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by

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Finite Elements on Degenerate Meshes: Inverse-type Inequalities and Applications

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Abstract

In this paper we obtain a range of inverse-type inequalities which are applicable to finite element functions on general classes of meshes, including degenerate meshes obtained by anisotropic refinement. These are obtained for Sobolev norms of positive, zero and negative order. In contrast to classical inverse estimates, negative powers of the minimum mesh diameter are avoided. We give two applications of these estimates in the context of boundary elements: (i) to the analysis of quadrature error in discrete Galerkin methods and (ii) to the analysis of the panel clustering algorithm. Our results show that degeneracy in the meshes yields no degradation in the approximation properties of these methods.

1 Introduction

For $d = 2$ or 3, let $\Omega \subset \mathbb{R}^3$ denote either a bounded domain ($d = 3$) or a bounded surface with or without boundary ($d = 2$). Suppose that Ω is decomposed into a mesh of tetrahedra/bricks ($d = 3$) or curvilinear triangles/quadrilaterals ($d = 2$). Then classical inverse estimates give

$$\|u\|_{H^s(\Omega)} \lesssim h_{\min}^{-s} \|u\|_{L^2(\Omega)} \lesssim h_{\min}^{-2s} \|u\|_{H^{-s}(\Omega)} , \quad s \geq 0, \tag{1.1}$$

for all functions $u \in H^s(\Omega)$ which are piecewise polynomials of degree $\leq m$ with respect to this mesh. (Here the notation $A \lesssim B$ means that A/B is bounded by a constant independent of the mesh and independent of u - for a more precise statement, see §2.) The quantity $h_{\min}$ is the minimum diameter of all the elements of the mesh and (1.1) holds under the assumption of *shape regularity*, i.e. $\rho_\tau \gtrsim h_\tau$ for each τ, where ρ_τ is the diameter of the largest inscribed sphere (see Definition 2.1). Such estimates are regularly used in finite element analysis. When the mesh is quasiuniform ($h \lesssim h_{\min}$, where h is the maximum diameter of all the elements), they can be used to obtain convergence rates in powers of h for various quantities in various norms. However, practical meshes are often non-quasiuniform and then the negative powers of $h_{\min}$ in (1.1) may give rise to overly pessimistic convergence rates. In the recent paper [4], less pessimistic replacements for (1.1) have been derived, a particular case being

$$\|u\|_{H^s(\Omega)} \lesssim \|h^{-s}u\|_{L^2(\Omega)} \lesssim \|h^{-2s}u\|_{H^{-s}(\Omega)}, \quad s \geq 0, \tag{1.2}$$

where $h : \Omega \to \mathbb{R}$ is now a continuous piecewise linear *mesh function* whose value on each element τ reflects the diameter of that element (i.e. $h_\tau \lesssim h|_\tau \lesssim h_\tau$, where h_τ is the diameter of τ).

Estimates (1.2) have several applications, e.g. to the analysis of quadrature errors in discrete Galerkin boundary element methods [8] and to the analysis of the mortar element method [4]. In fact [4] contains more general versions of (1.2), e.g. in the Sobolev space $W^{s,p}(\Omega)$ and in related Besov spaces. While the left-hand inequality in (1.2) is well-known, at least in the Sobolev space case, the right-hand inequality requires rather delicate analysis.

In this paper we obtain more general versions of (1.2) which do not require the mesh sequence to be shape-regular. A typical estimate is

$$\|u\|_{H^s(\Omega)} \lesssim \|\rho^{-s}u\|_{L^2(\Omega)} \lesssim \|\rho^{-2s}u\|_{H^{-s}(\Omega)}, \quad s \geq 0, \tag{1.3}$$

where the mesh function $\rho : \Omega \to \mathbb{R}$ is now a continuous piecewise linear function whose value on each element τ reflects the diameter of the largest inscribed sphere, introduced in Definition 2.1. Estimates (1.3) hold under the rather weak assumptions that (i) the quantities h_τ and ρ_τ are locally quasiuniform (i.e. $h_\tau/h_{\tau'} \lesssim 1$ and $\rho_\tau/\rho_{\tau'} \lesssim 1$ for all neighbouring elements τ, τ') and (ii) the number of neighbouring elements of any element remains bounded as the mesh is refined (see Assumption 2.6). These assumptions admit degenerate meshes, containing long thin "stretched" elements, which are typically used for approximating edge singularities or boundary layers in solutions of PDEs. It is expected that these estimates will have a range of applications similar to those already identified above for (1.2). In particular we already used a special case of (1.3) to analyse quadrature errors for a Galerkin boundary element discretisation of a model screen problem in [10]. In this paper we give as applications a more general Galerkin quadrature error analysis, as well as an error analysis of the panel clustering algorithm in the presence of degenerate meshes.

Our inverse estimates are proved in §3. We briefly introduce the well-known Galerkin boundary element method in §4. The analysis of Galerkin quadrature is given in §5. Quadrature almost always has to be employed in practical computations; a general analysis for shape-regular meshes was included in [8]. In §5, with the help of (1.3), we generalise the results of [8] to degenerate meshes. The results turn out to be qualitatively the same as those in [8]: in the far field the degeneracy of the mesh has no effect on the required precision of the quadrature needed to preserve the accuracy of the Galerkin method. The error analysis of the panel clustering algorithm is given in §6. This algorithm [15, 20] provides an alternative representation of the finite-dimensional Galerkin operator which has the same order of accuracy as the standard representation. The multiplication of the panel clustering representation with any vector has complexity $\mathcal{O}(N \log^\kappa N)$, for some (small) κ, where N is the number of degrees of freedom. This should be compared with the complexity $\mathcal{O}(N^2)$ of the standard matrix representation. Up till now the accuracy and complexity analysis for this algorithm was obtained only for quasiuniform meshes. In §6 we extend the accuracy analysis to the case of much more general (including degenerate) meshes using (1.3). Again we find the error estimate is qualitatively the same as in the quasiuniform case.

It turns out, however, that when the conventional panel clustering algorithm is applied in practice to some discretisations on degenerate mesh sequences, it has a complexity higher than the $\mathcal{O}(N \log^\kappa N)$ mentioned above. In the subsequent paper [11] we shall elaborate on this and we shall propose a new variant of the panel clustering algorithm which is optimal for this type of mesh. The results here, depending on (1.3), are crucial for the analysis which will be given in [11].

2 Meshes and Finite Elements

Throughout the paper, Ω will denote a bounded d-dimensional subset of $\mathbb{R}^3$, for $d = 2$ or 3. More precisely, when $d = 3$, Ω will denote a bounded Euclidean domain in $\mathbb{R}^3$ and for $d = 2$, Ω will denote a bounded 2-dimensional piecewise smooth Lipschitz manifold in $\mathbb{R}^3$ which may or may not have a boundary. The case when Ω is a bounded 2-dimensional Euclidean domain is then included as a special case, by trivially embedding it into $\mathbb{R}^3$.

We define the Sobolev space $H^s(\Omega)$, $s \geq 0$, in the usual way (see, e.g., [12]). Note that in the case $d = 2$ the range of s for which $H^s(\Omega)$ is defined may be limited, depending on the global smoothness of the surface Ω. Throughout, we let $[-k, k]$ denote the range of Sobolev indices for which we are going to prove the inverse estimates, and we assume that $H^s(\Omega)$ is defined for all $s \in [-k, k]$, with the negative order spaces defined by duality in the usual way. We assume that Ω is decomposed into a mesh $\mathcal{T}$ of open pairwise-disjoint finite elements $\tau \subset \Omega$ with the property $\overline{\Omega} = \cup\{\overline{\tau} : \tau \in \mathcal{T}\}$.

Definition 2.1 (Mesh Parameters) *For each $\tau \in \mathcal{T}$, $|\tau|$ denotes its d-dimensional measure, h_τ denotes its diameter and ρ_τ is the diameter of the largest sphere centred at a point in τ whose intersection with $\overline{\Omega}$ lies entirely inside $\overline{\tau}$. For any other simplex or cube $t \in \mathbb{R}^d$ (not necessarily an element of $\mathcal{T}$) we define h_t and ρ_t in the same way.*

In order to impose a simple geometric character on the mesh τ, we assume that each $\tau \in \mathcal{T}$ is diffeomorphic to a simple unit element. More precisely, let $\hat{\sigma}^3$ denote the unit simplex with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, let $\hat{\kappa}^3$ denote the unit cube with vertices $\{(x,y,z) : x,y,z \in \{0,1\}\}$, and let $\hat{\sigma}^2$, $\hat{\kappa}^2$ denote the orthogonal projections of these onto the xy plane.

Then we assume that for each $\tau \in \mathcal{T}$, there exists a unit element $\hat{\tau} = \hat{\sigma}^d$ or $\hat{\kappa}^d$ and a bijective map $\chi_\tau : \hat{\tau} \to \tau$, with χ_τ and χ_τ^{-1} both smooth. (Here, for simplicity, "smooth" means $\mathcal{C}^\infty$.) We also let $|\hat{\tau}|$ denote the $d-$dimensional measure of $\hat{\tau}$ and $h_{\hat{\tau}}$ denote its diameter. Since χ_τ is smooth, each element $\tau \in \mathcal{T}$ is either a curvilinear tetrahedron/brick ($d = 3$) or a curvilinear triangle/rectangle ($d = 2$). The mesh $\mathcal{T}$ is allowed to contain both types of elements. Each element has nodes and edges and also (when $d = 3$) has faces. For suitable index sets $\mathcal{N}_0, \mathcal{N}_1$, we let $\{c_p : p \in \mathcal{N}_0\}$ denote the set of all centroids[1] of elements of $\mathcal{T}$ and $\{\mathbf{x}_p : p \in \mathcal{N}_1\}$ denote the set of all nodes of $\mathcal{T}$. We assume the mesh is *conforming*, i.e. that for each $\tau, \tau' \in \mathcal{T}$ with $\tau \neq \tau'$, $\tau \cap \tau'$ is allowed to be either empty, a node, an edge or (when

[1] When τ is curved, we replace the centroid of τ with the image of the centroid of the corresponding unit element under the map χ_τ.

$d = 3$) a face. The requirement χ_τ is smooth ensures that edges of Ω ($d = 2$) and edges of $\partial\Omega$ ($d = 3$) are confined to edges of elements $\tau \in \mathcal{T}$. Let J_τ denote the $3 \times d$ Jacobian of χ_τ. Then

$$g_\tau := \{\det J_\tau^T J_\tau\}^{1/2}$$

is the Gram determinant of the map χ_τ, which appears in the change of variable formula: $\int_\tau f(\mathbf{x})dx = \int_{\hat{\tau}} f(\chi_\tau(\hat{\mathbf{x}}))g_\tau(\hat{\mathbf{x}})d\hat{x}$. To ensure that the map χ_τ is sufficiently regular we shall make the following assumptions on J_τ:

Assumption 2.2 (Mapping Properties)

$$D^{-1}|\tau|^2 \le \det\{J_\tau(\hat{\mathbf{x}})^T J_\tau(\hat{\mathbf{x}})\} \le D|\tau|^2, \tag{2.1a}$$

$$E\rho_\tau^2 \le \lambda_{\min}\{J_\tau(\hat{\mathbf{x}})^T J_\tau(\hat{\mathbf{x}})\} \tag{2.1b}$$

uniformly in $\hat{\mathbf{x}} \in \hat{\tau}$*, with positive constants* D, E *independent of* τ*.*

(Throughout this section, for a symmetric matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote respectively the minimum and maximum eigenvalues of A.) Assumption 2.2 is satisfied in a number of standard cases.

Example 2.3 *Suppose either* $d = 2$ *and* Ω *is a planar polygon (assumed without loss of generality to lie in the plane* $x_3 = 0$*) or* $d = 3$*. Suppose also that* χ_τ *is an affine map. Then the Jacobian* J_τ *can be identified with a* $d \times d$ *constant matrix and it is well-known (e.g., [3]) that* $\det J_\tau = |\tau|/|\hat{\tau}|$ *and that* $\|J_\tau^{-1}\|_2 \le h_{\hat{\tau}}\rho_\tau^{-1}$*, from which the estimates (2.1a,b) follow.*

Proceeding to the case when Ω is a surface we have:

Example 2.4 *Suppose* $d = 2$ *and let* Ω *be the surface of a polyhedron. Let* τ *be a triangle with nodes* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ *and choose* χ_τ *to be the affine map:* $\chi_\tau(\hat{\mathbf{x}}) = \mathbf{x}_1 + \hat{x}_1\mathbf{a} + \hat{x}_2\mathbf{b}$*, where* $\mathbf{a} = \mathbf{x}_2 - \mathbf{x}_1$*,* $\mathbf{b} = \mathbf{x}_3 - \mathbf{x}_1$*. Then*

$$J_\tau^T J_\tau = \begin{bmatrix} |\mathbf{a}|^2 & \mathbf{a}^T\mathbf{b} \\ \mathbf{a}^T\mathbf{b} & |\mathbf{b}|^2 \end{bmatrix}, \ \det J_\tau^T J_\tau = |\mathbf{a} \times \mathbf{b}|^2 = 4|\tau|^2, \tag{2.2}$$

from which (2.1a) follows. If we denote the eigenvalues of $J_\tau^T J_\tau$ *by* $0 < \lambda_- < \lambda_+$*, then we can easily obtain the relations* $\lambda_+ \le \lambda_- + \lambda_+ = |\mathbf{a}|^2 + |\mathbf{b}|^2 \le 2h_\tau^2$ *and* $\lambda_-\lambda_+ = 4|\tau|^2$ *which imply (2.1b).*

Finite elements on curved surfaces can similarly be shown to satisfy Assumption 2.2, for example when the map χ_τ *is sufficiently close to affine.*

In the case of a planar quadrilateral element ($\hat{\tau} = \hat{\kappa}^2 = (0,1)^2$), the assumption that χ_τ is affine forces τ to be a parallelogram. In that case, the results in Example 2.4 for triangles carries over verbatim.

More general four-sided quadrilaterals can be obtained by a bilinear mapping χ_τ. In the following example we will consider a planar convex quadrilateral τ with straight edges and choose χ_τ to be the usual bilinear map from $\hat{\tau}$ to τ. We assume that $\mathbf{x}_0, \mathbf{x}_0{+}\mathbf{a}, \mathbf{x}_0{+}\mathbf{b}, \mathbf{x}_0{+}\mathbf{a}{+}\mathbf{b}{+}\mathbf{c}$ are the vertices of τ. Without loss of generality (after a suitable permutation of the vertices), we may assume that $|\mathbf{a}| \le |\mathbf{b}|$ and that the parallelogram $\pi = (\mathbf{x}_0, \mathbf{x}_0 + \mathbf{a}, \mathbf{x}_0 + \mathbf{b}, \mathbf{x}_0 + \mathbf{a} + \mathbf{b})$ is contained in τ (see Figure 1).

Example 2.5 *For a convex planar quadrilateral τ, the constants D, E, $\lambda_{\min}$ in the estimates (2.1a,b) depend only on the ratios $|\tau| / |\pi|$ and $|\pi| / |\tau|$. If these ratios remain bounded for all elements τ as the mesh is refined then (2.1a,b) hold. Note that these ratios can be moderately bounded even if the quadrilaterals are degenerate, i.e., if either $|\mathbf{a}| \ll |\mathbf{b}|$ or $\sin^2(\mathbf{a},\mathbf{b}) \ll 1$. In both cases, $|\tau| \ll |\mathbf{b}|^2$. In particular, a flat rhombus is degenerate but still satisfies $|\tau| / |\pi| = |\pi| / |\tau| = 1$.*

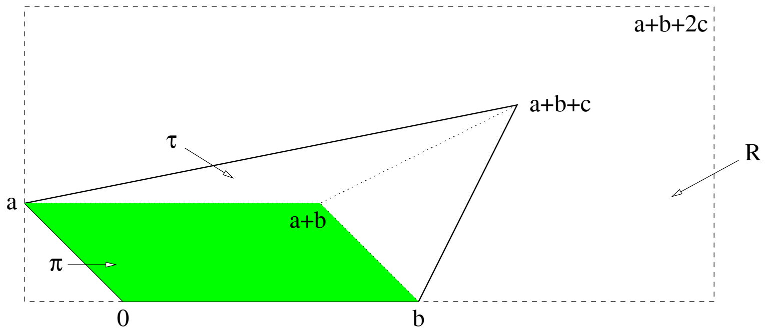

Figure 1: Quadrilateral τ (with $\mathbf{x}_0 := 0$), parallelogram π and bounding rectangle R

Proof. We give the proof of the above assertion in three steps.

a) $|\tau| \sim |\pi|$ implies $|\mathbf{c}| \lesssim |\mathbf{b}|$ and $|P^{\perp}\mathbf{c}| \lesssim |P^{\perp}\mathbf{a}|$, where $P^{\perp}\mathbf{x} = \mathbf{x} - \mathbf{b}\langle\mathbf{b},\mathbf{x}\rangle / \langle\mathbf{b},\mathbf{b}\rangle$ is the projection orthogonal to $\mathbf{b}$.

b) Let R be the smallest rectangle containing π, τ, and the shifted vertices $\{\mathbf{x}_0 + \mathbf{a} + \mathbf{c}, \mathbf{x}_0 + \mathbf{b} + \mathbf{c}, \mathbf{x}_0 + \mathbf{a} + \mathbf{b} + 2\mathbf{c}\}$. The length of R is bounded by $|\mathbf{b}| + |\mathbf{a}| + 2|\mathbf{c}| \sim |\mathbf{b}|$, while the height is $\leq |P^{\perp}\mathbf{a}| + 2|P^{\perp}\mathbf{c}| \sim |P^{\perp}\mathbf{a}|$ due to part a). Hence, $|\tau| \sim |\pi| = |\mathbf{b}| |P^{\perp}\mathbf{a}| \sim |R|$ holds.

c) The bilinear map is $\chi_\tau(\hat{\mathbf{x}}) = \mathbf{x}_0 + \hat{x}_1\mathbf{a} + \hat{x}_2\mathbf{b} + \hat{x}_1\hat{x}_2\mathbf{c}$. With $J_\tau(\hat{\mathbf{x}})$ denoting the Jacobian, it is easily seen that $J_\tau^T J_\tau$ has the form (2.2) with $\mathbf{a},\mathbf{b}$ replaced by $\mathbf{a}'(\hat{\mathbf{x}}) := \mathbf{a} + \hat{x}_2\mathbf{c}$, $\mathbf{b}' := \mathbf{b} + \hat{x}_1\mathbf{c}$. Hence, $\det J_\tau^T J_\tau(\hat{\mathbf{x}}) = 4|\pi(\hat{\mathbf{x}})|^2$, where $\pi(\hat{\mathbf{x}})$ is the parallelogram with vertices $\mathbf{x}_0, \mathbf{x}_0 + \mathbf{a}'(\hat{\mathbf{x}}), \mathbf{x}_0 + \mathbf{b}'(\hat{\mathbf{x}}), \mathbf{x}_0 + \mathbf{a}'(\hat{\mathbf{x}}) + \mathbf{b}'(\hat{\mathbf{x}})$. Since $\pi(\hat{\mathbf{x}}) \subset R$ for all $0 \leq \hat{x}_1, \hat{x}_2 \leq 1$, the right inequality in (2.1a) follows from $\det J_\tau^T J_\tau(\hat{\mathbf{x}}) = 4|\pi(\hat{\mathbf{x}})|^2 \leq 4R^2 \lesssim 4|\tau|^2$ as in the case of Example 2.4. The left inequality in (2.1a) uses the fact that $|\pi(\hat{\mathbf{x}})| \geq |\pi|$ (for a proof use the fact that $\mathbf{a},\mathbf{b},\mathbf{c}$ satisfy $\det(\mathbf{c},\mathbf{a}) \geq 0$ and $\det(\mathbf{b},\mathbf{c}) \geq 0$. Hence, $|\pi(\hat{\mathbf{x}})| = \det(\mathbf{b}'(\hat{\mathbf{x}}),\mathbf{a}'(\hat{\mathbf{x}})) = \det(\mathbf{b},\mathbf{a}) + \hat{x}_1\det(\mathbf{b},\mathbf{c}) + \hat{x}_2\det(\mathbf{c},\mathbf{a}) + \hat{x}_1\hat{x}_2\det(\mathbf{c},\mathbf{c}) \geq \det(\mathbf{b},\mathbf{a}) = |\pi|$). Also the proof of (2.1b) is analogous to the proof of (2.1b) in Example 2.4. ■

Assumption 2.2 describes the quality of the maps which take the unit element $\hat{\tau}$ to each τ. We also need assumptions on how the size and shape of neighbouring elements in our mesh may vary. Here we impose *only very weak local conditions which require the meshes to be neither quasi-uniform nor shape-regular*. Throughout the rest of this paper we make the following assumption.

Assumption 2.6 (Mesh Properties) *For some $K, L \in \mathbb{R}^+$ and $M \in \mathbb{N}$, we assume that, for all $\tau, \tau' \in \mathcal{T}$ with $\overline{\tau} \cap \overline{\tau'} \neq \emptyset$,*

$$h_\tau \leq K h_{\tau'}, \qquad \rho_\tau \leq L \rho_{\tau'}, \tag{2.3a}$$

$$\max_{i \in \mathcal{N}_1} \#\{\tau \in \mathcal{T} : \mathbf{x}_i \in \overline{\tau}\} \leq M. \tag{2.3b}$$

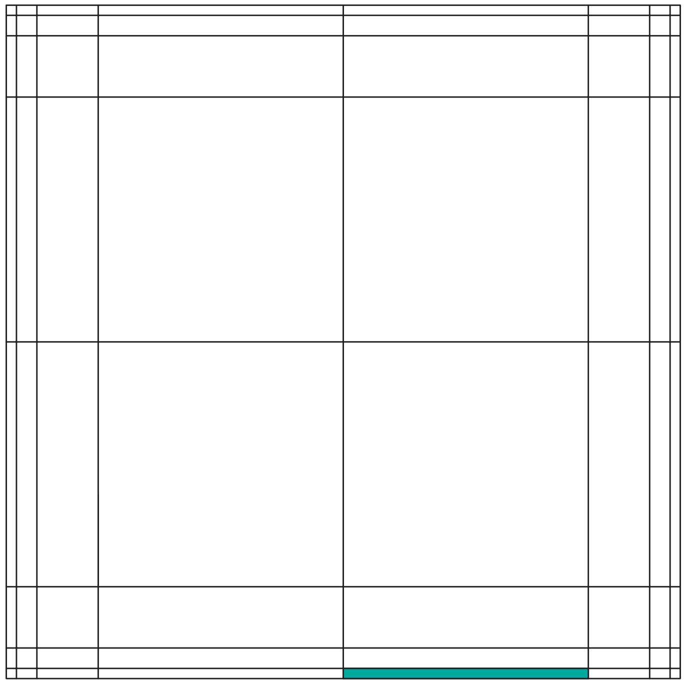

Figure 2: Illustration of a graded mesh. Some elements become very long and thin as, e.g., the shaded one in the figure. We have chosen here a smaller grading exponent g<5 for illustration purpose only.

We denote the class of meshes which satisfy this assumption as $\mathcal{M}_{K,L,M}$. Note that condition (2.3a) require that h_τ and ρ_τ do not vary too rapidly across neighbouring elements. This allows elements with large aspect ratio, provided their immediate elements have comparable aspect ratio. From now on, if $A(\mathcal{T})$ and $B(\mathcal{T})$ are two mesh-dependent quantities, then the inequality $A(\mathcal{T}) \lesssim B(\mathcal{T})$ will mean that there is a constant C independent of $\mathcal{T} \in \mathcal{M}_{K,L,M}$, such that $A(\mathcal{T}) \leq CB(\mathcal{T})$. (The class of meshes $\mathcal{M}_{K,L,M}$ depends on K, L, M, and we do not claim that C is independent of K, L, M.) Also the notation $A(\mathcal{T}) \sim B(\mathcal{T})$ will mean that $A(\mathcal{T}) \lesssim B(\mathcal{T})$ and $B(\mathcal{T}) \lesssim A(\mathcal{T})$.

Example 2.7 *Shape-regular meshes are easily shown to lie in the class $\mathcal{M}_{K,L,M}$, with moderate K, L, M. Also, meshes which are anisotropically graded towards an edge typically lie in this class. A classical example of these arises in the approximation of boundary integral formulations of screen problems for elliptic PDEs, where the screen is a polygon. Near an edge, but away from the corners, the solution typically is badly behaved only in the direction orthogonal to the edge and efficient approximations require meshes which are anisotropically graded.*

For example, for the square screen $[0,1] \times [0,1]$, a typical tensor product anisotropic mesh would be: $\mathbf{x}_{i,j} = (t_i, t_j)$, where $t_i = (i/n)^g/2$ and $t_{2n-i} = 1 - (i/n)^g/2$ for $i = 0, \ldots, n$, for some grading exponent $g \geq 1$. (For example, see [18], [19], [6], [10].) An illustration of such a graded mesh is given in Figure 2. In this case the elements become very long and thin near smooth parts of edges. In the hp version of the finite element method similar meshes but with more extreme grading may be used (e.g. [22]) and these also satisfy Assumption 2.6.

Now we introduce finite element spaces on the mesh $\mathcal{T}$.

Definition 2.8 (Finite Element Spaces) *For $m \geq 0$ and $\hat{\tau} \in \{\hat{\sigma}^d, \hat{\kappa}^d\}$, we define*

$$\mathbb{P}^m(\hat{\tau}) = \begin{cases} \text{polynomials of total degree } \leq m \text{ on } \hat{\tau} & \text{if } \hat{\tau} = \hat{\sigma}^d, \\ \text{polynomials of coordinate degree } \leq m \text{ on } \hat{\tau} & \text{if } \hat{\tau} = \hat{\kappa}^d. \end{cases}$$

Then, for $i \in \{0, 1\}$ and $m \geq i$, we set

$$\mathcal{S}_0^m(\mathcal{T}) = \{u \in L^\infty(\Omega) : u \circ \chi_\tau \in \mathbb{P}^m(\hat{\tau}), \tau \in \mathcal{T}\},$$
$$\mathcal{S}_1^m(\mathcal{T}) = \{u \in C^0(\Omega) : u \circ \chi_\tau \in \mathbb{P}^m(\hat{\tau}), \tau \in \mathcal{T}\}.$$

We finish this section with a generalisation of a standard scaling argument which is used several times in later proofs.

Proposition 2.9 *Let $\tau \in \mathcal{T}$ and let $\hat{t}$ be any simplex which is contained in the associated unit element $\hat{\tau} \in \mathbb{R}^d$. Let $\hat{P}$ denote any d-variate polynomial on $\hat{t}$ and define $t = \chi_\tau(\hat{t})$, $P = \hat{P} \circ \chi_\tau^{-1}$. Then for all $0 \leq s \leq k$,*

$$\|P\|_{H^s(t)} \lesssim \rho_{\hat{t}}^{-s} \rho_\tau^{-s} \|P\|_{L^2(t)}. \tag{2.4}$$

The constant of proportionality in (2.4) depends on $\hat{P}$ only through its degree.

Proof. The proof is a refinement of standard scaling arguments (e.g., [3]).

Consider first the case $d = 3$. Then $\Omega \subset \mathbb{R}^3$ is a bounded Euclidean domain and by the chain rule we have $\widehat{\nabla} \widehat{P}(\hat{\mathbf{x}}) = J_\tau(\hat{\mathbf{x}})^T (\nabla P)(\chi_\tau(\hat{\mathbf{x}}))$, where $\widehat{\nabla}$ denotes the gradient with respect to $\hat{\mathbf{x}} \in \hat{\tau}$ and ∇ denotes gradient with respect to $\mathbf{x} \in \tau$. By (2.1a), $J_\tau^T J_\tau$ is invertible and

$$|P|_{H^1(t)}^2 = \int_{\hat{t}} (\widehat{\nabla} \widehat{P})^T (J_\tau^T J_\tau)^{-1} (\widehat{\nabla} \widehat{P}) g_\tau,$$

where $|\cdot|_{H^1(t)}$ denotes the usual seminorm. Using (2.1b) we get

$$|P|_{H^1(t)}^2 \lesssim \rho_\tau^{-2} \int_{\hat{t}} |\widehat{\nabla} \widehat{P}|^2 g_\tau.$$

We can also introduce an affine map $\nu : \hat{\tau} \to \hat{t}$, introduce a new function $\widehat{P} \circ \nu$ and repeat the previous argument, using also Example 2.3 to obtain:

$$|P|_{H^1(t)}^2 \lesssim \rho_\tau^{-2} \rho_{\hat{t}}^{-2} \int_{\hat{\tau}} |\widetilde{\nabla}(\widehat{P} \circ \nu)|^2 g g_\tau,$$

where g is the Gram determinant for ν and $\widetilde{\nabla}$ denotes the gradient with respect to $\tilde{\mathbf{x}} := \nu^{-1}(\hat{\mathbf{x}})$. Then, by equivalence of norms on finite-dimensional spaces,

$$|P|_{H^1(t)}^2 \lesssim \rho_\tau^{-2} \rho_{\hat{t}}^{-2} \int_{\hat{\tau}} |\widehat{P} \circ \nu|^2 g g_\tau = \rho_\tau^{-2} \rho_{\hat{t}}^{-2} \|P\|_{L^2(t)}^2.$$

This proves the result for $s = 1$. The result for higher integer s is obtained similarly and non-integer s is obtained by interpolation.

Turn now to the case $d = 2$. When Ω is a bounded 2-dimensional Euclidean domain, the proof is entirely analogous to that given above. When Ω is a general surface in $\mathbb{R}^3$, the element τ (since it forms a smooth part of a surface) can be written $\tau = \eta(\tilde{\tau})$ where $\tilde{\tau} \subset \mathbb{R}^2$ is a planar, curvilinear triangle lying in one of the charts which determine Ω and η is a smooth bijective map with smooth inverse. We consider η as the transformation of the surface metric to a planar metric which is independent of the size of τ. We may write the mapping χ_τ as the composition $\chi_\tau = \eta \circ \chi_{\tilde{\tau}}$, where $\chi_{\tilde{\tau}}$ is now a scaling from the unit element $\hat{\tau}$ to $\tilde{\tau}$. Introduce the set $\tilde{t} := \eta^{-1}(t) \subset \tilde{\tau}$ and the function $\tilde{P} := P \circ \eta$ on $\tilde{\tau}$. The above result on two-dimensional Euclidean domains shows

$$\|\tilde{P}\|_{H^s(\tilde{t})} \lesssim \rho_{\tilde{t}}^{-s} \rho_{\tilde{\tau}}^{-s} \|\tilde{P}\|_{L^2(\tilde{t})}.$$

Since the constants in $\rho_{\tilde{\tau}} \sim \rho_\tau$ only depend on the mapping η, and since we also have

$$\|\tilde{P}\|_{H^s(\tilde{t})} \sim \|P\|_{H^s(t)}, \qquad \|\tilde{P}\|_{L^2(\tilde{t})} \sim \|P\|_{L^2(t)} ,$$

the result follows. ■

3 Inverse Estimates

In this section we prove our inverse estimates, which were motivated in the Introduction (see (1.3)). To define the scaling function ρ, recall the parameters ρ_τ introduced in Definition 2.1. From these we construct a continuous mesh function $\rho \in \mathcal{S}_1^1$ on Ω as follows.

Definition 3.1 (Mesh Function) *For each $p \in \mathcal{N}_1$, set $\rho_p = \max\{\rho_\tau : \mathbf{x}_p \in \overline{\tau}\}$. The mesh function ρ is the unique function in $\mathcal{S}_1^1(\mathcal{T})$ such that $\rho(\mathbf{x}_p) = \rho_p$, for each $p \in \mathcal{N}_1$.*

Clearly ρ is a positive, continuous function on Ω and, by Assumption 2.6, it follows that $\rho(\mathbf{x}) \sim \rho_\tau$ for $\mathbf{x} \in \tau$, and all $\tau \in \mathcal{T}$. The main results of this section are Theorems 3.2, 3.4 and 3.6. The first two of these provide inverse estimates in positive Sobolev norms for functions $u \in \mathcal{S}_i^m(\mathcal{T})$ with continuity index $i = 1, 0$ respectively. The third theorem provides inverse estimates in negative norms.

Theorem 3.2 *Let $0 \le s \le 1$ and $-\infty < \underline{\alpha} < \overline{\alpha} < \infty$. Then*

$$\|\rho^\alpha u\|_{H^s(\Omega)} \lesssim \|\rho^{\alpha-s} u\|_{L^2(\Omega)} ,$$

uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $u \in \mathcal{S}_1^m(\mathcal{T})$.

Remark 3.3 *Since $\mathcal{S}_1^m(\mathcal{T}) \subset H^s(\Omega)$ for all $s < 3/2$, it may be expected that the range of Sobolev indices for which Theorem 3.2 holds may be extended. Such an extension has been obtained in [4] for shape-regular meshes at the expense of working in Besov norms. We have avoided such extensions here in order to simplify the present paper.*

Proof. It is a generalisation of [4, Theorem 4.1]. First write

$$\nabla(\rho^\alpha u) = \alpha \rho^{\alpha-1} u \nabla\rho + \rho^\alpha \nabla u .$$

Using this, Assumption 2.6 and Proposition 2.9, we have

$$\begin{aligned}\|\nabla(\rho^\alpha u)\|_{L^2(\tau)}^2 &\lesssim \|\rho^{\alpha-1} u\|_{L^\infty(\tau)}^2 \|\nabla\rho\|_{L^2(\tau)}^2 + \|\rho^\alpha \nabla u\|_{L^2(\tau)}^2 \\ &\lesssim \rho_\tau^{2\alpha-4} \|u\|_{L^\infty(\tau)}^2 \|\rho\|_{L^2(\tau)}^2 + \rho_\tau^{2\alpha-2} \|u\|_{L^2(\tau)}^2 \\ &\lesssim \rho_\tau^{2\alpha-2} \|u\|_{L^\infty(\tau)}^2 |\tau| + \rho_\tau^{2\alpha-2} \|u\|_{L^2(\tau)}^2 .\end{aligned}$$

Now a simple scaling argument shows that

$$\|u\|^2_{L^\infty(\tau)}|\tau| \sim \|u\|^2_{L^2(\tau)} \qquad \text{uniformly in } u \in S^m_i(\mathcal{T}), \quad i = 0, 1. \tag{3.1}$$

Hence

$$\|\nabla(\rho^\alpha u)\|^2_{L^2(\tau)} \lesssim \rho_\tau^{2\alpha-2}\|u\|^2_{L^2(\tau)} \sim \|\rho^{\alpha-1}u\|^2_{L^2(\tau)}$$

and the proof for $s = 1$ follows by summation over $\tau \in \mathcal{T}$. The proof for $s \in [0, 1]$ follows by interpolation. ∎

Theorem 3.4 *Let $0 \le s < 1/2$ and $-\infty < \underline{\alpha} < \overline{\alpha} < \infty$. Then*

$$\|\rho^\alpha u\|_{H^s(\Omega)} \lesssim \|\rho^{\alpha-s}u\|_{L^2(\Omega)},$$

uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $u \in \mathcal{S}^m_0(\mathcal{T})$.

Proof. We give the proof for $d = 2$. It is a generalisation of [4, Theorem 4.2]. The proof for $d = 3$ follows similar lines. By a result of B. Faermann [7, Lemma 3.1], the fractional order Sobolev norm $\|\cdot\|_{H^s(\Omega)}$ admits an estimate in terms of local norms which yields

$$\|\rho^\alpha u\|^2_{H^s(\Omega)} \lesssim \sum_{\tau\in\mathcal{T}} \left\{ \rho_\tau^{2(\alpha-s)}\|u\|^2_{L^2(\tau)} + \sum_{\substack{\tau'\in\mathcal{T}\\ \overline{\tau'}\cap\overline{\tau}\neq\emptyset}} \int_\tau\int_{\tau'} \frac{|(\rho^\alpha u)(\mathbf{x}) - (\rho^\alpha u)(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2s}}\,dx\,dy \right\}. \tag{3.3}$$

Because of the local quasiuniformity, Assumption 2.6, the proof is finished, provided we can show

$$\sum_{\tau\in\mathcal{T}} \sum_{\substack{\tau'\in\mathcal{T}\\ \overline{\tau'}\cap\overline{\tau}\neq\emptyset}} \int_\tau\int_{\tau'} \frac{|(\rho^\alpha u)(\mathbf{x}) - (\rho^\alpha u)(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2s}}\,dx\,dy \quad\lesssim\quad \sum_{\tau\in\mathcal{T}} \rho_\tau^{2(\alpha-s)}\|u\|^2_{L^2(\tau)}. \tag{3.4}$$

To prove this, we decompose the left-hand side of (3.4) as

$$\sum_{\tau\in\mathcal{T}} \sum_{\substack{\tau'\in\mathcal{T}\setminus\{\tau\}\\ \overline{\tau'}\cap\overline{\tau}\neq\emptyset}} \int_\tau\int_{\tau'} \frac{|(\rho^\alpha u)(\mathbf{x}) - (\rho^\alpha u)(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2s}}\,dx\,dy + \sum_{\tau\in\mathcal{T}} \int_\tau\int_\tau \frac{|(\rho^\alpha u)(\mathbf{x}) - (\rho^\alpha u)(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2s}}\,dx\,dy. \tag{3.5}$$

By definition of the Aronszajn-Slobodeckij norm on $H^s(\tau)$ (see, e.g., [7]) and by using Proposition 2.9, the second term in (3.5) may be bounded by

$$\sum_{\tau\in\mathcal{T}} \|\rho^\alpha u\|^2_{H^s(\tau)} \quad\lesssim\quad \sum_{\tau\in\mathcal{T}} \rho_\tau^{2(\alpha-s)}\|u\|^2_{L^2(\tau)}. \tag{3.6}$$

Finally, following the proof of [4, Theorem 4.2], the first term in (3.5) may be bounded by

$$\sum_{\tau\in\mathcal{T}} \sum_{\substack{\tau'\in\mathcal{T}\setminus\{\tau\}\\ \overline{\tau'}\cap\overline{\tau}\neq\emptyset}} \left\{\|\rho^\alpha u\|^2_{L^\infty(\tau)} + \|\rho^\alpha u\|^2_{L^\infty(\tau')}\right\} J_{\tau,\tau'}, \quad \text{where } J_{\tau,\tau'} = \int_\tau\int_{\tau'} |\mathbf{x}-\mathbf{y}|^{-2-2s}\,dx\,dy. \tag{3.7}$$

Moreover, for all $\overline{\tau} \neq \overline{\tau'}$ with $\overline{\tau} \cap \overline{\tau'} \neq \emptyset$ the proof of Theorem 4.2 in [4] shows that $J_{\tau,\tau'} \lesssim \min\left\{\rho_\tau^{-2s}|\tau|, \rho_{\tau'}^{-2s}|\tau'|\right\}$. Inserting this into (3.5) and using again Assumption 2.6 shows that the first term of (3.5) may be bounded by a constant times $\sum_{\tau\in\mathcal{T}} \|u\|^2_{L^\infty(\tau)} |\tau| \rho_\tau^{2\alpha-2s}$. Using (3.1), this can be bounded analogously to (3.6), completing the proof. ∎

The final theorem in this section (Theorem 3.6) provides estimates in negative Sobolev norms for finite element functions. Before we prove this, we require the following technical lemma.

Lemma 3.5 *Let $\hat{\tau}$ and $\mathbb{P}^m(\hat{\tau})$ be as in Definition 2.8. Then for each integer $m \geq 0$, there exists $\delta = \delta(m) \in (0,1)$ with the following property:*

For each $u \in \mathbb{P}^m(\hat{\tau})$, there exists a simplex $\hat{t} \subset \hat{\tau}$ (which may depend on u and m), such that

$$\rho_{\hat{t}} \geq \delta \quad \textit{and} \quad \inf_{\mathbf{x}\in\hat{t}} |u(\mathbf{x})| \geq \delta \|u\|_{L^\infty(\hat{\tau})}, \tag{3.8}$$

where $\rho_{\hat{t}}$ is as defined in Definition 2.1.

Proof. By equivalence of norms on finite-dimensional spaces, there exists $\gamma = \gamma(m) > 0$ such that, for all $u \in \mathbb{P}^m(\hat{\tau})$,

$$\|\nabla u\|_{L^\infty(\hat{\tau})} \leq \gamma \|u\|_{L^\infty(\hat{\tau})}. \tag{3.9}$$

Now, by choosing $\delta_0 = \delta_0(m)$ such that $0 < \delta_0 \leq (1+\gamma)^{-1} < 1$, it follows that

$$0 < \delta_0 \leq (1 + \|\nabla u\|_{L^\infty(\hat{\tau})}/\|u\|_{L^\infty(\hat{\tau})})^{-1} = \frac{\|u\|_{L^\infty(\hat{\tau})}}{\|\nabla u\|_{L^\infty(\hat{\tau})} + \|u\|_{L^\infty(\hat{\tau})}} \tag{3.10}$$

for all $u \in \mathbb{P}^m(\hat{\tau})$.

For any $\mathbf{x} \in \mathbb{R}^3$ and $\rho > 0$, let $B_\rho(\mathbf{x})$ denote the open ball centred at $\mathbf{x}$ with radius ρ. We shall establish the statement: For all $u \in \mathbb{P}^m(\hat{\tau})$, there exists $\rho \geq \delta_0$ and $\mathbf{x}^* \in \overline{\hat{\tau}}$ (both of which may depend on u and m), such that

$$\inf_{\mathbf{x}\in B_\rho(\mathbf{x}^*)\cap\overline{\hat{\tau}}} |u(\mathbf{x})| \geq \delta_0 \|u\|_{L^\infty(\hat{\tau})}. \tag{3.11}$$

Then, with a suitable choice of $\alpha \in (0,1)$, (depending only on the unit element $\hat{\tau}$), there is always a simplex $\hat{t} \subset B_\rho(\mathbf{x}^*) \cap \overline{\hat{\tau}}$ with $\rho_{\hat{t}} \geq \alpha\delta_0$. The required result follows with $\delta = \alpha\delta_0$.

To establish (3.11), consider any $u \in \mathbb{P}^m(\hat{\tau})$. Suppose that $\|\nabla u\|_{L^\infty(\hat{\tau})} \neq 0$. Then $\|u\|_{L^\infty(\hat{\tau})} \neq 0$ and we can choose $\rho = \rho(u,m) > 0$ by setting

$$\rho = (1-\delta_0)\|u\|_{L^\infty(\hat{\tau})}/\|\nabla u\|_{L^\infty(\hat{\tau})}. \tag{3.12}$$

By (3.10), we then have $\delta_0 \leq \rho$. Moreover, if we now choose any $\mathbf{x}^* \in \overline{\hat{\tau}}$ such that

$$|u(\mathbf{x}^*)| = \|u\|_{L^\infty(\hat{\tau})},$$

then, for any $\mathbf{x} \in B_\rho(\mathbf{x}^*) \cap \overline{\hat{\tau}}$, we have

$$|u(\mathbf{x}) - u(\mathbf{x}^*)| \leq \|\nabla u\|_{L^\infty(\hat{\tau})} |\mathbf{x} - \mathbf{x}^*| < \|\nabla u\|_{L^\infty(\hat{\tau})}\rho = (1-\delta_0)\|u\|_{L^\infty(\hat{\tau})}.$$

This implies that $|u(\mathbf{x})| \geq |u(\mathbf{x}^*)| - |u(\mathbf{x}) - u(\mathbf{x}^*)| > \delta_0\|u\|_{L^\infty(\hat{\tau})}$. This establishes the statement (3.11) when $\|\nabla u\|_{L^\infty(\hat{\tau})} \neq 0$. On the other hand, if $\|\nabla u\|_{L^\infty(\hat{\tau})} = 0$, then u is constant on $\hat{\tau}$, and (3.11) holds trivially with $\rho = \delta_0$ and any $\mathbf{x}^* \in \overline{\hat{\tau}}$. ∎

Theorem 3.6 *Let $i \in \{0,1\}$, $m \geq i$, $0 \leq s \leq k$ and $-\infty < \underline{\alpha} < \overline{\alpha} < \infty$. Then the inequality*

$$\|\rho^{s+\alpha}u\|_{L^2(\Omega)} \lesssim \|\rho^\alpha u\|_{H^{-s}(\Omega)},$$

holds uniformly in $u \in \mathcal{S}_i^m(\mathcal{T})$ and $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.

Proof. The result is clear for $s = 0$. We prove it for $s = k$ and the theorem follows by interpolation. Suppose $u \in \mathcal{S}_i^m(\mathcal{T})$. The case $u \equiv 0$ is trivial, so from now on we assume that $u \not\equiv 0$. Then, for any $w \in H^k(\Omega)$, we have, by definition,

$$\|\rho^\alpha u\|_{H^{-k}(\Omega)} \geq \frac{|(\rho^\alpha u, w)|}{\|w\|_{H^k(\Omega)}}.$$

We shall construct $w \in H^k(\Omega)$ such that

$$|(\rho^\alpha u, w)| \gtrsim \|\rho^{k+\alpha}u\|^2_{L^2(\Omega)} \tag{3.13}$$

and

$$\|w\|_{H^k(\Omega)} \lesssim \|\rho^{k+\alpha}u\|_{L^2(\Omega)}, \tag{3.14}$$

from which the result follows immediately.

The construction of w is a generalisation of the argument used to prove [4, Theorem 4.7]. For any $\tau \in \mathcal{T}$, we have $u \circ \chi_\tau \in \mathbb{P}^m(\hat{\tau})$, and by Lemma 3.5, there exists a simplex $\hat{t}(\tau) \subseteq \hat{\tau}$ such that:

$$\rho_{\hat{t}(\tau)} \gtrsim 1 \qquad \text{and} \qquad \inf_{\mathbf{x}\in\hat{t}(\tau)} |u \circ \chi_\tau(\mathbf{x})| \geq \delta \|u \circ \chi_\tau\|_{L^\infty(\hat{\tau})} \gtrsim \|u \circ \chi_\tau\|_{L^\infty(\hat{\tau})}. \tag{3.15}$$

(Recall that the constant δ in Lemma 3.5 was independent of u, hence $\delta \gtrsim 1$.) It is clear from this that $u \circ \chi_\tau$ does not change sign on $\hat{t}(\tau)$ and that

$$|\hat{t}(\tau)| \sim 1. \tag{3.16}$$

Using the Bernstein representation of polynomials (as described, for example in [4, §4.3]), we can construct a non-negative function $\hat{P}_{\hat{t}(\tau)}$ in $H_0^k(\hat{\tau})$ such that $\mathrm{supp}\,\hat{P}_{\hat{t}(\tau)} = \hat{t}(\tau)$, $\hat{P}_{\hat{t}(\tau)}$ is a polynomial on $\hat{t}(\tau)$ and such that

$$C_2|\hat{t}(\tau)|^{1/p} \geq \|\hat{P}_{\hat{t}(\tau)}\|_{L^p(\hat{t}(\tau))} \geq C_1|\hat{t}(\tau)|^{1/p}, \tag{3.17}$$

with C_1, C_2 independent of p and of $\hat{t}(\tau)$. (This is done by constructing a positive-valued polynomial on $\hat{t}(\tau)$ to vanish with sufficiently high order on the boundary of $\hat{t}(\tau)$.) Combining this with (3.16), we have

$$\int_{\hat{t}(\tau)} \hat{P}_{\hat{t}(\tau)} \sim |\hat{t}(\tau)| \sim 1. \tag{3.18}$$

Now set $t(\tau) = \chi_\tau(\hat{t}(\tau)) \subseteq \tau$ and define a corresponding non-negative function $P_{t(\tau)} \in H^k(\Omega)$ by setting $P_{t(\tau)} = \hat{P}_{\hat{t}(\tau)} \circ \chi_\tau^{-1}$ on τ and $P_{t(\tau)} = 0$ on $\Omega \backslash \tau$. It follows that

$$\mathrm{supp}\,P_{t(\tau)} = t(\tau) \qquad \text{and} \qquad \int_{t(\tau)} P_{t(\tau)} \sim |\tau|, \tag{3.19}$$

the proof of the second relation making use of (2.1a) and (3.18).

For each $\tau \in \mathcal{T}$, we introduce scalars

$$b_\tau = \rho_\tau^{k+\alpha} \operatorname{sign}(u|_{t(\tau)}) \inf_{\mathbf{x}\in t(\tau)} |u(\mathbf{x})| \,, \tag{3.20}$$

and we define $w \in H^k(\Omega)$ by

$$w = \sum_{\tau\in\mathcal{T}} b_\tau \rho_\tau^k P_{t(\tau)} \,. \tag{3.21}$$

Then, using (3.21), (3.19), we obtain

$$(\rho^\alpha u, w) = \sum_{\tau\in\mathcal{T}} \int_{t(\tau)} (\rho_\tau/\rho)^k \left\{\rho^{k+\alpha} b_\tau u\right\} P_{t(\tau)} \,.$$

By (3.20), (2.3a) and the non-negativity of $P_{t(\tau)}$, we have,

$$|(\rho^\alpha u, w)| \gtrsim \sum_{\tau\in\mathcal{T}} \rho_\tau^{2(k+\alpha)} \left\{\inf_{\mathbf{x}\in t(\tau)} |u(\mathbf{x})|\right\}^2 \int_{t(\tau)} P_{t(\tau)} \,.$$

Then, by (3.15) and (3.19)

$$|(\rho^\alpha u, w)| \gtrsim \sum_{\tau\in\mathcal{T}} \rho_\tau^{2(k+\alpha)} \|u\|^2_{L^\infty(\tau)} |\tau| \,,$$

which, using (3.1), readily yields (3.13).

To obtain (3.14), we first obtain the estimate

$$\|w\|^2_{H^k(\Omega)} = \sum_{\tau\in\mathcal{T}} \|w\|^2_{H^k(\tau)} \le \sum_{\tau\in\mathcal{T}} \rho_\tau^{2k} |b_\tau|^2 \|P_{t(\tau)}\|^2_{H^k(t(\tau))} \lesssim \sum_{\tau\in\mathcal{T}} |b_\tau|^2 \|P_{t(\tau)}\|^2_{L^2(t(\tau))} \,, \tag{3.22}$$

where the final inequality follows from Proposition 2.9 and (3.15). Since

$$\left\|P_{t(\tau)}\right\|^2_{L^2(t(\tau))} = \int_{\hat{t}(\tau)} \left|\hat{P}_{\hat{t}(\tau)}\right|^2 g_\tau \sim |\tau| \int_{\hat{t}(\tau)} \left|\hat{P}_{\hat{t}(\tau)}\right|^2 ,$$

(3.17) yields $|\tau| \left\|P_{t(\tau)}\right\|^2_{L^2(t(\tau))} \sim |\tau| \left|\hat{t}\,(\tau)\right| \sim |t\,(\tau)|$. Using this together with the definition (3.20) of b_τ, we finally obtain

$$\|w\|^2_{H^k(\Omega)} \lesssim \sum_{\tau\in\mathcal{T}} |b_\tau|^2 |t(\tau)| \le \sum_{\tau\in\mathcal{T}} \rho_\tau^{2(k+\alpha)} \left\{\inf_{\mathbf{x}\in t(\tau)} |u(\mathbf{x})|\right\}^2 |t(\tau)| \lesssim \|\rho^{k+\alpha} u\|^2_{L^2(\Omega)}, \tag{3.23}$$

i.e., (3.14). ■

Remark 3.7 *When $i = k = 1$, a simpler construction for w can be given in terms of a suitable element in $\mathcal{S}_1^m(\mathcal{T})$ (see [8] for the case $m = 1$).*

4 Galerkin Boundary Element Method

In this section we review briefly the Galerkin boundary element method for elliptic PDEs, which forms the basis of the applications in the proceeding sections. We consider a 2D surface Ω in $\mathbb{R}^3$ (i.e. the case $d = 2$ above). To conform with more usual notation in boundary integral equations, we rename this surface Γ. To avoid technicalities, we assume that Γ is a Lipschitz surface in $\mathbb{R}^3$, consisting of planar pieces joined at corners and edges. (The extension to a piecewise smooth curvilinear surface is standard.) Consider the general linear integral equation

$$(\lambda I + \mathcal{K})u(\mathbf{x}) := \lambda u(\mathbf{x}) + \int_\Gamma k(\mathbf{x}, \mathbf{y})u(\mathbf{y})dy = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma,$$

for some given scalar $\lambda \in \mathbb{R}$ kernel function k and sufficiently smooth right-hand side f. The corresponding weak form is

$$\text{Find } u \in H^\mu(\Gamma) \text{ such that } a(u, v) := ((\lambda I + \mathcal{K})u, v) = (f, v) \quad \text{for all } v \in H^\mu(\Gamma), \tag{4.1}$$

and we assume that $a(\cdot, \cdot)$ is elliptic in the energy space $H^\mu(\Gamma)$ for some $\mu \in \mathbb{R}$. (The bracket $(\cdot, \cdot)$ denotes the continuous extension of the $L^2(\Gamma)$ scalar product to the $H^{-\mu}(\Gamma) \times H^\mu(\Gamma)$ duality pairing.) Typical examples are: the classical single layer, double layer and hypersingular operators for the Laplacian:

$$\textbf{Single layer potential}: \quad k(\mathbf{x}, \mathbf{y}) = 1/(4\pi |\mathbf{x} - \mathbf{y}|), \tag{4.2a}$$

$$\textbf{Double layer potential}: \quad k(\mathbf{x}, \mathbf{y}) = \partial/\partial n(\mathbf{y}) \{1/(4\pi |\mathbf{x} - \mathbf{y}|)\}, \tag{4.2b}$$

$$\textbf{Hypersingular operator}: \quad k(\mathbf{x}, \mathbf{y}) = \partial/\partial n(\mathbf{x})\partial/\partial n(\mathbf{y}) \{1/(4\pi |\mathbf{x} - \mathbf{y}|)\}. \tag{4.2c}$$

For the operators (4.2a-b) we have $\mu = -1/2$, 0 and $\lambda = 0$, $\lambda \neq 0$ respectively. (Note however that for the double layer potential in $L^2(\Gamma)$ the ellipticity, in the sense of inf-sup conditions, is proved for smooth surfaces while the generalisation to other classes of surfaces is still open - see [5].)

For the hypersingular operator (4.2c), we have $\lambda = 0$ and the bilinear form $a(\cdot, \cdot)$ in (4.1) is elliptic and continuous in the subspace $\hat{H}^{1/2}(\Gamma) := H^\mu(\Gamma)/\mathbb{R}$. In this case, the space $H^{1/2}(\Gamma)$ in (4.1) and throughout the rest of this paper has to be replaced by $\hat{H}^{1/2}(\Gamma)$ and the approximating space $\mathcal{S}_i^m(\mathcal{T})$ by $\mathcal{S}_i^m(\mathcal{T})/\mathbb{R}$. To avoid notational complexity we shall, however, simply write $H^\mu(\Gamma)$ for the energy space throughout, except in Section 6.2 where this operator is considered in isolation.

In the standard, conforming Galerkin method we select a subspace $\mathcal{S}_i^m(\mathcal{T}) \subset H^\mu(\Gamma)$ and approximate (4.1) by seeking $U \in \mathcal{S}_i^m(\mathcal{T})$, such that

$$a(U, V) = (f, V) \qquad \text{for all } V \in \mathcal{S}_i^m(\mathcal{T}). \tag{4.3}$$

In order to realise (4.3) numerically, we need to introduce a basis for the approximating space $\mathcal{S}_i^m(\mathcal{T})$. We denote this basis by $\{\phi_p : p \in \mathcal{F}\}$, where $\mathcal{F}$ is the set of *freedoms*. We shall assume (for convenience) that the ϕ_p constitute a local nodal basis, i.e., there is a sequence of nodes $(\mathbf{x}_p)_{p\in\mathcal{F}}$ so that

$$\phi_p(\mathbf{x}_q) = \delta_{p,q}, \qquad \text{and} \qquad \operatorname{supp}\phi_p \subseteq \bigcup \{\overline{\tau} : \mathbf{x}_p \in \overline{\tau}\}. \tag{4.4}$$

The sequence of nodes can contain repeated entries, thus allowing discontinuous elements (which may be used when $\mu \leq 0$).

Moreover we assume that for each $u \in \mathcal{S}_i^m(\mathcal{T})$, and $\tau \in \mathcal{T}$, the restriction $u|_\tau$ is determined uniquely by its values at the local nodal points $\{\mathbf{x}_p : p \in \mathcal{F}_\tau\}$, where $\mathcal{F}_\tau := \{p \in \mathcal{F} : \mathbf{x}_p \in \overline{\tau}\}$. If we introduce $\hat{\mathbf{x}}_p = \chi_\tau^{-1}(\mathbf{x}_p)$, $p \in \mathcal{F}_\tau$, then a convenient criterion to ensure this is

$$(\hat{u} = 0) \Leftrightarrow (\forall p \in \mathcal{F}_\tau : \hat{u}(\hat{\mathbf{x}}_p) = 0) \tag{4.5}$$

for all $\hat{u} \in \mathbb{P}^m(\hat{\tau})$.

Note that (4.5) implies that

$$\text{the functional} \quad \hat{u} \to \left\{ \sum_{p \in \mathcal{J}_\tau} |\hat{u}(\hat{\mathbf{x}}_p)|^2 \right\}^{1/2} \quad \text{is a norm on } \mathbb{P}^m(\hat{\tau}). \tag{4.6}$$

Writing $U = \sum_{p \in \mathcal{F}} U_p \phi_p$, (4.3) is equivalent to the linear system

$$\sum_{q \in \mathcal{F}} (\lambda M_{p,q} + K_{p,q}) U_q = f_p, \qquad p \in \mathcal{F}, \tag{4.7}$$

where $f_p = (f, \phi_p)$, $M_{p,q} = (\phi_q, \phi_p)$ is the mass matrix and

$$K_{p,q} = \int_\Gamma \int_\Gamma k(\mathbf{x}, \mathbf{y}) \phi_q(\mathbf{y}) \phi_p(\mathbf{x}) dy dx, \quad p, q \in \mathcal{F}. \tag{4.8}$$

is the stiffness matrix. The mass matrix M is sparse and can be easily computed. The stiffness matrix is dense and generally has to be approximated by quadrature. Replacing $K_{p,q}$ by an approximation $\tilde{K}_{p,q}$ leads to the discrete counterpart of (4.3): Find $\tilde{U} \in \mathcal{S}_i^m(\mathcal{T})$, such that

$$\tilde{a}(\tilde{U}, V) = (f, V) \qquad \text{for all } V \in \mathcal{S}_i^m(\mathcal{T}), \tag{4.9}$$

where

$$\tilde{a}(V, W) := \sum_{p \in \mathcal{N}_i} \sum_{q \in \mathcal{N}_i} W_p \left(\lambda M_{p,q} + \tilde{K}_{p,q} \right) V_q. \tag{4.10}$$

The stability and convergence of the corresponding solution $\tilde{U}$ is provided by the first "Strang Lemma" [3]. The lemma allows to make use of the possible regularity of the exact solution u and we introduce the relevant parameters first.

- Assume the continuous problem (4.1) has regularity $\delta \geq 0$, i.e.

$$f \in H^{-\mu+\delta}(\Gamma) \Longrightarrow \text{The solution } u \text{ of (4.1) is in } H^{\mu+\delta}(\Gamma) \tag{4.11}$$

  (Note that ellipticity and continuity of $a(\cdot, \cdot)$ in $H^\mu(\Gamma)$ imply regularity for $\delta = 0$).

- We introduce a measure for stability of the perturbed bilinear form by

$$r_{\text{stab}} := \sup_{V, W \in \mathcal{S}_i^m(\mathcal{T}) \setminus \{0\}} \frac{|a(V, W) - \tilde{a}(V, W)|}{\|V\|_{H^\mu(\Gamma)} \|W\|_{H^\mu(\Gamma)}}. \tag{4.12}$$

- A measure of consistency depends on some parameter ν:

$$\mu \leq \nu \leq \mu + \delta \quad \text{with} \quad \mathcal{S}_i^m(\mathcal{T}) \subset H^\nu(\Gamma) \tag{4.13}$$

  and is given by

$$r_{\text{conv}} := \sup_{V, W \in \mathcal{S}_i^m(\mathcal{T}) \setminus \{0\}} \frac{|a(V, W) - \tilde{a}(V, W)|}{\|V\|_{H^\nu(\Gamma)} \|W\|_{H^\mu(\Gamma)}}. \tag{4.14}$$

- For $\lambda > 0$, the *distance* of $\mathcal{S}_i^m(\mathcal{T})$ from $H^{\nu+\lambda}(\Gamma)$ is defined by

$$d_{\nu,\lambda}\left(\mathcal{S}_i^m(\mathcal{T})\right) := \sup_{w\in H^{\nu+\lambda}(\Gamma)\backslash\{0\}} \inf_{Z\in\mathcal{S}_i^m(\mathcal{T})} \frac{\|w-Z\|_{H^\nu(\Gamma)}}{\|w\|_{H^{\nu+\lambda}(\Gamma)}}.$$

Lemma 4.1 *Consider the problem (4.1), where $a(\cdot,\cdot)$ is a continuous and elliptic bilinear form on $H^\mu(\Gamma)$. Assume that $f \in H^{-\mu+\delta}$ and assume (4.11) holds. Choose ν such that (4.13) holds and define r_{conv} as in (4.14). Suppose*

$$r_{\text{stab}} \to 0 \quad \textit{as} \quad h := \max\{h_\tau : \tau \in \mathcal{T}\} \to 0. \tag{4.15}$$

Then, for sufficiently small h, the approximate Galerkin method (4.9) has a unique solution $\tilde{U} \in \mathcal{S}_i^m(\mathcal{T})$ and we have the error estimate

$$\|u-\tilde{U}\|_{H^\mu(\Gamma)} \lesssim d_{\nu,\nu-\mu}\left(\mathcal{S}_i^m(\mathcal{T})\right) \inf_{Z\in\mathcal{S}_i^m(\mathcal{T})} \|u-Z\|_{H^\nu(\Gamma)} + r_{\text{conv}}\|u\|_{H^\nu(\Gamma)}. \tag{4.16}$$

The possible choice $\nu = \mu$ and $\delta = 0$ in (4.13) shows that r_{conv} can be replaced by r_{stab} in (4.16) and the assumption on the regularity is not used. However, for any ν in (4.13), there holds $r_{\text{conv}} \leq r_{\text{stab}}$ and a proper choice of ν could lead to $r_{\text{conv}} \ll r_{\text{stab}}$, i.e., to an improved error estimate provided the first term of (4.16) is also of higher order.
Proof. The proof is a simple consequence of the first Strang lemma (cf. [3, Theorem 4.1.1]). The solvability for sufficiently small h follows directly from the lemma. For the error estimate we employ, from the lemma,

$$\|u-\tilde{U}\|_{H^\mu(\Gamma)} \leq C\left(\inf_{Z\in\mathcal{S}_i^m(\mathcal{T})}\|u-Z\|_{H^\mu(\Gamma)} + \sup_{W\in\mathcal{S}_i^m(\mathcal{T})\backslash\{0\}} \frac{|a(Z,W)-\tilde{a}(Z,W)|}{\|W\|_{H^\mu(\Gamma)}}\right).$$

The second term in the bracket can be estimated from above by $r_{\text{conv}}\|Z\|_{H^\nu(\Gamma)}$.

Let ν be such that (4.13) holds. We introduce the $H^\nu(\Gamma)$-orthogonal projection $P : H^\nu(\Gamma) \to \mathcal{S}_i^m(\mathcal{T})$, i.e., $(Pu,w)_{H^\nu(\Gamma)} = (u,w)_{H^\nu(\Gamma)}$ for all $w \in \mathcal{S}_i^m(\mathcal{T})$.

The choice $Z = Pu$ leads to

$$\|u-\tilde{U}\|_{H^\mu(\Gamma)} \leq C\left\{\|u-Pu\|_{H^\mu(\Gamma)} + r_{\text{conv}}\|Pu\|_{H^\nu(\Gamma)}\right\}. \tag{4.17}$$

Since P is the $H^\nu(\Gamma)$-orthogonal projection we obtain $\|Pu\|_{H^\nu(\Gamma)} \leq \|u\|_{H^\nu(\Gamma)}$.

In order to estimate the first term on the right-hand side in (4.17) we obtain by duality and the orthogonality of P

$$\begin{aligned}\|u-Pu\|_{H^\mu(\Gamma)} &= \sup_{w\in H^{2\nu-\mu}(\Gamma)\backslash\{0\}} \frac{\left|(u-Pu,w)_{H^\nu(\Gamma)}\right|}{\|w\|_{H^{2\nu-\mu}(\Gamma)}} = \sup_{w\in H^{2\nu-\mu}(\Gamma)\backslash\{0\}} \frac{\left|(u-Pu,w-Pw)_{H^\nu(\Gamma)}\right|}{\|w\|_{H^{2\nu-\mu}(\Gamma)}} \\ &\leq \|u-Pu\|_{H^\nu(\Gamma)} \sup_{w\in H^{2\nu-\mu}(\Gamma)\backslash\{0\}} \frac{\|w-Pw\|_{H^\nu(\Gamma)}}{\|w\|_{H^{2\nu-\mu}(\Gamma)}} \\ &= d_{\nu,\nu-\mu}\left(\mathcal{S}_i^m(\mathcal{T})\right) \inf_{Z\in\mathcal{S}_i^m(\mathcal{T})} \|u-Z\|_{H^\nu(\Gamma)}.\end{aligned}$$

■

Remark 4.2 *In our applications (quadrature and panel clustering for the Galerkin boundary element method), we can derive, as a first step, estimates of the form*

$$\left|a\left(V,W\right)-\tilde{a}\left(V,W\right)\right|\leq C_h\left\|V\right\|_{L^2(\Gamma)}\left\|W\right\|_{L^2(\Gamma)}\ ,\qquad \text{for all } V,W\in\mathcal{S}_i^m\left(\mathcal{T}\right),$$

where $C_h\to 0$ *as* $h:=\max\left\{h_\tau:\tau\in\mathcal{T}\right\}\to 0$. *If the underlying energy space* $H^\mu\left(\Gamma\right)$ *satisfies* $\mu\geq 0$*, then the estimate*

$$\left|a\left(V,W\right)-\tilde{a}\left(V,W\right)\right|\leq C_h\left\|V\right\|_{H^\mu(\Gamma)}\left\|W\right\|_{H^\mu(\Gamma)}\qquad \text{for all } V,W\in\mathcal{S}_i^m\left(\mathcal{T}\right)$$

is a trivial consequence. It also turns out that we cannot gain additional powers of h *by replacing* $\|V\|_{H^\mu(\Gamma)}$ *by* $\|V\|_{H^\nu(\Gamma)}$*, for any* $\nu>\mu$*. Thus, for the double layer potential and the hypersingular operator the simplest choice* $\nu=\mu$ *and* $\delta=0$ *is optimal in Lemma 4.1, leading to* $r_{\text{conv}}=r_{\text{stab}}$*, independent of any regularity in the problem.*

The situation is different for the single layer operator. There $\mu=-1/2$ *and one has to apply inverse inequalities for* V ***and*** W *to estimate* r_{stab}*. If we assume* L^2*-regularity, i.e.,* $\delta=1/2$*, we obtain optimal convergence rates by choosing* $\nu=0$*. Under moderate assumptions on the mesh we have*

$$d_{0,1/2}\left(\mathcal{S}_i^m\left(\mathcal{T}\right)\right)\leq Ch^{1/2}$$

and the error estimate (4.16) takes the form

$$\|u-\tilde{U}\|_{H^{-1/2}(\Gamma)}\lesssim h^{1/2}\inf_{Z\in\mathcal{S}_i^m(\mathcal{T})}\|u-Z\|_{L^2(\Gamma)}+r_{\text{conv}}\left\|u\right\|_{L^2(\Gamma)}.$$

Since the estimate of r_{conv} *only requires* ***one*** *application of inverse estimates (for the function* W*), the term* r_{conv} *converges faster to zero than* r_{stab} *and this, exactly, is the gain from the use of different measures for stability and consistency.*

5 Galerkin Method with Quadrature

The effect of quadrature errors in Galerkin methods is analysed in [8], under the assumption of shape-regular meshes. The following theory generalises these results, allowing also the treatment of degenerate mesh sequences, provided they satisfy Assumption 2.6.

Theorem 5.1 *Suppose the assumptions of Lemma 4.1 hold and suppose, for all* $p,q\in\mathcal{F}$ *the approximate matrix entries* $\tilde{K}_{p,q}$ *are constructed so that the following error estimate holds:*

$$|K_{p,q}-\tilde{K}_{p,q}|\leq h^{\chi+1}|\operatorname{supp}\phi_p|\ |\operatorname{supp}\phi_q| \tag{5.1}$$

for some $\chi\geq 0$ *where* $h=\max\left\{h_\tau:\tau\in\mathcal{T}\right\}$ *is the global mesh diameter. Then, for all* $\nu_1,\nu_2\in[-k,k]$ *such that* $\mathcal{S}_i^m(\mathcal{T})\subset H^{\max\{\nu_1,\nu_2\}}(\Gamma)$,

$$\frac{|a(V,W)-\tilde{a}(V,W)|}{\|V\|_{H^{\nu_1}(\Gamma)}\|W\|_{H^{\nu_2}(\Gamma)}}\quad\lesssim\quad h^{\chi+1}\sqrt{\sum_{\tau\in\mathcal{T}}\rho_\tau^{2\nu_1^-}|\tau|}\ \sqrt{\sum_{\tau\in\mathcal{T}}\rho_\tau^{2\nu_2^-}|\tau|}\,,$$

uniformly in $V,W\in\mathcal{S}_i^m(\mathcal{T})$ *where* $\nu_i^-:=\min\{\nu_i,0\}$.

Proof. By the definitions (4.1) and (4.10) of a and $\tilde{a}$, we have

$$|a(V,W)-\tilde{a}(V,W)| \le \sum_{p\in\mathcal{F}}\sum_{q\in\mathcal{F}} |V_p||K_{p,q}-\tilde{K}_{p,q}||W_q| \le h^{\chi+1}\left\{\sum_{p\in\mathcal{F}} s_p|V_p|\right\}\left\{\sum_{p\in\mathcal{F}} s_p|W_p|\right\}, \quad (5.2)$$

where $s_p = |\operatorname{supp}\phi_p|$. Now, using the Cauchy-Schwarz inequality, we obtain

$$\sum_{p\in\mathcal{F}} s_p|V_p| = \sum_{p\in\mathcal{F}}\{\rho_p^{\nu_1^-} s_p^{1/2}\}\{\rho_p^{-\nu_1^-} s_p^{1/2}|V_p|\} \le \sqrt{\sum_{p\in\mathcal{F}} \rho_p^{2\nu_1^-} s_p}\,\sqrt{\sum_{p\in\mathcal{F}} \rho_p^{-2\nu_1^-} s_p|V_p|^2}, \quad (5.3)$$

where $\rho_p = \rho(\mathbf{x}_p)$, $p \in \mathcal{F}$ and ρ is as in Definition 3.1. Now, using (4.4) and then Assumption 2.6 and , we have

$$\sum_{p\in\mathcal{F}} \rho_p^{2\nu_1^-} s_p \le \sum_{p\in\mathcal{F}} \rho_p^{2\nu_1^-} \sum_{\substack{\tau\in\mathcal{T}\\ \mathbf{x}_p\in\overline{\tau}}} |\tau| = \sum_{\tau\in\mathcal{T}}\sum_{p\in\mathcal{F}_\tau} \rho_p^{2\nu_1^-}|\tau| \lesssim \sum_{\tau\in\mathcal{T}} \rho_\tau^{2\nu_1^-}|\tau|, \quad (5.4)$$

where the constant of proportionality in the last inequality depends on the polynomial degree m, but not on the mesh $\mathcal{T}$ or on ν_1.

A similar argument shows

$$\sum_{p\in\mathcal{F}} \rho_p^{-2\nu_1^-} s_p|V_p|^2 \lesssim \sum_{\tau\in\mathcal{T}} \rho_\tau^{-2\nu_1^-}\left\{\sum_{p\in\mathcal{F}_\tau}|V_p|^2\right\}|\tau|.$$

Thus, by a simple scaling argument based on (4.6),

$$\sum_{p\in\mathcal{F}} \rho_p^{-2\nu_1^-} s_p|V_p|^2 \lesssim \sum_{\tau\in\mathcal{T}} \left\|\rho^{-2\nu_1^-}V\right\|_{L^2(\tau)}^2. \quad (5.5)$$

Combining (5.3) with (5.4) and (5.5) we obtain

$$\sum_{p\in\mathcal{F}} s_p|V_p| \lesssim \sqrt{\sum_{\tau\in\mathcal{T}} \rho_\tau^{2\nu_1^-}|\tau|}\,\|\rho^{-\nu_1^-}V\|_{L^2(\Gamma)} \lesssim \sqrt{\sum_{\tau\in\mathcal{T}} \rho_\tau^{2\nu_1^-}|\tau|}\,\|V\|_{H^{\nu_1}(\Gamma)},$$

where the final relation follows from Theorem 3.6 when $\nu_1 \le 0$ and trivially otherwise. Using this and an analogous estimate for $\sum_{p\in\mathcal{F}} s_p|W_p|$ in (5.2), we obtain the theorem. ∎

Remark 5.2 *(i) In Theorem 5.1 the parameter χ can be chosen as required, in order that the resulting estimates for r_{stab} and r_{conv} are suitable for the required application of Lemma 4.1. Typically for kernels $k(\mathbf{x},\mathbf{y})$ which blow up at $\mathbf{x}=\mathbf{y}$, the value of χ has to be increased when* $\operatorname{supp}\phi_p$ *and* $\operatorname{supp}\phi_q$ *get closer together (see, e.g. [8]). Special transformation methods are used when* $\operatorname{supp}\phi_p$ *and* $\operatorname{supp}\phi_q$ *intersect (cf. [14], [23]).*

(ii) In [8, 9] we give a number of different quadrature schemes which can achieve (5.1) in the case when the approximating space is $\mathcal{S}_1^1(\mathcal{T})$. This analysis can be easily extended to more general approximating spaces.

A general theory of quadrature approximation of Galerkin methods follows by combining Lemma 4.1 and Theorem 5.1.

Corollary 5.3 *Under the conditions of Theorem 5.1, we have*

$$r_{\text{stab}} \lesssim h^{\chi+1} \left\{ \sum_{\tau\in\mathcal{T}} \rho_\tau^{2\mu^-+1} h_\tau \right\}, \qquad r_{\text{conv}} \lesssim h^{\chi+1} \sqrt{\sum_{\tau\in\mathcal{T}} \rho_\tau^{2\mu^-+1} h_\tau} \sqrt{\sum_{\tau\in\mathcal{T}} \rho_\tau^{2\nu^-+1} h_\tau}.$$

Proof. It is a trivial application of Theorem 5.1, with $\nu_1 = \mu = \nu_2$ for r_{stab} and respectively $\nu_1 = \nu$, $\nu_2 = \mu$ for r_{conv}. ∎

When the energy space of the Galerkin method is $H^\mu(\Gamma)$, for $\mu \geq -1/2$, we have $2\nu^- + 1 \geq 2\mu^- + 1 \geq 0$, and no negative exponent appears in the estimates in Corollary 5.3. Hence the degeneracy has no effect on the stability and consistency estimates. This holds for all the standard boundary integral equations for second-order elliptic PDEs. In particular, for the three standard integral equations given by (4.2a-c), we have the following corollary, the proof of which follows directly from Corollary 5.3.

Corollary 5.4 *For the single layer potential $\mu = -1/2$ and*

$$r_{\text{stab}} \lesssim h^{\chi+1} \{ \sum_{\tau\in\mathcal{T}} h_\tau \} \leq h^{\chi+2} \ \{\#\mathcal{T}\}.$$

With the regularity assumption $\delta = 1/2$ we may set $\nu = 0$ to obtain

$$r_{\text{conv}} \lesssim h^{\chi+1} \{ \sum_{\tau\in\mathcal{T}} h_\tau \}^{1/2} \left\{ \sum_{\tau\in\mathcal{T}} |\tau| \right\}^{1/2} \lesssim h^{\chi+3/2} \{\#\mathcal{T}\}^{1/2}.$$

To see why r_{conv} may be smaller than r_{stab}, assume, as is often the case, that $(\#\mathcal{T}) \leq Ch^{-2}$. Then, $r_{\text{stab}} \lesssim h^\chi$, while $r_{\text{conv}} \lesssim h^{\chi+1/2}$ and we see the gain of using different quantities for measuring the stability and consistency.

For the double layer potential and hypersingular operator, we choose $\delta = 0$ and $\nu = \mu$ to obtain in this case

$$r_{\text{stab}} = r_{\text{conv}} \lesssim h^{\chi+1} \sum_{\tau\in\mathcal{T}} |\tau| \lesssim h^{\chi+1}.$$

6 Galerkin Method with Panel Clustering

The panel clustering algorithm provides an alternative representation of the finite-dimensional Galerkin operator described in §4 , so that multiplication of any vector by the corresponding matrix representation has complexity $\mathcal{O}(N \log^\kappa N)$, for some (small) κ, where N $(= \#\mathcal{F})$ is the number of degrees of freedom. This should be compared with the N^2 complexity required for multiplication by the exact matrix. Approximations of this sort are at the heart of many fast methods for dense systems. As well as providing a fast multiplication, the approximation needs also to be sufficiently accurate and so far this has only been shown for quasi-uniform meshes. The purpose of this section is to extend the error analysis to (possibly) degenerate meshes. Our results show that the panel clustering approximation satisfies stability and consistency estimates which are independent of mesh degeneracy.

First, we will analyse standard formulations of integral operators in a unified setting. In the final subsection we consider a special formulation of the hypersingular operator.

6.1 Panel Clustering in the general case

To obtain this result we need to introduce the following concepts. (For a more complete introduction, see [16, 14, 13, 21, 1, 2]).

Definition 6.1 (Cluster Tree) *A cluster tree* $\mathbb{T}$ *is a tree*[2] *whose vertices (called "clusters") consist of unions* $\sigma = \cup\{\overline{\tau} : \tau \in \mathcal{T}'\}$ *for certain subsets* $\mathcal{T}' \subset \mathcal{T}$. *These are required to satisfy the following properties:*

(i) $\Gamma = \cup_{\tau\in\mathcal{T}}\overline{\tau}$ *is the root of* $\mathbb{T}$.

(ii) $\mathcal{L}(\mathbb{T}) = \mathcal{T}$, *where* $\mathcal{L}(\mathbb{T})$ *denotes the set of leaves of* $\mathbb{T}$.

(iii) If $\sigma \in \mathbb{T}$ *is not a leaf, there is a set of vertices of* $\mathbb{T}$ *(denoted* $\mathrm{sons}(\sigma)$*) such that:*
(a) $\sigma = \cup\{\sigma' : \sigma' \in \mathrm{sons}(\sigma)\}$;
(b) If $\sigma', \sigma'' \in \mathrm{sons}(\sigma)$ *and* $\sigma' \neq \sigma''$, *then* σ', σ'' *intersect at most by their boundaries.*

There are standard procedures for constructing cluster trees (see for example [1, Example 2.1]). Once $\mathbb{T}$ has been constructed, a second tree, $\mathbb{T}_2$, whose vertices are pairs of clusters may be constructed with the following properties:

Definition 6.2 $\mathbb{T}_2$ *is uniquely defined by*
(i) $(\Gamma,\Gamma) \in \mathbb{T}_2$ *is the root of* $\mathbb{T}_2$,
(ii) For $b = (\sigma', \sigma'') \in \mathbb{T}_2$, *the set of sons is defined as follows:*

$$\mathrm{sons}\,(b) := \begin{cases} \mathrm{sons}\,(\sigma') \times \mathrm{sons}\,(\sigma'') & \textit{if } \sigma', \sigma'' \in \mathbb{T}\backslash\mathcal{L}\,(\mathbb{T})\,, \\ \{\sigma'\} \times \mathrm{sons}\,(\sigma'') & \textit{if } b \in \mathcal{L}\,(\mathbb{T}) \times \mathbb{T}\backslash\mathcal{L}\,(\mathbb{T})\,, \\ \mathrm{sons}\,\{\sigma'\} \times \{\sigma''\} & \textit{if } b \in \mathbb{T}\backslash\mathcal{L}\,(\mathbb{T}) \times \mathcal{L}\,(\mathbb{T})\,, \\ \emptyset & \textit{if } b \in \mathcal{L}\,(\mathbb{T}) \times \mathcal{L}\,(\mathbb{T})\,. \end{cases}$$

The key point in the panel clustering algorithm is to select pairs of clusters $(\sigma', \sigma'') \in \mathbb{T}_2$ and to approximate the corresponding integrals by replacing the kernel k in (4.8) with some suitable separable expansion. This cannot be done on all pairs of clusters, but only on pairs which are sufficiently far apart relative to their diameters. This leads to the definition of an admissible pair of clusters:

Definition 6.3 (Admissible Pair) *For* $\eta > 0$, *a pair* $(\sigma', \sigma'') \in \mathbb{T}_2$ *is called* η-admissible *if*

$$\eta\ \mathrm{dist}(\sigma', \sigma'') \geq \max\{\mathrm{diam}\,\sigma',\, \mathrm{diam}\,\sigma''\}\,.$$

Using the concept of admissibility, the integration domain $\Gamma\times\Gamma$ in (4.8) is split into a near field and a far field, characterised by the subsets P_{far} ("far field") and P_{near} ("near field") of $\mathbb{T}_2$, defined as follows.

First set $P_{\mathrm{near}} = \emptyset = P_{\mathrm{far}}$, and then initiate a call **divide**(Γ,Γ) to the following recursive procedure:

[2] Usually a tree is a graph (V, E) with vertices V and edges E having a certain structure. Here the structure will be given by the sons of the vertices (defined below), while V is identified with $\mathbb{T}$.

procedure divide(σ', σ'');
begin **if** (σ', σ'') is η-admissible **then** $P_{\text{far}} := P_{\text{far}} \cup \{(\sigma', \sigma'')\}$
else **if** (σ', σ'') is a leaf **then** $P_{\text{near}} := P_{\text{near}} \cup \{(\sigma', \sigma'')\}$
else **for all** $(c', c'') \in \text{sons}(\sigma', \sigma'')$ **do divide**(c', c'')
end;

As a result of this call, $P := P_{\text{near}} \cup P_{\text{far}}$ describes a non-overlapping covering of $\Gamma \times \Gamma$ in the sense that $\cup\{\sigma' \times \sigma'' : (\sigma', \sigma'') \in P\} = \Gamma \times \Gamma$ and all contributions $\sigma' \times \sigma''$ in this union intersect at most by their boundaries.

Now we describe how the matrix K is approximated, using this decomposition. For simplicity, we assume that the integration in P_{near} is done exactly although this could be approximated by quadrature. For the integration in P_{far}, we approximate the kernel $k(\mathbf{x}, \mathbf{y})$ as follows. Let $b = (\sigma', \sigma'') \in P_{\text{far}}$. For $\mathbf{x} \in \sigma'$, $\mathbf{y} \in \sigma''$, we use a separable approximation $k_b(\mathbf{x}, \mathbf{y}) \approx k(\mathbf{x}, \mathbf{y})$ of the form:

$$k_b(\mathbf{x}, \mathbf{y}) := \sum_{i \in I_{\sigma'},\ j \in I_{\sigma''}} \kappa_{i,j}(b) \Phi_{\sigma'}^{(i)}(\mathbf{x}) \Psi_{\sigma''}^{(j)}(\mathbf{y}) \tag{6.1}$$

with appropriate function systems $\{\Phi_{\sigma'}^{(i)} : i \in I_{\sigma'}\}$ and $\{\Psi_{\sigma''}^{(j)} : j \in I_{\sigma''}\}$ and expansion coefficients $\kappa_{i,j}(b)$.

For kernel functions which are related to linear elliptic PDEs of second order with constant coefficients one can prove (cf. [15], [23]) the exponential convergence estimate

$$|k(\mathbf{x}, \mathbf{y}) - k_b(\mathbf{x}, \mathbf{y})| \leq C_1 \frac{(\eta')^\ell}{\text{dist}(\sigma', \sigma'')^s}, \tag{6.2}$$

for all $\mathbf{x} \in \sigma', \mathbf{y} \in \sigma''$ and $b = (\sigma', \sigma'') \in P_{\text{far}}$, where $\eta' = C_2 \eta$ for some constant C_2 and s is the blow-up rate of the kernel

$$|k(\mathbf{x}, \mathbf{y})| \leq C_3 |\mathbf{x} - \mathbf{y}|^{-s}, \qquad \mathbf{x}, \mathbf{y} \in \Gamma,\ \mathbf{x} \neq \mathbf{y}. \tag{6.3}$$

Note that the constants C_1 and C_2 are independent of ℓ while the cardinality of the index sets $I_{\sigma'}, I_{\sigma''}$ depends on ℓ. In the following, we assume that (6.2) holds.

The panel-clustering approximation of the bilinear form a in (4.1) acting on the finite-dimensional space $\mathcal{S}_i^m(\mathcal{T}) \times \mathcal{S}_i^m(\mathcal{T})$ is given by

$$\tilde{a}(V, W) = ((\lambda I + \tilde{\mathcal{K}})V, W), \text{ with } \tilde{\mathcal{K}}v(\mathbf{x}) = \int_\Gamma \tilde{k}(\mathbf{x}, \mathbf{y}) V(\mathbf{y}) dy, \text{ for } V, W \in \mathcal{S}_i^m(\mathcal{T}), \tag{6.4}$$

and

$$\tilde{k}(\mathbf{x}, \mathbf{y}) := \begin{cases} k(\mathbf{x}, \mathbf{y}) & \mathbf{x} \in \sigma', \mathbf{y} \in \sigma'' \text{ with } b = (\sigma', \sigma'') \in P_{\text{near}}, \\ k_b(\mathbf{x}, \mathbf{y}) & \mathbf{x} \in \sigma', \mathbf{y} \in \sigma'' \text{ with } b = (\sigma', \sigma'') \in P_{\text{far}}. \end{cases} \tag{6.5}$$

Since we are concerned here only with error estimates for this approximation, we do not discuss its implementation, but instead refer readers to [16, 14] for details.

Analogously to Theorem 5.1 we then have

Theorem 6.4 *Suppose that the assumptions of Lemma 4.1 hold and suppose we use the panel clustering algorithm described above to obtain an approximate bilinear form $\tilde{a}$. Then*

$$\frac{|a(V, W) - \tilde{a}(V, W)|}{\|V\|_{H^{\nu_1}(\Gamma)} \|W\|_{H^{\nu_2}(\Gamma)}} \lesssim (\eta')^{\ell+s} \{\#\mathcal{T}\} \max_{t, \tau \in \mathcal{T}} \Lambda_{t,\tau}^s \tag{6.6}$$

uniformly in $V, W \in \mathcal{S}_i^m(\mathcal{T})$, where

$$\Lambda_{t,\tau}^s := \max\{h_t, h_\tau\}^{1-s} \left\{\rho_t^{\nu_1^- + 1/2} \rho_\tau^{\nu_2^- + 1/2}\right\} \qquad \text{and} \quad \nu_i^- := \min\{\nu_i, 0\}\,,\ i = 1, 2.$$

The asymptotic constant in (6.6) may depend on m.

Proof. By (6.4), (6.5) and (6.2),

$$\begin{aligned} |a(V,W) - \tilde{a}(V,W)| &= \left| \sum_{b=(\sigma',\sigma'')\in P_{\text{far}}} \int_{\sigma'} \int_{\sigma''} V(\mathbf{x})(k(\mathbf{x},\mathbf{y}) - k_b(\mathbf{x},\mathbf{y}))W(\mathbf{y})dydx \right| \\ &\lesssim \sum_{b=(\sigma',\sigma'')\in P_{\text{far}}} \frac{(\eta')^\ell}{\text{dist}\,(\sigma',\sigma'')^s} \int_{\sigma'} |V| \int_{\sigma''} |W| \\ &= \sum_{b=(\sigma',\sigma'')\in P_{\text{far}}} \frac{(\eta')^\ell}{\text{dist}\,(\sigma',\sigma'')^s} \sum_{\substack{t,\tau\in\mathcal{T} \\ t\subset\sigma',\tau\subset\sigma''}} \int_t |V| \int_\tau |W|\,. \end{aligned} \tag{6.7}$$

Now if $t, \tau \in \mathcal{T}$, $t \subset \sigma'$, $\tau \subset \sigma''$ and $(\sigma', \sigma'') \in P_{\text{far}}$, then (σ', σ'') is η-admissible and we have

$$\eta\,\text{dist}(t,\tau) \geq \eta\,\text{dist}(\sigma',\sigma'') \geq \max\{\text{diam}\,\sigma', \text{diam}\,\sigma''\} \geq \max\{\text{diam}\,t, \text{diam}\,\tau\}\,, \tag{6.8}$$

which shows that $(t, \tau) \in \mathbb{T}_2$ is η-admissible. Since the procedure **divide** implies that each such η-admissible (t, τ) belongs to a unique far field block $(\sigma_t', \sigma_\tau'') \in P_{\text{far}}$, we can rewrite (6.7) as

$$|a(V,W) - \tilde{a}(V,W)| \quad \lesssim \sum_{\substack{t,\tau\in\mathcal{T} \\ (t,\tau)\ \eta-\text{admissible}}} \frac{(\eta')^\ell}{\text{dist}\,(\sigma_t',\sigma_\tau'')^s} \int_t |V| \int_\tau |W|\,. \tag{6.9}$$

Because of the properties of $(\sigma_t', \sigma_\tau'')$,

$$\text{dist}(\sigma_t',\sigma_\tau'') \geq \eta^{-1}\max\{\text{diam}(\sigma_t'), \text{diam}(\sigma_\tau'')\} \geq \eta^{-1}\max\{h_t, h_\tau\} \gtrsim (\eta')^{-1}\max\{h_t, h_\tau\}, \tag{6.10}$$

where the constant of proportionality is independent of η. Moreover, for any $V \in \mathcal{S}_i^m(\mathcal{T})$ and any $\tau \in \mathcal{T}$, we have, by Assumption 2.6, for any $\nu \in \mathbb{R}$,

$$\int_\tau |V| \quad \sim \quad \int_\tau \rho_\tau^{\nu^-} |\rho^{-\nu^-} V| \quad \leq \quad \sqrt{|\tau|\rho_\tau^{2\nu^-}}\, \|\rho^{-\nu^-} V\|_{L^2(\tau)}\,,$$

where $\nu^- = \min\{\nu, 0\}$. Inserting these last two results into (6.9), we obtain

$$\begin{aligned} &|a(V,W) - \tilde{a}(V,W)| \\ &\quad\lesssim \sum_{\substack{t,\tau\in\mathcal{T} \\ (t,\tau)\ \eta-\text{admissible}}} \frac{(\eta')^{\ell+s}}{\max\{h_t, h_\tau\}^s} \sqrt{|t|\rho_t^{2\nu_1^-}\,|\tau|\rho_\tau^{2\nu_2^-}}\ \|\rho^{-\nu_1^-} V\|_{L^2(t)} \|\rho^{-\nu_2^-} W\|_{L^2(\tau)} \\ &\quad\lesssim (\eta')^{\ell+s} \max_{t,\tau\in\mathcal{T}} \frac{\sqrt{|t|\rho_t^{2\nu_1^-}\,|\tau|\rho_\tau^{2\nu_2^-}}}{\max\{h_t, h_\tau\}^s} \sum_{t\in\mathcal{T}} \|\rho^{-\nu_1^-} V\|_{L^2(t)} \sum_{\tau\in\mathcal{T}} \|\rho^{-\nu_2^-} W\|_{L^2(\tau)}\,. \end{aligned} \tag{6.11}$$

Now observe that (since here Γ is a two-dimensional manifold), for all $t, \tau \in \mathcal{T}$,

$$\frac{\sqrt{|t|\rho_t^{2\nu_1^-}|\tau|\rho_\tau^{2\nu_2^-}}}{\max\{h_t, h_\tau\}^s} \sim \frac{\sqrt{h_t h_\tau}}{\max\{h_t, h_\tau\}^s}\left\{\rho_t^{\nu_1^-+1/2}\rho_\tau^{\nu_2^-+1/2}\right\} \leq \max\{h_t, h_\tau\}^{1-s}\left\{\rho_t^{\nu_1^-+1/2}\rho_\tau^{\nu_2^-+1/2}\right\}.$$

Hence the result follows from (6.11) on application of the Cauchy-Schwarz inequality and using Theorem 3.6. ■

In Theorem 6.4, ℓ and η are parameters which should be chosen to ensure the required overall stability and consistency estimate in the Strang Lemma (Lemma 4.1). (They also have to be chosen so that the complexity of the panel-clustering approximation is optimised.) With respect to the former requirement, Theorem 6.4 leads to the following corollary.

Corollary 6.5 *Under the assumptions of Theorem 6.4, we have*

(i) For the single layer potential: $r_{\text{stab}} \lesssim (\eta')^{\ell+1}\{\#\mathcal{T}\}$,

(ii) For the double layer potential: $r_{\text{stab}} \lesssim (\eta')^{\ell+2}\{\#\mathcal{T}\}$,

(iii) For the hypersingular operator: $r_{\text{stab}} \lesssim (\eta')^{\ell+3}\{\#\mathcal{T}\}\max_{\tau\in\mathcal{T}}\{h_\tau^{-1}\}$.

According to Remark 4.2, the choice $\nu = \mu$ and $\delta = 0$ is optimal for the double layer potential and the hypersingular operator. In this case $r_{\text{stab}} = r_{\text{conv}}$.

If we assume for the single layer potential L^2-regularity, i.e., $\delta = 1/2$ in (4.11) we may choose $\nu = 0$ to obtain

$$r_{\text{conv}} \lesssim h^{1/2}(\eta')^{\ell+1}(\#\mathcal{T}).$$

Proof. Putting $\nu_1 = \mu = \nu_2$ in the result of Theorem 6.4, we obtain

$$r_{\text{stab}} \lesssim (\eta')^{\ell+s}\{\#\mathcal{T}\}\max_{t,\tau\in\mathcal{T}}\left\{\max\{h_t, h_\tau\}^{1-s}\{\rho_t\rho_\tau\}^{\mu^-+1/2}\right\}. \tag{6.12}$$

The estimate (i) follows easily since, for the single layer potential, $s = 1$ and $\mu = -1/2$.

The estimate for r_{conv} in the case $\delta = 1/2$ and $\nu = 0$ follows from (6.6) with $s = 1$, $\nu_1 = 0$, $\nu_2 = -1/2$:

$$\begin{aligned} r_{\text{conv}} &\lesssim (\eta')^{\ell+s}\{\#\mathcal{T}\}\max_{t,\tau\in\mathcal{T}}\left\{\rho_t^{\nu_1^-+1/2}\rho_\tau^{\nu_2^-+1/2}\max\{h_t, h_\tau\}^{1-s}\right\} \\ &= (\eta')^{\ell+1}\{\#\mathcal{T}\}\left(\max_{t\in\mathcal{T}}\rho_t^{1/2}\right) \lesssim h^{1/2}(\eta')^{\ell+1}\{\#\mathcal{T}\}. \end{aligned}$$

For the double layer potential (on a polyhedron) we have $s = 2$ and $\mu = 0$. Then (6.12) leads to

$$r_{\text{stab}} \lesssim (\eta')^{\ell+2}\{\#\mathcal{T}\}\max_{t,\tau\in\mathcal{T}}\left\{\sqrt{\rho_t\rho_\tau}\max\{h_t, h_\tau\}^{-1}\right\}.$$

Since $\max\{h_t, h_\tau\}^{-1} \leq \{h_t h_\tau\}^{-1/2}$, and since $\rho_\tau \leq h_\tau$, for all $\tau \in \mathcal{T}$, the result (ii) follows.

In the hypersingular case $s = 3$ and $\mu = 1/2$ and (6.12) readily yields

$$r_{\text{stab}} \lesssim (\eta')^{\ell+3}\{\#\mathcal{T}\}\max_{t,\tau\in\mathcal{T}}\max\{h_t, h_\tau\}^{-1},$$

which yields the required result. ■

Example 6.6 *Consider the screen problem on* $\Gamma = (0,1)^2 \subset \mathbb{R}^3$ *with the single layer potential operator: For given* $f \in H^{1/2}(\Gamma)$, *find* $u \in H^{-1/2}(\Gamma)$ *such that*

$$\int_\Gamma \frac{u(\mathbf{y})\, v(\mathbf{x})}{4\pi |\mathbf{x}-\mathbf{y}|}\, dx\, dy = \int_\Gamma f(\mathbf{x})\, v(\mathbf{x})\; dx \qquad \forall v \in H^{-1/2}(\Gamma).$$

The Galerkin discretisation with discontinuous, piecewise linear boundary elements on the graded mesh $\mathcal{T}$ *as described in Example 2.7 leads to a solution* $U \in \mathcal{S}_0^1(\mathcal{T})$ *which satisfies the quasi-optimal error estimate (cf. [19], [23])*

$$\|u - U\|_{H^{-1/2}(\Gamma)} \leq C N^{-\min\{g-\varepsilon,5\}/4},$$

where $0 < \varepsilon < 1$ *is arbitrary but fixed,* $N = \dim \mathcal{S}_0^1(\mathcal{T})$, *and* g *denotes the grading exponent. The choice* $g > 5$ *leads to the optimal convergence rate of* $N^{-5/4}$.

For this problem, Corollary 6.5 (i) tells us that the condition $(\eta')^{\ell+1} N \overset{!}{\lesssim} N^{-5/4}$ *is sufficient for optimal convergence. Hence, the Galerkin solution converges with optimal rate if the expansion order for the panel clustering algorithm is chosen according to*

$$\ell = \left\lceil \frac{9}{4} \frac{\log N}{|\log \eta'|} \right\rceil.$$

The estimates given in Corollary 6.5 for the single and double layer potentials are clearly unaffected by any mesh degeneracy. However in the case of the hypersingular operator, a negative power of the minimum diameter occurs. This is not a severe deficiency, but nevertheless it can be removed if we reformulate the hypersingular equation using the concept of partial integration. This we describe in the following final subsection.

6.2 Hypersingular Operator with Partial Integration

The integral of the kernel of the hypersingular operator in (4.2c) does not exist as an improper integral and has to be defined as a *finite part integral.* Various regularisation methods for hypersingular integrals exist in the literature and we choose here the method of partial integration (cf. [17]). Since the constant functions span the kernel of the hypersingular operator, we introduce the quotient space $\hat{H}^{1/2}(\Gamma) := H^{1/2}(\Gamma)/\mathbb{R}$ with norm $\|u\|_{\hat{H}^{1/2}(\Gamma)} := \inf_{c\in\mathbb{R}} \|u - c\|_{H^{1/2}(\Gamma)}$. The bilinear form $a : \hat{H}^{1/2}(\Gamma) \times \hat{H}^{1/2}(\Gamma) \to \mathbb{R}$ which is associated with the hypersingular operator can be written in the form

$$a(u,v) = \int_{\Gamma\times\Gamma} \frac{\left\langle \overrightarrow{\operatorname{curl}}_\Gamma v(\mathbf{x}), \overrightarrow{\operatorname{curl}}_\Gamma u(\mathbf{y}) \right\rangle}{4\pi |\mathbf{x}-\mathbf{y}|}\, dx\, dy, \tag{6.13}$$

where the *tangential rotation* $\overrightarrow{\operatorname{curl}}_\Gamma$ is defined as follows (cf. [17]). For functions $u \in H^{1/2}(\Gamma)$ and surface vector fields having componentwise differentiable extensions $\tilde{u}$ and $\tilde{v}$, respectively, in $H^1(\mathcal{U})$, where $\mathcal{U}$ is some three-dimensional neighbourhood of Γ, we define the *tangential gradient* $\nabla_\Gamma u$ as the restriction of the Euclidean gradient to the surface Γ

$$\nabla_\Gamma u := (\nabla \tilde{u})|_\Gamma .$$

This enables us to introduce the tangential rotation of u as

$$\overrightarrow{\operatorname{curl}}_\Gamma u := -n \times \nabla_\Gamma u.$$

Since the energy space for (6.13) is $\hat{H}^{1/2}(\Gamma)$ we must use continuous piecewise polynomials $\widehat{\mathcal{S}}_1^m(\mathcal{T}) := \mathcal{S}_1^m(\mathcal{T})/\mathbb{R}$ for its discretisation. To control the effect of the approximation of the bilinear form $a(\cdot,\cdot)$ by the panel clustering algorithm, the estimate of the quantities

$$r_{\text{stab}} := \frac{|a(V,W) - \tilde{a}(V,W)|}{\|V\|_{\hat{H}^{1/2}(\Gamma)} \|W\|_{\hat{H}^{1/2}(\Gamma)}} \quad \text{and} \quad r_{\text{conv}} := \sup_{V,W \in \mathcal{S}_i^m(\mathcal{T})\setminus\{0\}} \frac{|a(V,W) - \tilde{a}(V,W)|}{\|V\|_{\hat{H}^{\nu}(\Gamma)} \|W\|_{\hat{H}^{1/2}(\Gamma)}} \tag{6.14}$$

for all $V, W \in \mathcal{S}_1^m(\mathcal{T}) \setminus \{0\}$ plays the essential rôle. The index ν of differentiation must satisfy (4.13) with $\mu = 1/2$ and $\hat{H}^{\nu}(\Gamma) := \hat{H}^{1/2}(\Gamma) \cap H^{\nu}(\Gamma)$.

For simplicity, we assume that Γ is the surface of a polyhedron

$$\Gamma = \bigcup_{i=1}^{q} \Gamma_i \quad \text{where every } \Gamma_i, 1 \le i \le q, \text{ is planar.} \tag{6.15}$$

As a consequence, the normal n is constant on every panel $\tau \in \mathcal{T}$ and

$$\overrightarrow{\operatorname{curl}_\Gamma}\, \widehat{\mathcal{S}}_1^m(\mathcal{T}) = \overrightarrow{\operatorname{curl}_\Gamma}\, \mathcal{S}_1^m(\mathcal{T}) := \left\{ \overrightarrow{\operatorname{curl}_\Gamma}\, u : \ u \in \mathcal{S}_1^m(\mathcal{T}) \right\} \subset \left(\mathcal{S}_0^{m-1}(\mathcal{T}) \right)^3. \tag{6.16}$$

Theorem 6.7 *Let r_{stab}, r_{conv} be defined as in (6.14) with respect to the bilinear form $a(\cdot,\cdot)$ as in (6.13) and denote its panel-clustering approximation by $\tilde{a}(\cdot,\cdot)$. Assume that Γ satisfies (6.15). Then the stability estimate:*

$$r_{\text{stab}} \quad \lesssim \quad (\eta')^{\ell+1} \{\#\mathcal{T}\}$$

holds uniformly in $V, W \in \mathcal{S}_1^m(\mathcal{T}) \setminus \{0\}$. The choice $\nu = 1/2$ leads to $r_{\text{conv}} = r_{\text{stab}}$.

Assume that the continuous problem has regularity $\delta = 1/2$. In this case, we have

$$r_{\text{conv}} \quad \lesssim \quad h^{1/2} (\eta')^{\ell+1} (\#\mathcal{T}).$$

Proof. It is well known that the bilinear form $a(\cdot,\cdot)$ is elliptic and continuous in $\hat{H}^{1/2}(\Gamma)$ (cf. [17]). In view of the inclusion in (6.16) we may use the inverse inequality from Theorem 3.4 for $s = 1$, $\alpha = 1/2$ and interpolate with the trivial identity $\|\rho^{1/2} V\|_{H^1(\Omega)} = \|\rho^{1/2} V\|_{H^1(\Omega)}$ to obtain

$$\left\| \rho^{1/2} \overrightarrow{\operatorname{curl}_\Gamma}\, V \right\|_{(L^2(\Gamma))^3} \le \left\| \rho^{1/2} V \right\|_{H^1(\Gamma)} \lesssim \|V\|_{H^{1/2}(\Gamma)} \qquad \text{for all } V \in \mathcal{S}_1^m.$$

Since $\overrightarrow{\operatorname{curl}_\Gamma} V = 0$ for every constant function V, the estimate

$$\left\| \rho^{1/2} \overrightarrow{\operatorname{curl}_\Gamma}\, V \right\|_{L^2(\Gamma)} \quad \lesssim \quad \|V\|_{\hat{H}^{1/2}(\Gamma)} \qquad \text{for all } V \in \mathcal{S}_1^m$$

follows. Hence,

$$\frac{|a(V,W) - \tilde{a}(V,W)|}{\|V\|_{\hat{H}^{1/2}(\Gamma)} \|W\|_{\hat{H}^{1/2}(\Gamma)}} \quad \lesssim \quad \frac{|a(V,W) - \tilde{a}(V,W)|}{\|\rho^{1/2} V^c\|_{L^2(\Gamma)} \|\rho^{1/2} W^c\|_{L^2(\Gamma)}}, \tag{6.17}$$

where $V^c := \overrightarrow{\operatorname{curl}_\Gamma}\, V$ and $W^c := \overrightarrow{\operatorname{curl}_\Gamma}\, W$. Now, by repeating the steps in the proof of Theorem 6.4, we obtain the estimate for $a - \tilde{a}$:

$$\begin{aligned} |a(V,W) - \tilde{a}(V,W)| &= \sum_{b=(\sigma',\sigma'') \in P_{\text{far}}} \int_{\sigma'} \int_{\sigma''} \langle V^c(\mathbf{x}), W^c(\mathbf{y}) \rangle \left| k(\mathbf{x},\mathbf{y}) - k_b(\mathbf{x},\mathbf{y}) \right| \, dx\, dy \\ &\lesssim \sum_{\substack{t,\tau \in \mathcal{T} \\ (t,\tau)\ \eta-\text{admissible}}} \frac{(\eta')^{\ell}}{\operatorname{dist}(\sigma'_t, \sigma''_\tau)} \int_t |V^c| \int_\tau |W^c|. \end{aligned} \tag{6.18}$$

Moreover, for any $V \in \mathcal{S}_0^{m-1}(\mathcal{T})$ and any $\tau \in \mathcal{T}$, we have, by Assumption 2.6,

$$\int_t |V^c| \sim \int_t \rho_t^{-1/2} \left|\rho^{1/2} V^c\right| \leq \left\{|t| \, \rho_t^{-1}\right\}^{1/2} \left\|\rho^{1/2} V^c\right\|_{L^2(t)} \lesssim h_t^{1/2} \left\|\rho^{1/2} V^c\right\|_{L^2(t)}$$

Inserting this and (6.10) into (6.18), we obtain

$$|a(V,W) - \tilde{a}(V,W)| \lesssim (\eta')^{\ell+1} \max_{\tau,t\in\mathcal{T}} \left\{ \frac{(h_t h_\tau)^{1/2}}{\max\{h_t, h_\tau\}} \right\} \sum_{t\in\mathcal{T}} \left\|\rho^{1/2} V^c\right\|_{L^2(t)} \sum_{\tau\in\mathcal{T}} \left\|\rho^{1/2} W^c\right\|_{L^2(\tau)}. \tag{6.19}$$

Since $\sqrt{h_t h_\tau} \leq \max\{h_t, h_\tau\}$, the maximum appearing in (6.19) is bounded from above by 1 and we obtain by the Cauchy-Schwarz inequality the final estimate

$$|a(V,W) - \tilde{a}(V,W)| \lesssim (\eta')^{\ell+1} (\#\mathcal{T}) \left\|\rho^{1/2} V^c\right\|_{L^2(\Gamma)} \left\|\rho^{1/2} W^c\right\|_{L^2(\Gamma)}.$$

Combining this with (6.17) yields the estimate for r_{stab}.

Assume that the problem has regularity $\delta = 1/2$. Repeating the proof for r_{stab} but applying the inverse estimate only for the function W, we obtain

$$|a(V,W) - \tilde{a}(V,W)| \lesssim h^{1/2} (\eta')^{\ell+1} (\#\mathcal{T}) \|V\|_{\hat{H}^1(\Gamma)} \|W\|_{\hat{H}^{1/2}(\Gamma)}.$$

■

Remark 6.8 *Theorem 6.7 shows that the negative power of h in Corollary 6.5 (iii) can be avoided by applying the panel clustering algorithm to the kernel in (6.13) and not to the hypersingular kernel function in its original form (4.2c). In addition, we gain an additional factor* $h^{1/2}$ *in the error estimate by employing the regularity of the solution.*

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