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Partial differential equations

KAM for quasi-linear KdV

KAM pour KdV quasi-linéaire

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ABSTRACT

We prove the existence and stability of Cantor families of quasi-periodic, small-amplitude solutions of *quasi-linear* autonomous Hamiltonian perturbations of KdV.

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RÉSUMÉ

Nous prouvons l'existence de solutions quasi périodiques linéairement stables pour des perturbations hamiltoniennes autonomes *quasi linéaires* de l'équation KdV.

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1. Main result

The aim of this Note is to present the recent results in [3], concerning the existence and stability of Cantor families of small-amplitude quasi-periodic solutions for Hamiltonian *quasi-linear* (also called "strongly nonlinear", e.g. in [8]) perturbations of the KdV equation

$$u_t + u_{xxx} - 6uu_x + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \tag{1}$$

under periodic boundary conditions $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, where

$$\mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) := -\partial_x\big[(\partial_u f)(x, u, u_x) - \partial_x\big((\partial_{u_x} f)(x, u, u_x)\big)\big] \tag{2}$$

is the most general quasi-linear Hamiltonian (local) nonlinearity. Eq. (1) is the Hamiltonian PDE $u_t = \partial_x \nabla H(u)$ where $\nabla H(u)$ denotes the $L^2(\mathbb{T}_x)$ gradient of the Hamiltonian

$$H(u) = \int_{\mathbb{T}} \frac{u_x^2}{2} + u^3 + f(x, u, u_x)\,\mathrm{d}x \tag{3}$$

on the phase space $H_0^1(\mathbb{T}_x) := \{u(x) \in H^1(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} u(x)\,\mathrm{d}x = 0\}$.

We assume that the "Hamiltonian density" $f \in C^q(\mathbb{T} \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ for some q large enough, and that

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$$f = f_5(u, u_x) + f_{\geq 6}(x, u, u_x), \tag{4}$$

where $f_5(u, u_x)$ denotes the homogeneous component of f of degree 5 in (u, u_x) and $f_{\geq 6}$ collects all the higher-order terms. By (4), the nonlinearity $\mathcal{N}_4$ in (2) vanishes with order 4 at $u = 0$ and (1) may be seen, close to the origin, as a "small" perturbation of the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0, \tag{5}$$

which is completely integrable. Actually, the KdV equation (5) may be described by global analytic action-angle variables, see [6] and the references therein.

A natural question is to know whether the quasi-periodic solutions of (5) persist under small perturbations. This is the main content of KAM theory.

The first KAM results for KdV have been proved by Kuksin [7], and then by Kappeler and Pöschel [6], for semilinear Hamiltonian perturbations $\varepsilon\partial_x(\partial_u f)(x, u)$, namely when the density f is independent of u_x, so that (2) is a differential operator of order 1 (such perturbations are called "quasi-linear" in [8]). The key point is that the frequencies of KdV grow as $\sim j^3$ and the difference $|j^3 - i^3| \geq (j^2 + i^2)/2$, $i \neq j$, so that KdV gains (outside the diagonal $i = j$) two derivatives. This approach also works for Hamiltonian pseudo-differential perturbations of order 2 (in space), using the improved Kuksin's lemma in Liu and Yuan [9]. However it does *not* work for a quasi-linear perturbation as in (2), which is a nonlinear differential operator of the *same* order (i.e. 3) as the constant coefficient linear operator ∂_{xxx}. Such a strongly nonlinear perturbation makes the KAM question quite delicate because of the possible phenomenon of formation of singularities in finite time, see e.g. [8, section 1.5]. Concerning this issue, Kappeler and Pöschel [6, Remark 3, page 19] wrote:

"*It would be interesting to obtain perturbation results which also include terms of higher order, at least in the region where the KdV approximation is valid. However, results of this type are still out of reach, if true at all*".

Theorem 1.1, proved in [3], provides the first positive answer to this problem, at least for small-amplitude solutions. Note that (1) is a *completely resonant* PDE, namely the linearized equation at the origin is the linear Airy equation $u_t + u_{xxx} = 0$, which possesses only the 2π-periodic in time solutions

$$u(t, x) = \sum_{j \in \mathbb{Z} \setminus \{0\}} u_j e^{\mathrm{i}j^3 t} e^{\mathrm{i}jx}.$$

Thus the existence of quasi-periodic solutions of (1) is a purely nonlinear phenomenon (the diophantine frequencies in (7) are $O(|\xi|)$-close to integers and $\xi \to 0$).

The solutions that we find are localized in Fourier space close to finitely many "tangential sites"

$$S^+ := \{\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu\}, \quad \bar{\jmath}_i \in \mathbb{N} \setminus \{0\}, \quad \forall i = 1, \ldots, \nu, \qquad S := S^+ \cup \left(-S^+\right). \tag{6}$$

The set S is required to be even because the solutions u of (1) have to be real valued. Moreover, we also assume the following explicit hypotheses on S:

- (S1) $j_1 + j_2 + j_3 \neq 0$ for all $j_1, j_2, j_3 \in S$.
- (S2) $\nexists j_1, \ldots, j_4 \in S$ such that $j_1 + j_2 + j_3 + j_4 \neq 0$, $j_1^3 + j_2^3 + j_3^3 + j_4^3 - (j_1 + j_2 + j_3 + j_4)^3 = 0$.

Theorem 1.1. *Given $\nu \in \mathbb{N}$, let $f \in C^q$ (with $q := q(\nu)$ large enough) satisfy (4). Then, for all the tangential sites S as in (6) satisfying (S1)–(S2), the KdV equation (1) possesses small-amplitude quasi-periodic solutions with diophantine frequency vector $\omega := \omega(\xi) = (\omega_j)_{j \in S^+} \in \mathbb{R}^\nu$ of the form*

$$u(t, x) = \sum_{j \in S^+} 2\sqrt{\xi_j} \cos(\omega_j t + jx) + o\left(\sqrt{|\xi|}\right), \quad \omega_j := j^3 - 6\xi_j j^{-1}, \tag{7}$$

for a "Cantor-like" set of small amplitudes $\xi \in \mathbb{R}^\nu_+$ with density 1 at $\xi = 0$. The term $o(\sqrt{|\xi|})$ is small in some H^s-Sobolev norm, $s < q$. These quasi-periodic solutions are linearly stable.

Let us make some comments on this result.

(i) The set of tangential sites S satisfying (S1)–(S2) can be iteratively constructed in an explicit way. After fixing $\{\bar{\jmath}_1, \ldots, \bar{\jmath}_n\}$, in the choice of $\bar{\jmath}_{n+1}$ there are only finitely many forbidden values, while all the other infinitely many values are good choices for $\bar{\jmath}_{n+1}$. In this precise sense the set S is "generic".

(ii) The linear stability of a quasi-periodic solution $u(\omega t, x)$ means that there exists a set of symplectic coordinates (ψ, η, w), $\psi \in \mathbb{T}^\nu$, in which the linearized equation at u assumes the form

$$\dot{\psi} = K_{20}(\omega t)\eta + K_{11}^T(\omega t)w, \qquad \dot{\eta} = 0, \qquad \dot{w} - \partial_x K_{02}(\omega t)w = \partial_x K_{11}(\omega t)\eta.$$

The last PDE is a quasi-periodically forced Airy type equation that can be diagonalized into

$$\dot v_j + \mathrm{i}\mu_j^\infty v_j = f_j(\omega t), \quad j \in S^c, \quad \mu_j^\infty \in \mathbb{R}, \tag{8}$$

which is a sequence of uncoupled harmonic oscillators. Moreover, near the diophantine invariant torus, the Hamiltonian H assumes the (KAM) normal form (see [4])

$$K = \mathrm{const} + \omega \cdot \eta + \frac{1}{2} K_{20}(\psi)\eta \cdot \eta + \big(K_{11}(\psi)\eta, w\big)_{L^2(\mathbb{T})} + \frac{1}{2}\big(K_{02}(\psi)w, w\big)_{L^2(\mathbb{T})} + O\big(|\eta| + |w|\big)^3.$$

(iii) A similar result holds for perturbations of mKdV (both focusing and defocusing)

$$u_t + u_{xxx} \pm \partial_x u^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0$$

for tangential sites S satisfying $\frac{2}{2\nu-1}\sum_{i=1}^{\nu} \bar{j}_i^2 \notin \mathbb{Z}$. The KdV equation (1) is more difficult because the nonlinearity is quadratic and so its effects near the origin are stronger than for mKdV. An important point is that the fourth-order Birkhoff normal forms of both KdV and mKdV are completely integrable. The strategy in [3] for proving Theorem 1.1 could also be extended for generalized KdV equations with leading nonlinearity u^p by using the normal form techniques of Procesi–Procesi [11].

2. Strategy of the proof of Theorem 1.1

Weak Birkhoff normal form We decompose the phase space in the symplectic subspaces

$$H_0^1(\mathbb{T}_x) := H_S \oplus H_S^\perp, \quad H_S := \mathrm{span}\big\{e^{\mathrm{i}jx} : j \in S\big\}, \tag{9}$$

and, accordingly, we write $u = v + z$, where $v \in H_S$ is called the *tangential* variable and $z \in H_S^\perp$ the *normal* one. The dynamics of these two components is quite different. The variable v contains the largest oscillations of the quasi-periodic solution (7), while z remains much closer to the origin.

We write the KdV Hamiltonian (3) as $H = H_2 + H_3 + H_{\geq 5}$ where

$$H_2 := \int_{\mathbb{T}} \frac{v_x^2}{2}\,dx + \int_{\mathbb{T}} \frac{z_x^2}{2}\,dx, \qquad H_3 := \int_{\mathbb{T}} v^3\,dx + 3\int_{\mathbb{T}} v^2 z\,dx + 3\int_{\mathbb{T}} v z^2\,dx + \int_{\mathbb{T}} z^3\,dx, \tag{10}$$

and $H_{\geq 5} := \int_{\mathbb{T}} f(x, u, u_x)\,dx$. We perform a "weak" Birkhoff normal form (weak BNF), whose goal is to find an invariant manifold of solutions of the third-order approximation of Eq. (1), on which the dynamics is completely integrable. Thus we need to eliminate/normalize the monomials of H which are linear in z (this is the reason why we call this BNF only "weak"). Since the KdV nonlinearity is quadratic, two steps of weak BNF are required. We first remove the term $3\int_{\mathbb{T}} v^2 z\,dx$ of (10). Since v is Fourier supported on the finitely many sites S, these monomials are finitely many. As a consequence, the required Birkhoff map is the identity map plus a *finite-dimensional* nonlinear operator (with finite rank and which acts on a finite-dimensional space), see (11). The key advantage is that such a transformation modifies $\mathcal{N}_4$ only up to a finite dimensional remainder. Then a second step removes/normalizes also the monomials of order 4 that are linear in z. In order to construct a sufficiently good approximate solution such that the Nash–Moser iteration will converge, it is necessary to remove also the terms $O(v^5)$, $O(v^4 z)$ (these further steps of Birkhoff normal form are not required if the nonlinearity of the original PDE is yet cubic as for mKdV or in the KAM theorems [10]). This requires the hypothesis (S2) on the tangential sites. The following Birkhoff normal form proposition holds.

Proposition 2.1. *Assume* (S2). *Then there exists an analytic invertible symplectic map of the phase space $\Phi_B : H_0^1(\mathbb{T}_x) \to H_0^1(\mathbb{T}_x)$ of the form*

$$\Phi_B(u) = u + \Psi(u), \qquad \Psi(u) = \Pi_E \Psi(\Pi_E u), \tag{11}$$

where E is a finite-dimensional subspace such that the transformed Hamiltonian is

$$\mathcal{H} := H \circ \Phi_B = H_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \mathcal{H}_{\geq 6}, \tag{12}$$

where $H_2(u) := \frac{1}{2}\int_{\mathbb{T}} u_x^2\,dx$ and

$$\mathcal{H}_3 := \int_{\mathbb{T}} z^3\,dx + 3\int_{\mathbb{T}} v z^2\,dx, \qquad \mathcal{H}_4 := -\frac{3}{2}\sum_{j\in S}\frac{|u_j|^4}{j^2} + \mathcal{H}_{4,2} + \mathcal{H}_{4,3}, \qquad \mathcal{H}_5 := \sum_{q=2}^{5} R\big(v^{5-q} z^q\big),$$

$$\mathcal{H}_{4,2} := 6\int_{\mathbb{T}} v z \Pi_S\big((\partial_x^{-1} v)(\partial_x^{-1} z)\big)\,dx + 3\int_{\mathbb{T}} z^2 \pi_0\big(\partial_x^{-1} v\big)^2\,dx, \qquad \mathcal{H}_{4,3} := R\big(v z^3\big),$$

$R(v^n z^m)$ denotes a homogeneous polynomial of degree n in v and m in z, and $\mathcal{H}_{\geq 6}$ collects all the terms of order at least six in (v, z).

The weak normal form (12) does not remove (or normalize) the monomials $O(z^2)$. We do not give such stronger normal form (called "partial BNF" in Pöschel [10]) because the corresponding Birkhoff map is close to the identity only up to an operator of order $O(\partial_x^{-1})$, and so it would produce, in the transformed vector field $\mathcal{N}_4$, terms of order ∂_{xx} and ∂_x. A fortiori, we cannot either use the full BNF computed in [6], which normalizes all the fourth order monomials, because this Birkhoff map is only close to the identity up to a bounded operator. For the same reason, we do not use the global nonlinear Fourier transform in [6] (Birkhoff coordinates), which is close to the Fourier transform up to smoothing operators as $O(\partial_x^{-1})$.

Note that the Hamiltonian $\mathcal{H}$ in (12) possesses the invariant subspace $\{z = 0\}$ and the system restricted to $\{z = 0\}$ is completely integrable and non-isochronous (it is formed by ν decoupled rotators). The quasi-periodic solutions that we construct in (7) bifurcate from this invariant manifold.

Action-angle coordinates We introduce the rescaled symplectic action-angle variables on the tangential directions

$$u_j := \left(\varepsilon^2 \xi_j + \varepsilon^{2b}|j|y_j\right)^{1/2} \mathrm{e}^{\mathrm{i}\theta_j} \quad \text{if } j \in S, \qquad u_j := \varepsilon^b z_j \quad \text{if } j \in S^c, \quad b > 1. \tag{13}$$

After some calculations, (12) transforms into a Hamiltonian of the form

$$H_\varepsilon(\theta, y, z) = \alpha(\xi) \cdot y + \frac{1}{2}\left(N(\theta)z, z\right)_{L^2(\mathbb{T})} + P(\theta, y, z) \tag{14}$$

where $\alpha(\xi) = \bar{\omega} - \varepsilon^2 6\mathbb{A}\xi$, $\bar{\omega} := (j^3)_{j\in S^+}$, $\mathbb{A} := \mathrm{diag}\{j^{-1}\}_{j\in S^+}$, is the frequency-to-amplitude relation in (7), the normal form $\frac{1}{2}(N(\theta)z, z)_{L^2(\mathbb{T})}$ is quadratic in z and does not depend on the action variable y, and the Hamiltonian vector field of the perturbation P satisfies $X_P(\varphi, 0, 0) = O(\varepsilon^{6-2b})$, see [3, section 5].

Note that the normal form $N(\theta)$ in (14) depends on the angle θ, unlike those of the KAM theorems in [10,8]. This is because the weak BNF of Proposition 2.1 did *not* normalize the quadratic terms $O(z^2)$. These terms are dealt with two "linear BNF" transformations in the successive analysis of the linearized operator.

The nonlinear functional setting We look for an embedded invariant torus $i : \mathbb{T}^\nu \to \mathbb{T}^\nu \times \mathbb{R}^\nu \times H_S^\perp$, $\varphi \mapsto i(\varphi) := (\theta(\varphi), y(\varphi), z(\varphi))$ of the Hamiltonian vector field X_{H_ε} filled by quasi-periodic solutions with diophantine frequency ω,

$$|\omega \cdot l| \geq \gamma \langle l\rangle^{-\tau}, \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\}, \quad \gamma = o\left(\varepsilon^2\right). \tag{15}$$

The diophantine constant $\gamma = o(\varepsilon^2)$ because ω is ε^2-close to the integer vector $\bar{\omega}$ (the minimal condition is indeed $\gamma \leq c\varepsilon^2$ with c small). We shall also require that ω satisfies first- and second-order Melnikov non-resonance conditions. Actually, in this functional approach, the parameters are the frequencies ω. We choose in the Hamiltonian H_ε the unperturbed actions $\xi = \alpha^{-1}(\omega) = \varepsilon^{-2}\mathbb{A}^{-1}(\omega - \bar{\omega})$ and we look for a zero of the nonlinear operator

$$\mathcal{F}(i, \zeta, \omega) := \omega \cdot \partial_\varphi i - X_{H_{\varepsilon,\zeta}}(i) \tag{16}$$

where $X_{H_{\varepsilon,\zeta}}$ is the Hamiltonian vector field generated by $H_{\varepsilon,\zeta} := H_\varepsilon + \zeta \cdot \theta$ with $\zeta \in \mathbb{R}^\nu$. The unknowns in (16) are the embedded torus i and ζ. The frequency ω is a parameter. The auxiliary variable ζ is introduced in order to control the average in the y component of the linearized equation. If $\mathcal{F}(i, \zeta, \omega) = 0$, then $\zeta = 0$, and so $\varphi \mapsto i(\varphi)$ is an invariant torus for the Hamiltonian H_ε itself, see [4].

A solution of (16) is obtained by a Nash–Moser iterative scheme in Sobolev scales. The key step is to construct (for ω restricted to a suitable Cantor-like set) an approximate inverse (à la Zehnder) of the linearized operator at any approximate solution

$$d_{i,\zeta}\mathcal{F}(i_0)[\hat{\imath}, \hat{\zeta}] = \omega \cdot \partial_\varphi \hat{\imath} - d_i X_{H_\varepsilon}\left(i_0(\varphi)\right)[\hat{\imath}] + (0, \hat{\zeta}, 0). \tag{17}$$

This means to find a linear operator $\mathbf{T}_0$ such that $(d_{i,\zeta}\mathcal{F}(i_0) \circ \mathbf{T}_0 - I) = O(\gamma^{-1}\mathcal{F}(i_0, \zeta_0))$, see [3, Theorem 6.10]. Note in particular that $\mathbf{T}_0$ is an exact inverse of (17) at an exact solution $\mathcal{F}(i_0, \zeta_0) = 0$.

A major difficulty is that the tangential and the normal components of (17) are strongly coupled. This difficulty is overcome by implementing the abstract procedure in Berti–Bolle [4]: in a suitable set of symplectic variables, the "tangential" and the "normal" dynamics are almost decoupled and it remains to invert a Hamiltonian linear operator $\mathcal{L}_\omega$ of $H_S^\perp$. This is, up to a finite dimensional remainder, a quasi-periodic perturbed Airy operator with variable coefficients like

$$\mathcal{L}_\omega h = \Pi_S^\perp\left(\omega \cdot \partial_\varphi h + \partial_{xx}(a_1 \partial_x h) + \partial_x(a_0 h) - \varepsilon^2 \partial_x \mathcal{R}_2[h] - \partial_x \mathcal{R}_*[h]\right), \quad h \in H_S^\perp, \tag{18}$$

where $\Pi_S^\perp$ denotes the projection on $H_S^\perp$, the functions $a_1(\varphi, x), a_0(\varphi, x)$ are multiplicative coefficients, $\mathcal{R}_2, \mathcal{R}_*$ are finite-rank regularizing operators, and $\mathcal{R}_* = o(\varepsilon^2)$. The precise expression is in [3, section 7].

Reduction of the linearized operator in the normal directions The first task (obtained in [3, sections 8.1–8.6]) is to conjugate $\mathcal{L}_\omega$ to another Hamiltonian operator with constant coefficients

$$\mathcal{L}_6 := \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + m_1 \partial_x + R_6, \quad m_1, m_3 \in \mathbb{R}, \tag{19}$$

up to a small bounded remainder $R_6 = O(\partial_x^0)$. Such an expansion in "constant coefficients decreasing symbols" is similar to [2] and it is inspired by the work of Iooss, Plotnikov, Toland [5] in water waves theory, and [1] for Benjamin–Ono. The main perturbative effect to the spectrum of $\mathcal{L}_\omega$ is due to the term $a_1(\varphi, x)\partial_{xxx}$, which cannot be reduced to constants by the standard reducibility KAM techniques.

In order to eliminate the x-dependence from $a_1(\varphi, x)\partial_{xxx}$, we cannot use the symplectic transformation $\mathcal{A}(\varphi)u = (1 + \beta_x(\varphi, x))u(\varphi, x + \beta(\varphi, x))$ used in [2], because $\mathcal{L}_\omega$ acts on the normal subspace $H_S^\perp$ only, and not on the whole Sobolev space as in [2]. We need a symplectic diffeomorphism of $H_S^\perp$ near $\mathcal{A}_\perp := \Pi_S^\perp \mathcal{A} \Pi_S^\perp$ (which is *not* symplectic). The first observation is that, at each φ, $\mathcal{A}(\varphi)$ is the time 1-flow map of the linear Hamiltonian time-dependent transport PDE:

$$\partial_\tau u = \partial_x\big(b(\tau, x)u\big), \quad b(\tau, x) := \beta(x)\big(1 + \tau\beta_x(x)\big)^{-1}.$$

Hence we consider [3, section 8.1] the Hamiltonian flow map of the projected transport equation on $H_S^\perp$, which is Hamiltonian. This step may be seen as a quantitative application of the Egorov theorem which describes how the principal symbol of a pseudo-differential operator (here $a_1(\omega t, x)\partial_{xxx}$) transforms under the flow of a linear hyperbolic PDE. After a quasi-periodic reparameterization of time [3, section 8.2], we reduce to constant coefficients the term $O(\partial_{xxx})$ of $\mathcal{L}_\omega$ and we eliminate the term $O(\partial_{xx})$.

Since the weak BNF (12) did not normalize the quadratic terms $O(z^2)$, the operator $\mathcal{L}_\omega$ in (18) has variable coefficients also at the orders $O(\varepsilon)$ and $O(\varepsilon^2)$. These terms cannot be reduced to constants by the perturbative scheme in [2], which applies to operators R such that $O(R\gamma^{-1}) \ll 1$ where $\gamma = o(\varepsilon^2)$ is the diophantine constant of the frequency ω, see (15). These terms are reduced to constant coefficients in [3, sections 8.4 and 8.5] by means of purely algebraic arguments (linear BNF), which, ultimately, stem from the complete integrability of the fourth-order BNF of the KdV equation (5), see [6]. These Birkhoff transformations are symplectic maps of the form $I + \varepsilon O(\partial_x^{-1})$.

In [3, section 8.6] we complete the task of conjugating $\mathcal{L}_\omega$ to $\mathcal{L}_6$ in (19) via a symplectic transformation of the form $\exp(\Pi_S^\perp w \partial_x^{-1} \Pi_S^\perp)$. It is at this point that the assumption ($\mathtt{S}1$) on the tangential sites is used, see [3, Lemma 7.5]. If $f_5 = 0$ (see (4)) then ($\mathtt{S}1$) is not required. Finally, we apply the abstract reducibility [2, Theorem 4.2], which diagonalizes $\mathcal{L}_6$, and thus conjugate $\mathcal{L}_\omega$ to (see (8)):

$$\dot v_j + \mathrm{i}\mu_j v_j = 0, \quad j \notin S, \quad \mu_j := -m_3 j^3 + m_1 j + r_j \in \mathbb{R}, \quad m_3 - 1,\ m_1 = O\big(\varepsilon^4\big), \qquad \sup_j |r_j| = o\big(\varepsilon^2\big).$$

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