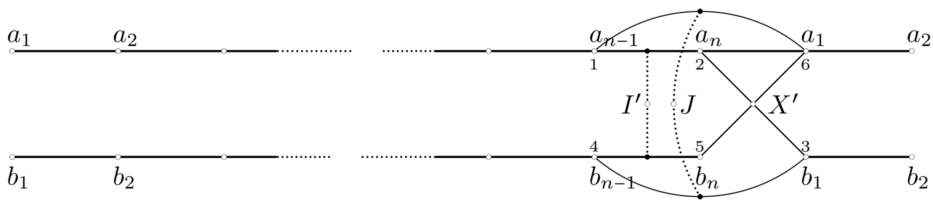
$$\begin{aligned} J &:= (a_{n-1} - a_1) \wedge (b_{n-1} - b_1) \\ X' &:= (a_n - b_1) \wedge (b_n - a_1) \end{aligned}$$



We apply now twice PASCAL'S THEOREM:



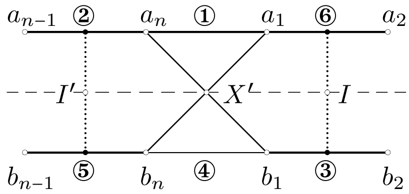
(a) By PASCAL'S THEOREM, the points $I - X - J$ are collinear.



(b) By PASCAL'S THEOREM, the points $I' - J - X'$ are collinear.

Since, by LEMMA 3.1, $I - X - I'$ are collinear, by (a) we get that $I - X - J - I'$ are collinear, and by (b) we finally get that $I - X' - I'$ are collinear.

For the last step, we apply CARNOT'S THEOREM*:



Since $I - X' - I'$ are collinear, we get (by CARNOT'S THEOREM*) that the six lines ①, …, ⑥ are tangent to some conic C'. Now, since a conic is determined by five tangents, and the five lines ①,②,③,⑤,⑥ are tangent to C, C' and C coincide. This implies that ④ is tangent to C, which is what we had to show. *q.e.d.*

4 Symmetries in Poncelet-Polygons

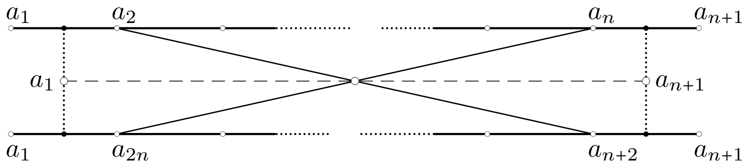
In this section we present some symmetries in $2n$-sided polygons which are inscribed in some conic K and circumscribed about another conic C. To keep the terminology short, we shall call such a polygon *$2n$-Poncelet-polygon* with respect to $K \,\&\, C$.

THEOREM 4.1. *Let K and C be smooth conics in general position which neither meet nor intersect and let $a_1, \ldots, a_{2n}$ be the vertices of a $2n$-Poncelet-polygon with respect to $K \,\&\, C$. Further let $t_1, \ldots, t_{2n}$ be the contact points of the n tangents $a_1 - a_2, \ldots, a_{2n} - a_1$.*

(a) All the n diagonals $a_1 - a_{n+1}, a_2 - a_{n+2}, \ldots, a_n - a_{2n}$ meet in a point H_0.

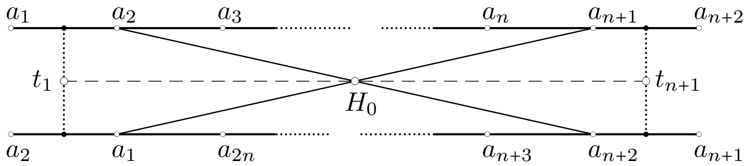
(b) All the n lines $t_1 - t_{n+1}, t_2 - t_{n+2}, \ldots, t_n - t_{2n}$ meet in the same point H_0.

Proof. (a) By the proof of LEMMA 3.1, we get that the three points a_1, a_{n+1}, and $(a_2 - a_{n+2}) \wedge (a_n - a_{2n})$ are collinear:



This is the same as saying that the three diagonals $a_1 - a_{n+1}$, $a_2 - a_{n+2}$, and $a_n - a_{2n}$ meet in a point, say H_0. Now, by cyclic permutation we get that all n diagonals meet in H_0.

(b) By the proof of LEMMA 3.1, we get that the three points $t_1 - H_0 - t_{n+1}$ are collinear:



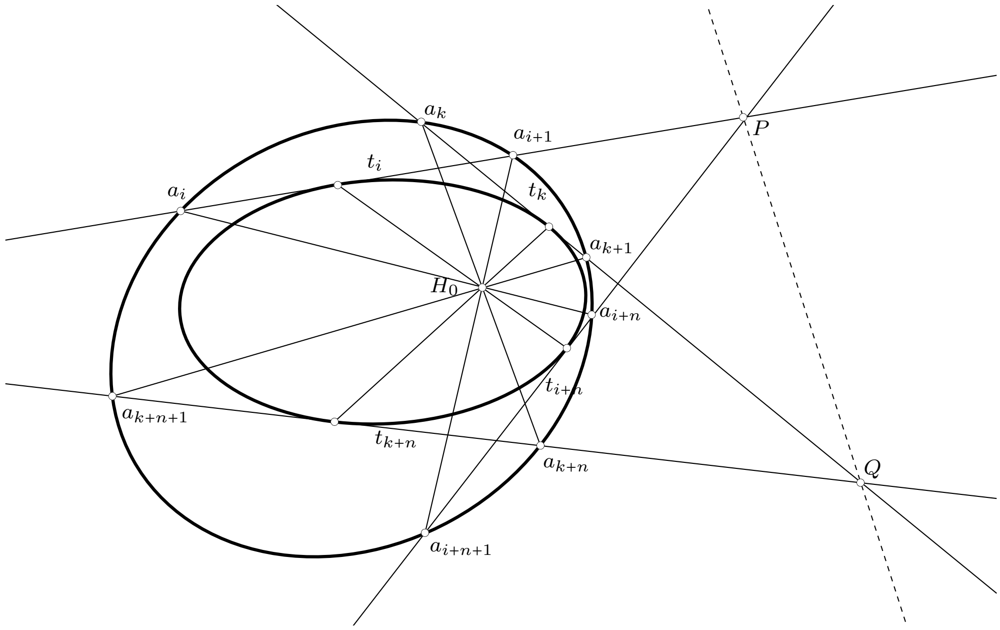
Thus, by cyclic permutation we get that all n lines $t_1 - t_{n+1}, t_2 - t_{n+2}, \ldots, t_n - t_{2n}$ pass through H_0, which implies that all n lines meet in H_0. *q.e.d.*

In the last result we show that the point H_0 is independent of the particular $2n$-Poncelet-polygon:

THEOREM 4.2. *Let K and C be smooth conics in general position which neither meet nor intersect and let $a_1, \ldots, a_{2n}$ and $b_1, \ldots, b_{2n}$ be the vertices of two $2n$-Poncelet-polygons with respect to K & C. Further let $t_1, \ldots, t_{2n}$ and $t'_1, \ldots, t'_{2n}$ be the contact points of the Poncelet-polygons. Then all $4n$ lines $a_1 - a_{n+1}, \ldots,$ $t_1 - t_{n+1}, \ldots,$ $b_1 - b_{n+1}, \ldots,$ $t'_1 - t'_{n+1}, \ldots$ meet in a point H_0. Moreover, opposite sides of the Poncelet-polygons meet on a fixed line h, where h is the polar of H_0, both with respect to C and K.*

Notice, that for $n = 3$, H_0 is the Brianchon point with respect to C of the Poncelet-hexagon, and h its Pascal line with respect to K.

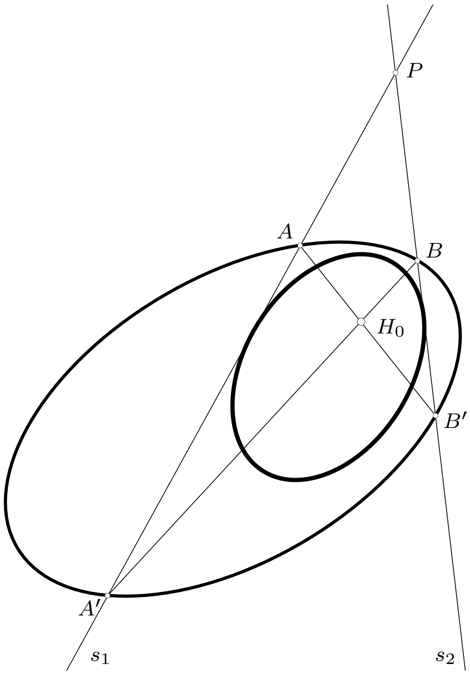
Proof. By THEOREM 4.1 we know that the $2n$ lines $a_1 - a_{n+1}, \ldots,$ $t_1 - t_{n+1}, \ldots$ meet in a point H_0. First we show that the polar h of the pole H_0 with respect to C is the same as the polar h' of H_0 with respect to K, and then we show that the point H_0 is independent of the choice of the $2n$-Poncelet-polygon.



First notice that in the figure above, H_0 is on the polar p of P with respect to the conic C and that H_0 is also on the polar p' of P with respect to the conic K (see for example Coxeter and Greitzer [3, Theorem 6.51]). Thus, P lies on the polar h of H_0 with respect to C, as well as on the polar h' of H_0 with respect ot K. Since the same applies to the point Q, the polars h and h' coincide, which shows that the pole H_0 has the same polar with respect to both conics.

The fact that H_0 is independent of the choice of the $2n$-Poncelet-polygon is just a consequence of the following

Claim. *Let H_0 be as above and let h be the polar of H_0 (with respect to K or C). Choose an arbitrary point P on h. Let $s_1 \,\&\, s_2$ be the two tangents from P to C and let $A \,\&\, A'$ and $B \,\&\, B'$ be the intersection points of s_1 and s_2 with K:*



Then $H_0 = (A - B') \wedge (B - A')$.

Proof of Claim. By a projective transformation, we may assume that h is the line at infinity. Then, the pole H_0 becomes the common center of both conics and the claim follows by symmetry. *q.e.d.*

Now, let $a_1, \ldots, a_{2n}$ and $b_1, \ldots, b_{2n}$ be the vertices of two $2n$-Poncelet-polygons with respect to K & C. Furthermore, let $H_0 = (a_1 - a_{n+1}) \wedge (a_2 - a_{n+2})$ and $H_0' = (b_1 - b_{n+1}) \wedge (b_2 - b_{n+2})$, and let h and h' be their respective polars. Choose any point P which lies on both h and h', and draw the two tangents from P to C which intersect K in the points A, A', B, B'. If the conics K and C do not meet (what we assume), then these points are pairwise distinct and by the CLAIM we get $H_0 = (A - B') \wedge (B - A') = H_0'$, which shows that $H_0 = H_0'$.
q.e.d.

References

[1] CHARLES JULIEN BRIANCHON, *Sur les surfaces courbes du second degré,* ***Journal de l'École Polytechnique,*** Cah. XIII, (1806).

[2] LAZARE CARNOT, ***Géométire de Position***, Crapelet, Paris, 1803.

[3] Harold Scott MacDonald Coxeter and Samuel L. Greitzer, ***Geometry Revisited***, The Mathematical Association of America, Washington D.C., 1967.

[4] Vladimir Dragović and Milena Radnović, ***Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics***, Frontiers in Mathematics, Springer, 2011.

[5] Leopold Flatto, ***Poncelet's Theorem***, American Mathematical Society, Providence, RI, 2009.

[6] Phillip Griffiths and Joseph Harris, *A Poncelet theorem in space*, ***Commentarii Mathematici Helvetici***, vol. 52 (1977), 145–160.

[7] ———, *On Cayley's explicit solution to Poncelet's porism*, ***L'Enseignement Mathématique. Revue Internationale. IIe Série***, vol. 24 (1978), 31–40.

[8] Blaise Pascal, *Essay pour les Coniques*, (1640).

[9] Jean-Victor Poncelet, ***Traité des propriétés projectives des figures***, Bachelier, Paris, 1822.