Bayesian inference for general Gaussian graphical models with application to multivariate lattice data

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Joint work with Adrian Dobra and Abel Rodriguez

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Let $Y_i$ be the count of lung cancer deaths in each state and $W$ the state adjacency matrix

\[
Y_i \sim \mathcal{P}(\eta_i)
\]

\[
\log(\eta_i) = \mu + \log(\text{Pop}_i) + X_i
\]

\[
X \sim N(0, K^{-1})
\]

\[
K \in P_{GW}
\]

where $P_{GW}$ is the cone of positive definite matrices according to the spatial graph $G_W$. 
To ensure $\mathbf{K} \in P_{G_W}$ we may consider the following Conditional Autoregressive (CAR) Model

$$
\mathbf{K} = \tau^{-2}(E_W - \rho W)
$$

posterior estimates focus on two hyperparameters.

“CAR models should most naturally appear as priors for the parameters in a model, not as a model for the observations themselves” (Banerjee, Carlin and Gelfand, 2004).
The G-Wishart Distribution

For $K \in P_G$ we consider the $G$-Wishart prior $\text{Wis}_G(\delta, D)$

$$pr \left( K \mid G, \delta, D \right) = \frac{1}{l_G(\delta, D)} (\det K)^{(\delta-2)/2} \exp \left\{ -\frac{1}{2} \langle K, D \rangle \right\} 1_{K \in P_G}.$$  

Note if $Q'Q = D^{-1}$, then the decomposition,

$$K = Q'\Psi'\Psi Q,$$

requires for $(r, s) \notin G$ that

$$\Psi_{rs} = -\sum_{j=r}^{s-1} \Psi_{rj} \frac{Q_{js}}{Q_{ss}} - \sum_{i=1}^{r-1} \left\{ \sum_{j=i}^{r} \frac{\Psi_{ij} Q_{jr}}{\Psi_{rr} Q_{rr}} \right\} \left[ \sum_{j=i}^{s} \frac{\Psi_{ij} Q_{js}}{Q_{ss}} \right].$$  

We may therefore focus on the incomplete matrix $\Psi^{(G)}$. 

Let \((K^s, G^s)\) be a current state of our model. If the candidate \(G'\) has the additional edge \((i_0, j_0)\) we first extract

\[
\left( \psi^s \right)^\nu(G^s)
\]

and create \(\Psi'\) such that

\[
\psi'_{ij} = \psi^{s}_{ij} \text{ for } (i, j) \in \nu(G^s)
\]

sample

\[
\gamma \sim N \left( \psi^{s}_{i_0 j_0}, \sigma^2_g \right)
\]

set \(\psi'_{i_0 j_0} = \gamma\) and complete relative to \(G'\).
Acceptance Probabilities

We then take

\[ K' = Q^T \left( (\psi')^T \psi' \right) Q \]

and are guaranteed \( K' \in P_{G'} \). The chain moves to \((K', G')\) with probability \( \min \{ R_g^+, 1 \} \), where

\[
R_g^+ = \frac{\text{pr}(D|K')}{\text{pr}(D|K[s])} \frac{\text{pr}(K'|G', \delta_0, D_0)}{\text{pr}(K[s]|G[s], \delta_0, D_0)} \times
\]

\[
J \left( K' \to (\psi')^\nu(G') \right) \frac{J \left( \left( (\psi[s])^\nu(G[s]), \gamma \right) \to (\psi')^\nu(G') \right)}{J \left( K[s] \to (\psi[s])^\nu(G[s]) \right)} \frac{1}{\sigma_g \sqrt{2\pi}} \exp \left( -\frac{\left( \psi'_{i0j0} - \psi_{i0j0}^{[s]} \right)^2}{2\sigma_g^2} \right).
\]
Regional Patterns of Cancer Mortality

(a) Colon
(b) Lung
(c) Breast
(d) Prostate
We extend our framework to the case when the observed data $x$ are associated with a $p_R \times p_C$ random matrix $X = (X_{ij})$ for which

$$\text{vec} \left( X^T \right) | K_R, K_C \sim N_{p_Rp_C} \left( 0, [K_R \otimes K_C]^{-1} \right)$$

Here $K_R$ is a $p_R \times p_R$ row precision matrix and $K_C$ is a $p_C \times p_C$ column precision matrix.
The MCAR and MGGM models

Gelfand and Vounatsou (2003) propose an extension of the CAR model (MCAR) to this data type

\[
K_R = V(\rho) \\
K_C \sim \text{Wis}(\delta, D_C) \\
\rho \sim \text{pr}(\rho)
\]

Where \( V(\rho) = E_W - \rho W \). Using our methods we can generalize this to the framework

\[
K_R \sim \text{Wis}_{GW}(\delta_R, (\delta_R - 2)V^{-1}(\rho)) \\
K_C \sim \text{Wis}_{GC}(\delta_C, D_C) \\
G_C \sim \text{pr}(G_C) \\
\rho \sim \text{pr}(\rho)
\]
Predictive Performance Results

Rank Probability Scores (RPS) from a 10-fold cross validation exercise

<table>
<thead>
<tr>
<th></th>
<th>RPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>MGGM</td>
<td>65.4</td>
</tr>
<tr>
<td>Full Cancer Graph</td>
<td>67.6</td>
</tr>
<tr>
<td>MCAR</td>
<td>76.4</td>
</tr>
</tbody>
</table>
Conclusions

Paper on the arXiv

- Dobra, Lenkoski and Rodriguez. *Bayesian inference for general Gaussian graphical models with application to multivariate lattice data*

For an example of an upcoming application please see the poster:

- Möller, Lenkoski & Thorarinsdottir. *Multivariate Probabilistic Weather Forecasting using Graphical Models*

Later today.