Pseudocodeword Weights and Stopping Sets

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Abstract — **We examine the structure of pseudocodewords in Tanner graphs and derive lower bounds of pseudocodeword weights. The weight of a pseudocodeword is related to the size of its support set, which forms a stopping set in the Tanner graph.**

I. Introduction

Recent works [1][2][3][4] have revealed that *pseudocodewords* of Tanner graph play analogous roles in determining convergence of an iterative decoder as *codewords* for a maximum likelihood decoder. The mimimal weight pseudocodeword [3] is more fundamental than the minimal weight codeword in the context of iterative decoding. In this note, we study the structure of pseudocodewords of an LDPC graph and derive lower bounds on the minimal pseudocodeword weight, assuming *min-sum* iterative decoding as in [1]. It has been observed that pseudocodewords are essentially stopping sets [3] in the case of the binary erasure channel (BEC), and hence the minimal pseudocodeword weight $w_{\min}$ is equal to the minimum stopping set size $s_{\min}$. This prompts us to examine how $w_{\min}$ and $s_{\min}$ relate over other channels such as the BSC and the AWGN channels.

II. Stopping Sets and Pseudocodewords

Let G be a bipartite graph representing a binary LDPC code C. Then a *stopping set* in G is a subset S of variable nodes whose neighbors are each connected to S at least twice. The smallest stopping set, with size denoted by $s_{\min}$, is called the *minimum* stopping set, and is not necessarily unique.

We refer to [4] for the definition of a degree ℓ cover (lift) $\hat{G}$ of G. A *pseudocodeword* $\mathbf{p} = [p_1, p_2, \ldots, p_n]$ is a vector of integer entries where p_i represents the number of variables nodes of value 1 in a lift $\hat{G}$ that are lifts of the node v_i of the base graph G, where the values assigned to the variable nodes in the lift correspond to a valid codeword configuration [2]. We will refer to [3] for the definition of pseudocodeword weights on different channels. The *minimal pseudocodeword weight* of G is the minimum weight over all pseudocodewords that occur over all possible lifts of G, and is denoted by $w_{\min}^{BSC/AWGN}$ for the BSC/AWGN channel.

III. Bounds on Minimal Pseudocodeword Weights

We first observe that the support of a pseudocodeword $\mathbf{p}$ forms a stopping set in G. The support size of a pseudocodeword $\mathbf{p}$ has been shown to upper bound its weight on the BSC/AWGN channel [3], implying $w_{\min}^{BSC/AWGN} \leq s_{\min}$. We establish the following lower bounds for the minimal pseudocodeword weight:

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Theorem III.1 *Let G be a d-left regular bipartite graph with girth g. Then the minimal pseudocodeword weight is lower bounded by*

$$w_{\min}^{BSC/AWGN} \geq \begin{cases} 1 + d + d(d-1) + .. + d(d-1)^{\frac{g-6}{4}}, & \frac{g}{2} \text{ odd} \\ 1 + d + .. + d(d-1)^{\frac{g-8}{4}} + (d-1)^{\frac{g-4}{4}}, & \frac{g}{2} \text{ even} \end{cases}$$

Note that this lower bound holds analogously for $s_{\min}$ and the minimum distance $d_{\min}$ of G. For generalized LDPC codes, wherein the right nodes in G of degree k represent constraints of a $[k, k', \epsilon k]$ sub-code, the above result is extended as:

Theorem III.2 *Let G be a (d,k)-regular bipartite graph with girth g and the right nodes represent constraints of a $[k, k', \epsilon k]$ subcode. Then:*

$$w_{\min} \geq \begin{cases} 1 + dx + d(d-1)x^2 + .. + d(d-1)^{\frac{g-6}{4}} x^{\frac{g-2}{4}}, & \frac{g}{2} \text{ odd} \\ 1 + dx + .. + d(d-1)^{\frac{g-8}{4}} x^{\frac{g-4}{4}} + (d-1)^{\frac{g-4}{4}} x^{\frac{g}{4}}, & \frac{g}{2} \text{ even} \end{cases}$$

for the BSC/AWGN channels, where $x = (\epsilon k - 1)$.

In the generalized case, a stopping set may be defined as a set of variable nodes S whose neighbors are each connected at least ϵk times to S in G. By this definition, a similar lower bound holds for $s_{\min}$ also.

Lemma III.1 *Suppose in an LDPC contraint graph G every irreducible pseudocodeword (generalizing the definition in [1]) $\mathbf{p} = [p_1, p_2, \ldots, p_n]$ with support set V has components $0 \leq p_i \leq t$, for $1 \leq i \leq n$, then: (a) $w^{AWGN}(\mathbf{p}) \geq \frac{2t^2}{(1+t^2)(t-1)}|V|$, and (b) $w^{BSC}(\mathbf{p}) \geq \frac{1}{t}|V|$.*

It is worth noting that for any pseudocodeword $\mathbf{p}$, $w^{BSC/AWGN}(\mathbf{p}) \geq w_{\texttt{max-frac}}(\mathbf{p})$, where $w_{\texttt{max-frac}}(\mathbf{p})$ is the max-fractional weight of $\mathbf{p}$ as introduced in [5]. Therefore, $w_{\min}^{BSC/AWGN} \geq d_{frac}^{\max}$, the max-fractional distance which is the minimum max-fractional weight over all $\mathbf{p}$. Consequently, the bounds established in [5] for $d_{frac}^{\max}$ are also lower bounds for $w_{\min}$. In addition,

Theorem III.3 *For an (n,k,d) code represented by an LDPC constraint graph G: (a) if $\mathbf{p}$ is a* good *pseudocodeword [2] of G, then $w^{BSC/AWGN}(\mathbf{p}) \geq w_{\texttt{max-frac}}(\mathbf{p}) \geq d_{\min}$, and (b) if $\mathbf{p}$ is a* bad *pseudocodeword [2] of G, then $w^{BSC/AWGN}(\mathbf{p}) \geq w_{\texttt{max-frac}}(\mathbf{p}) \geq \frac{2s_{\min}}{t}$, where t is as in the previous lemma.*

References

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