QUANTUM GIAMBELLI FORMULAS FOR ISOTROPIC GRASSMANNIANS

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Abstract. Let X be a symplectic or odd orthogonal Grassmannian which parametrizes isotropic subspaces in a vector space equipped with a nondegenerate (skew) symmetric form. We prove quantum Giambelli formulas which express an arbitrary Schubert class in the small quantum cohomology ring of X as a polynomial in certain special Schubert classes, extending the authors' cohomological Giambelli formulas.

0. Introduction

Let E be an even (respectively, odd) dimensional complex vector space equipped with a nondegenerate skew-symmetric (respectively, symmetric) bilinear form. Let X denote the Grassmannian which parametrizes the isotropic subspaces of E of some fixed dimension. The cohomology ring $\mathrm{H}^*(X, \mathbb{Z})$ is generated by certain special Schubert classes, which for us are (up to a factor of two) the Chern classes of the universal quotient vector bundle over X. These special classes also generate the small quantum cohomology ring $\mathrm{QH}(X)$, a q-deformation of $\mathrm{H}^*(X, \mathbb{Z})$ whose structure constants are the three point, genus zero Gromov-Witten invariants of X. In [BKT3], we proved a Giambelli formula in $\mathrm{H}^*(X, \mathbb{Z})$, that is, a formula expressing a general Schubert class as an explicit polynomial in the special classes. Our goal in the present work is to extend this result to a formula that holds in $\mathrm{QH}(X)$.

The quantum Giambelli formula for the usual type A Grassmannian was obtained by Bertram [Be], and is in fact identical to the classical Giambelli formula. In the case of maximal isotropic Grassmannians, the corresponding questions were answered in [KT1, KT2]. The main conclusions here are similar to those of loc. cit., provided that one uses the raising operator Giambelli formulas of [BKT3] as the classical starting point. For an odd orthogonal Grassmannian, we prove that the quantum Giambelli formula is the same as the classical one. The result is more interesting when X is the Grassmannian $\mathrm{IG}(n-k, 2n)$ parametrizing $(n-k)$-dimensional isotropic subspaces of a symplectic vector space E of dimension $2n$. Our theorem in this case states that the quantum Giambelli formula for $\mathrm{IG}(n-k, 2n)$ coincides with the classical Giambelli formula for $\mathrm{IG}(n+1-k, 2n+2)$, provided that the special Schubert class σ_{n+k+1} is replaced with $q/2$.

Although the two theorems in this article are analogous to those of [KT1, KT2], their proofs are quite different. We prove the quantum Giambelli formula by using the quantum Pieri rule of [BKT2], in a manner similar to [Bu] and [BKT1, Remark

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3]. However, unlike the previously known examples, for non-maximal isotropic Grassmannians no explicit recursion formula for the cohomological Giambelli polynomials is available, other than that given by the Pieri rule itself. We circumvent this difficulty by showing that a suitable recursion exists (Proposition 3). We also make essential use of a ring homomorphism from the stable cohomology ring of X to $\mathrm{QH}(X)$ that is the identity on Schubert classes coming from $\mathrm{H}^*(X,\mathbb{Z})$. The existence of this map (Propositions 4 and 5) may be of independent interest.

In a sequel to this paper, we will discuss the classical and quantum Giambelli formulas for even orthogonal Grassmannians.

1. Preliminary Results

1.1. **Classical Giambelli for IG.** Choose $k \geq 0$ and consider the Grassmannian $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ of isotropic $(n-k)$-dimensional subspaces of $\mathbb{C}^{2n}$, equipped with a symplectic form. A partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell)$ is *k-strict* if all of its parts greater than k are distinct integers. Following [BKT2], the Schubert classes on IG are parametrized by the k-strict partitions whose diagrams fit in an $(n-k)\times(n+k)$ rectangle, i.e. $\lambda_1 \leq n+k$ and $\ell(\lambda) \leq n-k$; we denote the set of all such partitions by $\mathcal{P}(k,n)$. Given any partition $\lambda \in \mathcal{P}(k,n)$ and a complete flag of subspaces

$$F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = \mathbb{C}^{2n}$$

such that $F_{n+i} = F_{n-i}^{\perp}$ for $0 \leq i \leq n$, we have a Schubert variety

$$X_\lambda(F_\bullet) := \{\Sigma \in \mathrm{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \ \ \forall\, 1 \leq j \leq \ell(\lambda)\},$$

where $\ell(\lambda)$ denotes the number of (non-zero) parts of λ and

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}.$$

This variety has codimension $|\lambda| = \sum \lambda_i$ and defines, via Poincaré duality, a Schubert class $\sigma_\lambda = [X_\lambda(F_\bullet)]$ in $\mathrm{H}^{2|\lambda|}(\mathrm{IG},\mathbb{Z})$. The Schubert classes σ_λ for $\lambda \in \mathcal{P}(k,n)$ form a free $\mathbb{Z}$-basis for the cohomology ring of IG. The *special Schubert classes* are defined by $\sigma_r = [X_r(F_\bullet)] = c_r(\mathcal{Q})$ for $1 \leq r \leq n+k$, where $\mathcal{Q}$ denotes the universal quotient bundle over IG.

The classical Giambelli formula for IG is expressed using Young's *raising operators* [Y, p. 199]. We first agree that $\sigma_0 = 1$ and $\sigma_r = 0$ for $r < 0$. For any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finite support and $i < j$, we set $R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots)$; a raising operator R is any monomial in these R_{ij}'s. Define $m_\alpha = \prod_i \sigma_{\alpha_i}$ and $R\, m_\alpha = m_{R\alpha}$ for any raising operator R. For any k-strict partition λ, we consider the operator

$$R^\lambda = \prod (1 - R_{ij}) \prod_{\lambda_i + \lambda_j > 2k+j-i} (1 + R_{ij})^{-1}$$

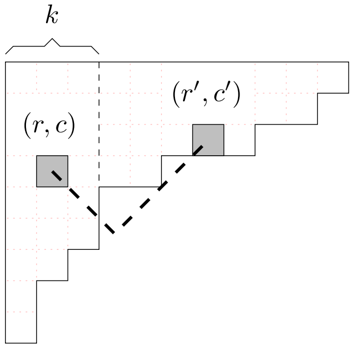
where the first product is over all pairs $i < j$ and second product is over pairs $i < j$ such that $\lambda_i + \lambda_j > 2k + j - i$. The main result of [BKT3] states that the *Giambelli formula*

$$\sigma_\lambda = R^\lambda\, m_\lambda \tag{1}$$

holds in the cohomology ring of $\mathrm{IG}(n-k, 2n)$.

1.2. **Classical Pieri for IG.** As is customary, we will represent a partition by its Young diagram of boxes; this is used to define the containment relation for partitions. Given two diagrams μ and ν with $\mu \subset \nu$, the skew diagram ν/μ (i.e., the set-theoretic difference $\nu \smallsetminus \mu$) is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row).

We say that the box $[r, c]$ in row r and column c of a k-strict partition λ is *k-related* to the box $[r', c']$ if $|c - k - 1| + r = |c' - k - 1| + r'$. For instance, the grey boxes in the following partition are k-related.



For any two k-strict partitions λ and μ, we write $\lambda \to \mu$ if μ may be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be k-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

Let $\mathbb{A}$ denote the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ which are not mentioned in (1) or (2) above, and define $N(\lambda, \mu)$ to be the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$. Here two boxes are connected if they share at least a vertex. In [BKT2, Thm. 1.1] we proved that the Pieri rule

$$\sigma_p \cdot \sigma_\lambda = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + p}} 2^{N(\lambda,\mu)} \, \sigma_\mu \tag{2}$$

holds in $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$, for any $p \in [1, n+k]$.

1.3. **A recursion formula for IG.** In the following sections we will work in the stable cohomology ring $\mathbb{H}(\mathrm{IG}_k)$, which is the inverse limit in the category of graded rings of the system

$$\cdots \leftarrow \mathrm{H}^*(\mathrm{IG}(n-k, 2n), \mathbb{Z}) \leftarrow \mathrm{H}^*(\mathrm{IG}(n+1-k, 2n+2), \mathbb{Z}) \leftarrow \cdots$$

The ring $\mathbb{H}(\mathrm{IG}_k)$ has a free $\mathbb{Z}$-basis of Schubert classes σ_λ, one for each k-strict partition λ, and may be presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \sigma_2, \ldots]$

modulo the relations

$$\sigma_r^2 + 2\sum_{i=1}^{r}(-1)^i \sigma_{r+i}\sigma_{r-i} = 0 \quad \text{for } r > k. \tag{3}$$

There is a natural surjective ring homomorphism $\mathbb{H}(\mathrm{IG}_k) \to \mathrm{H}(\mathrm{IG}(n-k, 2n), \mathbb{Z})$ that maps σ_λ to σ_λ, when $\lambda \in \mathcal{P}(k,n)$, and to zero, otherwise. The Giambelli formula (1) and Pieri rule (2) are both valid in $\mathbb{H}(\mathrm{IG}_k)$. We begin with some elementary consequences of these theorems.

For any k-strict partition λ of length ℓ, we define the sets of pairs

$$\mathcal{A}(\lambda) = \{(i,j) \mid \lambda_i + \lambda_j \leq 2k + j - i \ \text{ and } \ 1 \leq i < j \leq \ell\}$$

$$\mathcal{C}(\lambda) = \{(i,j) \mid \lambda_i + \lambda_j > 2k + j - i \ \text{ and } \ 1 \leq i < j \leq \ell\}$$

and two integer vectors $a = (a_1, \ldots, a_\ell)$ and $c = (c_1, \ldots, c_\ell)$ by setting

$$a_i = \#\{j \mid (i,j) \in \mathcal{A}(\lambda)\}, \quad c_i = \#\{j \mid (i,j) \in \mathcal{C}(\lambda)\}$$

for each i.

Proposition 1. *We have $\lambda_i - c_i \geq \lambda_j - c_j$ for each $i < j \leq \ell$.*

Proof. Observe that the desired inequality is equivalent to

$$\lambda_i - \lambda_j \geq \#\{r \leq \ell \mid (i,r) \in \mathcal{C}(\lambda)\} - \#\{r \leq \ell \mid (j,r) \in \mathcal{C}(\lambda)\}. \tag{4}$$

Let $j = i + r$ and let s (respectively t) be maximal such that $(i,s) \in \mathcal{C}(\lambda)$ (respectively, $(j,t) \in \mathcal{C}(\lambda)$). Assume first that t exists, hence s exists and $s \geq t$. The inequality (4) then becomes $\lambda_i - \lambda_{i+r} \geq s - t + r$. If $t = s$, this is true because $(j, j+1) \in \mathcal{C}(\lambda)$ and λ is k-strict, hence $\lambda_i > \lambda_{i+1} > \cdots > \lambda_{i+r}$. Otherwise we have $t < s \leq \ell$, $\lambda_i + \lambda_s \geq 2k + 1 + s - i$, and $\lambda_{i+r} + \lambda_{t+1} \leq 2k + t + 1 - i - r$. It follows that $\lambda_i - \lambda_{i+r} \geq s - t + r + (\lambda_{t+1} - \lambda_s) \geq s - t + r$.

Next we assume that t does not exist, so that either $j = \ell$ or the pair $(j, j+1)$ lies in $\mathcal{A}(\lambda)$ and

$$\lambda_j + \lambda_{j+1} \leq 2k + 1. \tag{5}$$

If s does not exist, the inequality is obvious. Otherwise, we must show that $\lambda_i - \lambda_j \geq s - i$, knowing that $(i,s) \in \mathcal{C}(\lambda)$, that is,

$$\lambda_i + \lambda_s \geq 2k + 1 + s - i. \tag{6}$$

Suppose first that $\lambda_s \geq \lambda_j$. If $\lambda_s > k$ then we have $\lambda_i > \lambda_{i+1} > \cdots > \lambda_s$ and hence $\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq s - i$. Otherwise $\lambda_s \leq k$ and (6) gives

$$\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq \lambda_i - k \geq s - i + 1 + (k - \lambda_s) \geq s - i.$$

Finally, suppose that $\lambda_s < \lambda_j$, so in particular $j + 1 \leq s$. Then (5) and (6) give

$$\begin{aligned} \lambda_i - \lambda_j &\geq \lambda_i + (\lambda_{j+1} - 2k - 1) \geq (2k + 1 + s - i - \lambda_s) + \lambda_{j+1} - 2k - 1 \\ &= (\lambda_{j+1} - \lambda_s) + (s - i) \geq s - i. \end{aligned}$$

□

Proposition 1 implies that for any λ, the composition $\lambda - c$ is a partition, while $\lambda + a$ is a strict partition.

Proposition 2. *For any k-strict partition λ, the Giambelli polynomial $R^\lambda\, m_\lambda$ for σ_λ involves only generators σ_p with $p \leq \lambda_1 + a_1 + \lambda_2 + a_2$.*

Proof. We have

$$R^\lambda\, m_\lambda = \prod_{1\le i<j\le \ell} \frac{1-R_{ij}}{1+R_{ij}} \prod_{(i,j)\in\mathcal{A}(\lambda)} (1+R_{ij})\, m_\lambda = \sum_{\nu\in N} \prod_{1\le i<j\le \ell} \frac{1-R_{ij}}{1+R_{ij}}\, m_\nu$$

where N is the multiset of integer vectors defined by

$$N = \left\{ \prod_{(i,j)\in S} R_{ij}\, \lambda \mid S \subset \mathcal{A}(\lambda) \right\}.$$

If $m>0$ is the least integer such that $2m \ge \ell$, then we have

$$\prod_{1\le i<j\le 2m} \frac{1-R_{ij}}{1+R_{ij}} = \text{Pfaffian}\left(\frac{1-R_{ij}}{1+R_{ij}}\right)_{1\le i<j\le 2m}. \tag{7}$$

Equation (7) follows from Schur's classical identity [S, Sect. IX]

$$\prod_{1\le i<j\le 2m} \frac{x_i-x_j}{x_i+x_j} = \text{Pfaffian}\left(\frac{x_i-x_j}{x_i+x_j}\right)_{1\le i,j\le 2m}.$$

Note that each single entry in the Pfaffian (7) expands according to the formula

$$\frac{1-R_{12}}{1+R_{12}}\, m_{c,d} = \sigma_c\,\sigma_d - 2\,\sigma_{c+1}\,\sigma_{d-1} + 2\,\sigma_{c+2}\,\sigma_{d-2} - \cdots + (-1)^d\, 2\,\sigma_{c+d}.$$

By Proposition 1, we know that $\lambda + a = (\lambda_1+a_1, \lambda_2+a_2, \ldots, \lambda_\ell+a_\ell)$ is a strict partition, hence $\lambda_i + a_i + \lambda_j + a_j \le \lambda_1 + a_1 + \lambda_2 + a_2$ for any distinct i and j. Since we furthermore have $\nu_i \le \lambda_i + a_i$, for any $\nu \in N$, the result follows. ☐

Corollary 1. *For any $\lambda \in \mathcal{P}(k,n)$ the stable Giambelli polynomial for σ_λ involves only special classes σ_p with $p \le 2n+2k-1$.*

Given any partition λ, we set $\lambda^* = (\lambda_2, \lambda_3, \ldots, \lambda_\ell)$.

Lemma 1. *Let λ and ν be k-strict partitions such that $\nu_1 > \max(\lambda_1, \ell(\lambda)+2k)$ and let $p, m \ge 0$. Then the coefficient of σ_ν in the Pieri product $\sigma_p \cdot \sigma_\lambda$ is equal to the coefficient of $\sigma_{(\nu_1+m,\nu^*)}$ in the product $\sigma_{p+m}\cdot\sigma_\lambda$.*

Proof. Since the box $[\ell(\lambda), 1]$ is k-related to $[1, \ell(\lambda)+2k]$ and $\nu_1 > \ell(\lambda)+2k$, it follows that $\lambda \to \nu$ if and only if $\lambda \to (\nu_1+m, \nu^*)$. In this case all of the boxes $[1,c]$ for $\max(\lambda_1, \ell(\lambda)+2k) < c \le \nu_1$ are contained in the rightmost component of the subset $\mathbb{A}$ of $\nu \smallsetminus \lambda$ defined in §1.2. Since replacing ν with (ν_1+m,ν^*) simply adds m boxes to this component, we deduce that $N(\lambda,\nu) = N(\lambda, (\nu_1+m,\nu^*))$. ☐

Proposition 3. *Let λ be a k-strict partition. Then there exist unique coefficients $a_{p,\mu} \in \mathbb{Z}$ for $p \ge \lambda_1$ and (p,μ) a k-strict partition, such that the recursive identity*

$$\sigma_\lambda = \sum_{p\ge\lambda_1} \sum_{\mu\,:\,(p,\mu)\ k\text{-strict}} a_{p,\mu}\, \sigma_p\, \sigma_\mu \tag{8}$$

holds in $\mathbb{H}(\mathrm{IG}_k)$. Furthermore, $a_{p,\mu} = 0$ whenever $\mu \not\subset \lambda^*$, or when $\lambda \in \mathcal{P}(k,n)$ and $p \ge 2n+2k$.

Proof. The Pieri rule (2) implies that

$$\sigma_\lambda = \sigma_{\lambda_1}\sigma_{\lambda^*} - \sum_{\substack{\lambda^*\to\nu\ne\lambda \\ |\nu|=|\lambda|}} 2^{N(\lambda^*,\nu)}\, \sigma_\nu\,.$$

Since all partitions ν in the sum satisfy $\nu_1 > \lambda_1$ and $\nu^* \subset \lambda^*$, the existence of the coefficients $a_{p,\mu}$ follows by descending induction on λ_1, and they satisfy $a_{(p,\mu)} = 0$ for $\mu \not\subset \lambda^*$. The uniqueness is true because the set of all products $\sigma_p \cdot \sigma_\mu$ for which (p, μ) is a k-strict partition is linearly independent in $\mathbb{H}(\mathrm{IG}_k)$. In fact, if the Schubert classes of $\mathbb{H}(\mathrm{IG}_k)$ are ordered by the dominance order of partitions, then the lowest term of the product $\sigma_p \cdot \sigma_\mu$ is the class $\sigma_{(p,\mu)}$.

On the other hand, Proposition 2 implies that there are coefficients $b_{p,\mu}$, indexed by integers $p \in [\lambda_1, \lambda_1 + a_1 + \lambda_2 + a_2]$ and k-strict partitions μ, such that

$$\sigma_\lambda = \sum_{p=\lambda_1}^{\lambda_1+a_1+\lambda_2+a_2} \sum_{|\mu|=|\lambda|-p} b_{p,\mu}\, \sigma_p\, \sigma_\mu\,.$$

In fact, if m_ν is any monomial appearing in the stable Giambelli formula $\sigma_\lambda = R^\lambda m_\lambda$, then $\lambda_1 \leq \max_i(\nu_i) \leq \lambda_1 + a_1 + \lambda_2 + a_2$. If $\lambda_1 > |\lambda^*|$, then the uniqueness of the coefficients $a_{p,\mu}$ implies that $b_{p,\mu} = a_{p,\mu}$. In particular, we have $a_{p,\mu} = 0$ for $p > \lambda_1 + a_1 + \lambda_2 + a_2$ in this case.

Now let $\lambda \in \mathcal{P}(k, n)$. Choose $m > |\lambda^*|$ and set $\lambda' = (\lambda_1 + m, \lambda^*)$. By the above discussion, there are coefficients $c_{p,\mu} \in \mathbb{Z}$ such that

$$\sigma_{\lambda'} = \sum_{p=\lambda_1+m}^{2n+2k-1+m} \sum_{\mu \subset \lambda^*} c_{p,\mu}\, \sigma_p\, \sigma_\mu\,. \tag{9}$$

We claim that the difference

$$\sigma_\lambda - \sum_{p=\lambda_1}^{2n+2k-1} \sum_{\mu \subset \lambda^*} c_{p+m,\mu}\, \sigma_p\, \sigma_\mu \tag{10}$$

is a linear combination of classes σ_ν for partitions $\nu \in \mathcal{P}(k, n)$ with $\nu_1 > \lambda_1$. To see this, notice that we must have $c_{\lambda_1+m,\lambda^*} = 1$, and hence the coefficient of σ_λ in the sum is equal to one. It follows that the difference (10) is equal to a linear combination of classes σ_ν for which $\nu_1 > \lambda_1$. Furthermore, if $\nu_1 > n + k$, then Lemma 1 implies that the coefficient of σ_ν in the sum in (10) is equal to the coefficient of $\sigma_{(\nu_1+m,\nu^*)}$ on the right hand side of (9), which is zero. This proves the claim. Finally, the proposition follows from the claim by descending induction on λ_1. □

Remark. One can be more precise about the recursion formula (8) in the case when the k-strict partition λ satisfies $\lambda_1 > \ell(\lambda) + 2k$. If the Pieri rule reads

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda^*} = \sum_{p \geq \lambda_1} \sum_{\mu \subset \lambda^*} 2^{n(p,\mu)}\, \sigma_{p,\mu}$$

then we have

$$\sigma_\lambda = \sum_{p \geq \lambda_1} \sum_{\mu \subset \lambda^*} (-1)^{p-\lambda_1}\, 2^{n(p,\mu)}\, \sigma_p\, \sigma_\mu.$$

This result is proved in [T].

2. Quantum Giambelli for $\mathrm{IG}(n-k, 2n)$

The quantum cohomology ring $\mathrm{QH}(\mathrm{IG})$ is a $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. The degree of the formal variable q here is $n+k+1$. We begin by recalling the quantum Pieri rule of [BKT2]. This states that for any k-strict partition $\lambda \in \mathcal{P}(k,n)$ and integer $p \in [1, n+k]$, we have

$$\sigma_p \cdot \sigma_\lambda = \sum_{\lambda \to \mu} 2^{N(\lambda,\mu)} \, \sigma_\mu + \sum_{\lambda \to \nu} 2^{N(\lambda,\nu)-1} \, \sigma_{\nu^*} \, q \tag{11}$$

in the quantum cohomology ring of $\mathrm{IG}(n-k, 2n)$. The first sum in (11) is over partitions $\mu \in \mathcal{P}(k,n)$ such that $|\mu| = |\lambda| + p$, and the second sum is over partitions $\nu \in \mathcal{P}(k, n+1)$ with $|\nu| = |\lambda| + p$ and $\nu_1 = n+k+1$.

Proposition 4. *There exists a unique ring homomorphism*

$$\pi : \mathbb{H}(\mathrm{IG}_k) \to \mathrm{QH}(\mathrm{IG}(n-k, 2n)) \otimes \mathbb{Q}$$

such that the following relations are satisfied:

$$\pi(\sigma_i) = \begin{cases} \sigma_i & \text{if } 1 \leq i \leq n+k, \\ q/2 & \text{if } i = n+k+1, \\ 0 & \text{if } n+k+1 < i \leq 2n+2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n+2k. \end{cases}$$

Furthermore, we have $\pi(\sigma_\lambda) = \sigma_\lambda$ for each $\lambda \in \mathcal{P}(k,n)$.

Proof. Recall that $\mathbb{H}(\mathrm{IG}_k)$ is the polynomial ring generated by all classes σ_i for $i \geq 1$, modulo the relations (3). These relations for $r > n+k$ uniquely specify the values $\pi(\sigma_i)$ for even integers $i > 2n+2k$. The quantum Pieri rule implies that the remaining relations (3) for $k < r \leq n+k$ are preserved by π.

We next prove that $\pi(\sigma_\lambda) = \sigma_\lambda$ for each $\lambda \in \mathcal{P}(k,n)$. This is clear when λ has only one part. When λ has more than one part, we apply the ring homomorphism π to both sides of (8) and use induction on $\ell(\lambda)$ to show that

$$\pi(\sigma_\lambda) = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a_{p,\mu} \, \sigma_p \, \sigma_\mu + \frac{q}{2} \sum_{\mu \subset \lambda^*} a_{n+k+1,\mu} \, \sigma_\mu \tag{12}$$

holds in $\mathrm{QH}(\mathrm{IG}(n-k, 2n)) \otimes \mathbb{Q}$. We also deduce from Proposition 3 that

$$\sigma_\lambda = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a_{p,\mu} \, \sigma_p \, \sigma_\mu + \sum_{\mu \subset \lambda^*} a_{n+k+1,\mu} \, \sigma_{(n+k+1,\mu)} \tag{13}$$

holds in the cohomology ring of $\mathrm{IG}(n+1-k, 2n+2)$. The quantum Pieri rule and (13) imply that the right hand side of (12) evaluates to σ_λ, as desired. ☐

Theorem 1 (Quantum Giambelli for IG)**.** *For every $\lambda \in \mathcal{P}(k,n)$, the quantum Giambelli formula for σ_λ in $\mathrm{QH}(\mathrm{IG}(n-k, 2n))$ is obtained from the classical Giambelli formula $\sigma_\lambda = R^\lambda \, m_\lambda$ in $\mathrm{H}^*(\mathrm{IG}(n+1-k, 2n+2), \mathbb{Z})$ by replacing the special Schubert class σ_{n+k+1} with $q/2$.*

Proof. This follows from Proposition 4 and Corollary 1. ☐

3. Quantum Giambelli for $\mathrm{OG}(n-k, 2n+1)$

3.1. **Classical Giambelli for OG.** For each $k \geq 0$, let $\mathrm{OG} = \mathrm{OG}(n-k, 2n+1)$ denote the odd orthogonal Grassmannian which parametrizes the $(n-k)$-dimensional isotropic subspaces in $\mathbb{C}^{2n+1}$, equipped with a non-degenerate symmetric bilinear form. The Schubert varieties in OG are indexed by the same set of k-strict partitions $\mathcal{P}(k,n)$ as for $\mathrm{IG}(n-k,2n)$. Given any $\lambda \in \mathcal{P}(k,n)$ and a complete flag of subspaces

$$F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n+1} = \mathbb{C}^{2n+1}$$

such that $F_{n+i} = F_{n+1-i}^\perp$ for $1 \leq i \leq n+1$, we define the codimension $|\lambda|$ Schubert variety

$$X_\lambda(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j(\lambda)}) \geq j \quad \forall\, 1 \leq j \leq \ell(\lambda)\},$$

where

$$\overline{p}_j(\lambda) = n + k + 1 + j - \lambda_j - \#\{i \leq j : \lambda_i + \lambda_j > 2k + j - i\}.$$

Let $\tau_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{OG}, \mathbb{Z})$ denote the cohomology class dual to the cycle given by $X_\lambda(F_\bullet)$.

Let $\ell_k(\lambda)$ be the number of parts λ_i which are strictly greater than k, and let $\mathcal{Q}_{\mathrm{IG}}$ and $\mathcal{Q}_{\mathrm{OG}}$ denote the universal quotient vector bundles over $\mathrm{IG}(n-k,2n)$ and $\mathrm{OG}(n-k,2n+1)$, respectively. It is known (see e.g. [BS, §3.1]) that the map which sends $\sigma_p = c_p(\mathcal{Q}_{\mathrm{IG}})$ to $c_p(\mathcal{Q}_{\mathrm{OG}})$ for all p extends to a ring isomorphism $\varphi : \mathrm{H}^*(\mathrm{IG}, \mathbb{Q}) \to \mathrm{H}^*(\mathrm{OG}, \mathbb{Q})$ such that $\varphi(\sigma_\lambda) = 2^{\ell_k(\lambda)}\tau_\lambda$ for all $\lambda \in \mathcal{P}(k,n)$.

We let $c_p = c_p(\mathcal{Q}_{\mathrm{OG}})$. The *special Schubert classes* on OG are related to the Chern classes c_p by the equations

$$c_p = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

For any integer sequence α, set $m_\alpha = \prod_i c_{\alpha_i}$. Then for every $\lambda \in \mathcal{P}(k,n)$, the classical Giambelli formula

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda\, m_\lambda$$

holds in $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$.

3.2. **From classical to quantum Giambelli.** Suppose $k \geq 1$. The quantum cohomology ring $\mathrm{QH}(\mathrm{OG}(n-k,2n+1))$ is defined similarly to that of IG, but the degree of q here is $n+k$. More notation is required to state the quantum Pieri rule for OG. For each λ and μ with $\lambda \to \mu$, we define $N'(\lambda,\mu)$ to be equal to the number (respectively, one less than the number) of connected components of $\mathbb{A}$, if $p \leq k$ (respectively, if $p > k$). Let $\mathcal{P}'(k,n+1)$ be the set of $\nu \in \mathcal{P}(k,n+1)$ for which $\ell(\nu) = n+1-k$, $2k \leq \nu_1 \leq n+k$, and the number of boxes in the second column of ν is at most $\nu_1 - 2k + 1$. For any $\nu \in \mathcal{P}'(k,n+1)$, we let $\widetilde{\nu} \in \mathcal{P}(k,n)$ be the partition obtained by removing the first row of ν as well as $n+k-\nu_1$ boxes from the first column. That is,

$$\widetilde{\nu} = (\nu_2, \nu_3, \ldots, \nu_r), \text{ where } r = \nu_1 - 2k + 1.$$

According to [BKT2, Thm. 2.4], for any k-strict partition $\lambda \in \mathcal{P}(k,n)$ and integer $p \in [1, n+k]$, the following quantum Pieri rule holds in $\mathrm{QH}(\mathrm{OG}(n-k,2n+1))$.

$$\tau_p \cdot \tau_\lambda = \sum_{\lambda \to \mu} 2^{N'(\lambda,\mu)}\, \tau_\mu + \sum_{\lambda \to \nu} 2^{N'(\lambda,\nu)}\, \tau_{\widetilde{\nu}}\, q \, + \sum_{\lambda^* \to \rho} 2^{N'(\lambda^*,\rho)}\, \tau_{\rho^*}\, q^2 \,. \tag{14}$$

Here the first sum is classical, the second sum is over $\nu \in \mathcal{P}'(k,n+1)$ with $\lambda \to \nu$ and $|\nu| = |\lambda| + p$, and the third sum is empty unless $\lambda_1 = n+k$, and over $\rho \in \mathcal{P}(k,n)$ such that $\rho_1 = n+k$, $\lambda^* \to \rho$, and $|\rho| = |\lambda| - n - k + p$.

Let $\delta_p = 1$, if $p \leq k$, and $\delta_p = 2$, otherwise. The stable cohomology ring $\mathbb{H}(\mathrm{OG}_k)$ has a free $\mathbb{Z}$-basis of Schubert classes τ_λ for k-strict partitions λ, and is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \tau_2, \ldots]$ modulo the relations

$$\tau_r^2 + 2\sum_{i=1}^{r}(-1)^i \delta_{r-i}\tau_{r+i}\tau_{r-i} = 0 \quad \text{for } r > k. \tag{15}$$

Proposition 5. *There exists a unique ring homomorphism*

$$\widetilde{\pi} : \mathbb{H}(\mathrm{OG}_k) \to \mathrm{QH}(\mathrm{OG}(n-k, 2n+1))$$

such that the following relations are satisfied:

$$\widetilde{\pi}(\tau_i) = \begin{cases} \tau_i & \text{if } 1 \leq i \leq n+k, \\ 0 & \text{if } n+k < i < 2n+2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n+2k. \end{cases}$$

Furthermore, we have $\widetilde{\pi}(\tau_\lambda) = \tau_\lambda$ for each $\lambda \in \mathcal{P}(k, n)$.

Proof. The relations (15) for $r \geq n+k$ uniquely specify the values $\widetilde{\pi}(\tau_i)$ for even integers $i \geq 2n+2k$. We must show that the remaining relations for $k < r < n+k$ are mapped to zero by $\widetilde{\pi}$. Observe that when $k < n-1$ the individual terms in these relations carry no q correction. Indeed, we are applying the quantum Pieri rule (14) to partitions of length one, hence the q term vanishes (since $1 < n-k$) and the q^2 term vanishes (since $\deg(q^2) = 2n+2k$). It remains only to consider the case $k = n-1$, which uses the quantum Pieri rule for the quadric $\mathrm{OG}(1, 2n+1)$. The computation is then done as in [BKT2, Thm. 2.5] (which treats the case $r = n$), and involves computing the coefficient c of $q\,\tau_{2(r-n)+1}$ in the corresponding expression. As in loc. cit., the result is $c = 1 - 2 + 2 - \cdots \pm 2 \mp 1$ when $r \leq (3n-2)/2$, and otherwise $c = 2 - 4 + 4 - \cdots \pm 4 \mp 2$; hence $c = 0$ in both cases.

To prove that $\widetilde{\pi}(\tau_\lambda) = \tau_\lambda$ for every $\lambda \in \mathcal{P}(k, n)$, we use an orthogonal analogue of Proposition 3, which follows from the isomorphism $\mathbb{H}(\mathrm{OG}_k) \otimes \mathbb{Q} \cong \mathbb{H}(\mathrm{IG}_k) \otimes \mathbb{Q}$. Arguing by induction on $\ell(\lambda)$ as in Proposition 4, we obtain that

$$\widetilde{\pi}(\tau_\lambda) = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a'_{p,\mu}\, \tau_p\, \tau_\mu \tag{16}$$

holds in $\mathrm{QH}(\mathrm{OG}(n-k, 2n+1)) \otimes \mathbb{Q}$, where $a'_{p,\mu} \in \mathbb{Q}$. The quantum Pieri rule (14) implies that any product $\tau_p\, \tau_\mu$ in (16) carries no q correction terms. It follows that the right hand side of (16) evaluates to τ_λ. □

Theorem 2 (Quantum Giambelli for OG)**.** *For every $\lambda \in \mathcal{P}(k, n)$, we have*

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda\, m_\lambda$$

in the quantum cohomology ring $\mathrm{QH}(\mathrm{OG}(n-k, 2n+1))$*. In other words, the quantum Giambelli formula for* OG *is the same as the classical Giambelli formula.*

Proof. This follows from Proposition 5 and the orthogonal version of Corollary 1. □

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