FLAG VARIETIES AND SCHUBERT CALCULUS

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Abstract. We discuss recent developments in Schubert calculus.

1. Introduction

In 1879, H. Schubert laid some foundations for enumerative geometry.

Question 1.1. What is the number of lines in 3-space incident to four given lines in general position?

Schubert's "Principle of Conservation of Number": In a degeneration in which the number of solutions remains finite, this number remains constant, provided that multiplicities are properly taken into account.

Notation 1.2. Schubert uses the following notation:

- g = lines incident to a given line
- g_e = lines contained in a given plane ("e" for "Ebene")
- g_p = lines through a given point ("p" for "Punkt")
- g_s = lines through a given point in a given plane ("s" for "Strahlenbüschel")

Example 1.3. In this notation, Question 1.1 asks for the number g^4. By the above principle, two of the lines in general position can be moved such that they intersect. Then for a line to be incident to both requires it to lie in the plan that they span, or to pass through the point where they intersect; hence $g^2 = g_e + g_p$. It is easy to see that $g_e^2 = g_p^2 = 1$ and $g_e \cdot g_p = 0$. Hence $g^4 = (g_e + g_p)^2 = 2$.

In modern language, Question 1.1 is a calculation in the cohomology ring of the *Grassmannian variety* (or manifold)

$$\begin{aligned} G(2,4) &= \{\text{lines in 3-space } (=\mathbb{P}^3_{\mathbb{C}})\} \\ &= \{\text{2-dimensional linear spaces in } \mathbb{C}^4\}. \end{aligned}$$

The modern notation for Schubert calculus is for example

$$\int_{G(2,4)} \sigma_1^4 = ? \qquad \sigma_1^2 = \sigma_{11} + \sigma_2 \qquad \dots \qquad \int_{G(2,4)} \sigma_1^4 = 2.$$

2. The Grassmannian $G(k,n)$

$$\begin{aligned} G(k,n) &= \{\Sigma \subset \mathbb{C}^n \mid \dim \Sigma = k\} \\ &= \{M \in \mathrm{Mat}_{k\times n} \mid \mathrm{Rank}\, M = k\}/(\text{row operations}) \\ &= P_k \backslash GL(n) \end{aligned}$$

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is the Grassmannian manifold of linear subspaces of dimension k in $\mathbb{C}^n$ which can also be written as a $k \times n$ matrix of k basis vectors in $\mathbb{C}^n$ (up to a change of basis) or as a quotient of $\mathrm{GL}(n)$ where P_k is the parabolic subgroup of $n \times n$ matrices with a 0-block of size $k \times (n-k)$ in the upper right corner:

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,k} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k,1} & \dots & a_{k,k} & 0 & \dots & 0 \\ a_{k+1,1} & \dots & a_{k+1,k} & a_{k+1,k+1} & \dots & a_{k+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,k} & a_{n,k+1} & \dots & a_{n,n} \end{pmatrix}$$

Fact 2.1. $G(k,n)$ is a nonsingular projective variety of (complex) dimension $k \cdot (n-k)$. It embeds in $\mathbb{P}^{\binom{n}{k}-1}$ via the *Plücker embedding*

$$\begin{aligned} G(k,n) &\to \mathbb{P}^{\binom{n}{k}-1} = \{[\cdots : z_{i_1,\dots,i_k} : \dots]\} \\ \mathrm{Mat}_{k\times n} \ni M &\mapsto [\cdots : \det(M_{i_1,\dots,i_k}) : \dots] \end{aligned}$$

where the coordinates are indexed by $1 \le i_1 < \cdots < i_k \le n$ and $M_{i_1,\dots,i_k}$ is the minor of columns $i_1, \dots, i_k$.

$G(k,n)$ is covered by the open sets with $z_{i_1,\dots,i_k} \neq 0$ for any $1 \le i_1 < \cdots < i_k \le n$ denoted by $U_{i_1,\dots,i_k}$. Bringing the corresponding minor into the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

we see that

$$U_{i_1,\dots,i_k} = \begin{pmatrix} & 0 & & 0 & & & & 0 & & 1 & \\ & 0 & & 0 & & & & 1 & & 0 & \\ * & \vdots & * & \vdots & * & \dots & * & \vdots & * & \vdots & * \\ & 0 & & 1 & & & & 0 & & 0 & \\ & 1 & & 0 & & & & 0 & & 0 & \end{pmatrix} \cong \mathbb{C}^{k(n-k)},$$

which explains that the dimension of $G(k,n)$ is $k(n-k)$.

Example 2.2. For $G(2,4)$, one open set is

$$U_{1,2} \ni \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & c & d \end{pmatrix} \mapsto [z_{1,2} : z_{1,3} : z_{1,4} : z_{2,3} : z_{2,4} : z_{3,4}] = [-1 : -a : -b : c : d : ad-bc].$$

The image in $\mathbb{P}^5$ satisfies the relation $z_{1,2}z_{3,4} - z_{1,3}z_{2,4} + z_{1,4}z_{2,3} = 0$, and $G(2,4)$ is the quadric defined by this.

In general, $G(k,n) \in \mathbb{P}^{\binom{n}{k}-1}$ is cut out by quadratic equations, but it is not a complete intersection.

Every matrix is row-equivalent to a unique matrix in row-echelon form, and its rank is the number of pivots. In our situation, every $M \in \mathrm{Mat}_{k\times n}$ of rank k can be

transformed by row transformations to a unique matrix of the form

$$\begin{pmatrix} * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & 0 & * & \dots & * & 1 & 0 & \dots & \dots & \dots & \dots & & 0 \\ * & \dots & * & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & 0 \end{pmatrix} \cong \mathbb{C}^{i_1+\dots+i_k-\frac{k(k+1)}{2}}$$

Definition 2.3. For any $1 \le i_1 < \dots < i_k \le n$, the *Schubert cell* $X^0_{i_1,\dots,i_k}$ is defined as

$$\{M \in \mathrm{Mat}_{k\times n} \mid \text{in echelon form, the pivots are in columns } i_1, \dots, i_k\}/\operatorname{GL}(k).$$

Fact 2.4.

$$G(k,n) = \bigsqcup_{1\le i_1<\dots<i_k\le n} X^0_{i_1,\dots,i_k}.$$

Definition 2.5. The *Schubert varieties* are defined as $X_{i_1,\dots,i_k} = \overline{X^0_{i_1,\dots,i_k}}$.

Fact 2.6. Schubert varieties are algebraic subvarieties of $G(k,n)$, each defined by vanishing of some set of the coordinates $z_{i_1,\dots,i_k}$.

There is a cellular structure on $G(k,n)$, given by the union of Schubert cells of real dimension not larger than j for $j = 0, 1, 2, \dots, 2k(n-k)$:

$$\{\text{point}\} = X^0 = X^1 \subset X^2 = X^3 \subset \dots \subset X^{2k(n-k)} = G(k,n)$$

For cellular structures in general, the condition $H_i(X^j, X^{j-1}) = 0$ must be satisfied if $i \neq j$. In our situation, as $X^0 = X^1, X^2 = X^3, \dots$, we have

$$H_*(G(k,n),\mathbb{Z}) = H_*(\mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z}^{m_2} \leftarrow \dots \leftarrow 0 \leftarrow \mathbb{Z}^{m_j} \leftarrow 0 \leftarrow \dots)$$

where

$$m_j = \#\{1 \le i_1 < \dots < i_k \le n \mid i_1 + \dots + i_k - \frac{k(k+1)}{2} = j\}.$$

Therefore

$$H_{2j}(G(k,n),\mathbb{Z}) = \mathbb{Z}^{m_j}, \qquad H_{2j+1}(G(k,n),\mathbb{Z}) = 0.$$

Cohomology is obtained from this by Poincare duality.

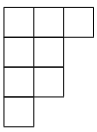
Theorem 2.7. *$G(k,n)$ has cohomology only in even degrees, and $H^{2j}(G(k,n),\mathbb{Z}) = \mathbb{Z}^{m_j}$ where m_j is the number of partitions of the integer j into at most k parts with each part less than or equal to $n-k$.*

Definition 2.8 (The language of partitions)**.** Consider a partition, usually denoted by λ, for example

$$8 = 3 + 2 + 2 + 1.$$

The order is irrelevant, each term is called a *part* and must be positive. By convention, the parts are usually given in decreasing order. We allow extra, irrelevant parts which are equal to 0.

The corresponding diagram is:



The weight of λ is denoted by $|\lambda|$ and is 8 in this case.

Denote the partition

$$k \cdot (n-k) = (n-k) + \cdots + (n-k)$$

consisting of k equal parts $n-k$ by $(n-k)^k$, and for two partitions $\lambda = a_1 + \cdots + a_k$ and $\lambda' = a'_1 + \cdots + a'_k$, we write $\lambda \subset \lambda'$ if the diagram of λ is contained in the diagram of λ', i.e. if $a_i \leq a'_i$ for all $i \in \{1, \ldots, k\}$.

In this language, the number m_j as above is given by the number of partitions of j contained in a rectangle of size $k \times (n-k)$.

$$\begin{aligned} m_j &= \#\{a_1 \geq \cdots \geq a_k \geq 0 \mid a_1 \leq n-k, \sum_i a_i = j\} \\ &= \#\{\text{partitions } \lambda \mid |\lambda| = j, \lambda \subset (n-k)^k\} \end{aligned}$$

The Schubert cells can also be defined as

$$X^0_{i_1,\ldots,i_k} = \{\Sigma \subset \mathbb{C}^n \mid \dim(\Sigma \cap \mathbb{C}^j) = \#(\{i_1, \ldots, i_k\} \cap \{1, \ldots, j\}) \text{ for } j = 1, \ldots, n\}$$

Fact 2.9. We have the following decomposition of a Schubert variety into Schubert cells:

$$\begin{aligned} X_{i_1,\ldots,i_k} &= \bigcup_{i'_1 \leq i_1, \ldots, i'_k \leq i_k} X^0_{i'_1,\ldots,i'_k} \\ &= \{\Sigma \subset \mathbb{C}^n \mid \dim(\Sigma \cap \mathbb{C}^j) \geq \#(\{i_1, \ldots, i_k\} \cap \{1, \ldots, j\}) \text{ for } j = 1, \ldots, n\} \end{aligned}$$

Using the language of partitions, we write

$$X_\lambda = X_{n-k+1-\lambda_1, n-k+2-\lambda_2, \ldots, n-\lambda_k}$$

where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$. Then

$$X_\lambda = \{\Sigma \subset \mathbb{C}^n \mid \dim(\Sigma \cap \mathbb{C}^{n-k+i-\lambda_i}) \geq i \text{ for } i = 1, \ldots, k\}.$$

The corresponding *Schubert classes* are defined as $\sigma_\lambda = [X_\lambda] \in H^{2|\lambda|}(G(k,n), \mathbb{Z})$. We have

$$H^*(G(k,n), \mathbb{Z}) = \bigoplus_{\lambda \subset (n-k)^k} \mathbb{Z} \cdot \sigma_\lambda.$$

Example 2.10. Explicitly, for $G(2,4)$, we have the following Schubert classes:

- $\sigma_\emptyset = \begin{pmatrix} * & * & 0 & 1 \\ * & * & 1 & 0 \end{pmatrix}$ corresponding to all $\Sigma \subset \mathbb{C}^4$ (lines in $\mathbb{P}^3$)
- $\sigma_1 = \begin{pmatrix} * & 0 & * & 1 \\ * & 1 & 0 & 0 \end{pmatrix}$ corresponding to Σ meeting $\mathbb{C}^2$ non-trivially (lines incident to a given line)
- $\sigma_{11} = \begin{pmatrix} * & 0 & 1 & 0 \\ * & 1 & 0 & 0 \end{pmatrix}$ corresponding to all $\Sigma \subset \mathbb{C}^3$ (lines in a plane)
- $\sigma_2 = \begin{pmatrix} 0 & * & * & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ corresponding to Σ containing $\mathbb{C}^1$ (lines through a given point)
- $\sigma_{21} = \begin{pmatrix} 0 & * & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ corresponding to $\mathbb{C}^1 \subset \Sigma \subset \mathbb{C}^3$ (lines in a plane through a point)
- $\sigma_{22} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ corresponding to $\Sigma = \mathbb{C}^2$ (a given line)

Schubert varieties can be defined with respect to any *complete flag*

$$F_\bullet = F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$$

Let $E_\bullet$ be the standard complete flag $E_1 = \mathbb{C}^1 \subset E_2 = \mathbb{C}^2 \subset \cdots \subset E_n = \mathbb{C}^n$.

Defining $X_\lambda(F_\bullet)$ in an obvious way, we have $X_\lambda(E_\bullet) = X_\lambda$ with X_λ defined as above. Any $X_\lambda(F_\bullet)$ is the translate of X_λ by a suitable element of $\mathrm{GL}(n)$.

Consider an intersection

$$X_{\lambda^1}(F_\bullet^1) \cap \cdots \cap X_{\lambda^j}(F_\bullet^j) \supset X^0_{\lambda^1}(F_\bullet^1) \cap \cdots \cap X^0_{\lambda^j}(F_\bullet^j).$$

In good situations, this is a dense open inclusion, and the variety on the right is non-singular and either empty or of dimension $k(n-k) - |\lambda^1| - \cdots - |\lambda^j|$.

By Kleiman's Bertini theorem (see [Har77], Theorem III.10.8, or the original paper [Kle74]), for a general tuple $(F_\bullet^1, \ldots, F_\bullet^j)$, we are in a good situation. When $|\lambda^1| + \cdots + |\lambda^j| = k(n-k)$, this tells us that the intersection is a finite set of points, each a point of transverse intersection, and hence

$$c_{\lambda^1,\ldots,\lambda^j} := \int_{G(k,n)} \sigma_{\lambda^1} \cup \cdots \cup \sigma_{\lambda^j} = \#(X_{\lambda^1}(F_\bullet^1) \cap \cdots \cap X_{\lambda^j}(F_\bullet^j)).$$

Here, $\int_{G(k,n)}$ denotes evaluation of the degree of a zero-dimensional cycle class:

$$\begin{aligned} \sigma_{(n-k)^k} &\mapsto 1 \\ \sigma_\lambda &\mapsto 0 \text{ if } |\lambda| < k(n-k) \end{aligned}$$

For $j = 2$, a result of Richardson (see [Ric92]) states that the intersection of two Schubert varieties in general position is irreducible.

As an exercise, one can prove that any general pair of flags can be mapped by a suitable element of $\mathrm{GL}(n)$ to a specific pair consisting of the standard flag $E_\bullet$ where $E_i = \mathrm{span}(e_1, \ldots, e_i)$ and the opposite flag $F_\bullet$ with $F_i = \mathrm{span}(e_{n+1-i}, \ldots, e_n)$.

Proposition 2.11. *$X_\lambda(E_\bullet) \cap X_\mu(F_\bullet)$ is empty if and only if $\lambda_i + \mu_{k+1-i} > n-k$ for some i. Otherwise, a dense open subset consists of element*

$$\begin{pmatrix}
0 & \ldots & & & & & & & 0 & * & \ldots & * & 1 & 0 & \ldots & 0 \\
0 & \ldots & & & & & & 0 & * & \ldots & * & 1 & 0 & \ldots & & 0 \\
\vdots & & & & & & & & & & & & & & & \vdots \\
0 & \ldots & & & & & 0 & * & \ldots & * & 1 & 0 & \ldots & & & 0 \\
0 & \ldots & 0 & * & \ldots & * & 1 & 0 & \ldots & & & & & & & 0
\end{pmatrix}$$

Here, the first $*$ in row i is in column $k+1-i+\mu_i$, and the 1 is in column $n+1-i-\lambda_{k+1-i}$ (i.e. the 1 are in the same positions as the pivots in σ_λ, and the positions of the first $*$'s are the pivots of σ_μ turned around). The leftmost $*$ in each row stands for a non-zero entry while the others can be arbitrary.

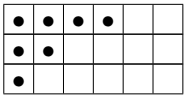
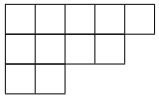
Proof. Suppose $\lambda_i + \mu_{k+1-i} > n-k$ for some i, and suppose there is a $\Sigma \in X_\lambda(E_\bullet) \cap X_\mu(F_\bullet)$. Then $\dim(\Sigma \cap E_{n-k+i-\lambda_i}) \geq i$, and $\dim(\Sigma \cap F_{n+1-i-\mu_{k+1-i}}) \geq k+1-i$. This implies $\dim(\Sigma \cap E_{n-k+i-\lambda_i} \cap F_{n+1-i-\mu_{k+1-i}}) \geq 1$, which gives us a contradiction since $E_{n-k+i-\lambda_i} \cap F_{n+1-i-\mu_{k+1-i}} = 0$.

For the second part, the matrix above describes a subset of the right dimension, and by Richardson's theorem, the intersection is irreducible. Therefore, this subset must be dense. □

Next, consider the special case $|\lambda| + |\mu| = k(n-k)$. Then

$$\int_{G(k,n)} \sigma_\lambda \cup \sigma_\mu = \delta_{\mu,\lambda^\vee}$$

where $\lambda^\vee = (n-k-\lambda_k, \ldots, n-k-\lambda_1)$

$\lambda =$  $\qquad \lambda^\vee =$ 

(Here, λ consists of the boxes marked by $\bullet$ inside the $k \times (n-k)$ box, and $\lambda^\vee$ is the complement of λ inside this box turned around.)

Therefore, in general,

$$\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu \text{ where } c^\nu_{\lambda\mu} = c_{\lambda\mu\nu^\vee}.$$

Hence, triple intersection numbers determine the multiplication in $H^*(G(k,n))$.

3. Combinatorial rules: Schubert calculus

Generally, *Schubert calculus* refers to Schubert's methods, involving symbolic manipulations, for the solution of enumerative problems. More specifically, it refers to methods, both classical and modern, for computing in $H^*(G(k,n))$ and similar rings.

The three main methods of Schubert calculus are:

(1) degeneration techniques (Schubert, Pieri, ...)
(2) triple intersections (Hodge–Pedoe, ...)
(3) algebraic methods

Definition 3.1. The Schubert class corresponding to the partition (i) for some i is called a *special Schubert class* and is denoted by σ_i.

Products of an arbitrary and a special Schubert class can be calculated easily:

Proposition 3.2 (Pieri's formula)**.** *We have*

$$\sigma_\lambda \cup \sigma_i = \sum_\mu \sigma_\mu,$$

where μ must fulfill the conditions that $\mu \supset \lambda$, $|\mu| = |\lambda| + i$, and μ/λ has at most one box in every column.

Here, μ/λ denotes the set of boxes in μ which are not in λ.

Example 3.3. In $H^*(G(2,4))$, we calculate $\sigma_1^2 = \sigma_1 \cup \sigma_1 = \sigma_{11} + \sigma_2$ where

$\sigma_1 =$  $\qquad \sigma_{11} =$  $\qquad \sigma_2 =$ 

Similarly, we compute (note that μ in Pieri's formula above must fit in the $k \times (n-k)$ box)

$$\sigma_1^3 = \sigma_{11} \cup \sigma_1 + \sigma_2 \cup \sigma_1 = \sigma_{21} + \sigma_{21} = 2\sigma_{21}$$

and

$$\sigma_1^4 = 2\sigma_{21} \cup \sigma_1 = 2\sigma_{22}.$$

Therefore,

$$\int_{G(2,4)} \sigma_1^4 = 2.$$

Proposition 3.4 (Giambelli's formula)**.** *We can write an arbitrary Schubert class σ_λ, where $\lambda = (\lambda_1, \dots, \lambda_\ell)$, in terms of the special Schubert classes $\sigma_1, \dots, \sigma_{n-k}$:*

$$\sigma_\lambda = \det \begin{pmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \sigma_{\lambda_1+2} & \cdots & \cdots \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \cdots \\ \sigma_{\lambda_3-2} & \sigma_{\lambda_3-1} & \sigma_{\lambda_3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \sigma_{\lambda_\ell} \end{pmatrix}$$

Here, the determinant is evaluated using the $\cup$-product, and we use the convention that $\sigma_0 = 1$ and $\sigma_i = 0$ if $i < 0$ or $i > n - k$.

Also we have a ring presentation $H^*(G(k,n), \mathbb{Z}) = \mathbb{Z}[\sigma_1, \dots, \sigma_{n-k}]/(R_{k+1}, \dots, R_n)$ where

$$R_i = \det \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \cdots \\ 1 & \sigma_1 & \sigma_2 & \cdots \\ 0 & 1 & \sigma_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \dots & 0 & 1 & \sigma_1 \end{pmatrix}$$

is the determinant of an $i \times i$-matrix.

4. Flag varieties

Having considered only Grassmannian varieties so far, we now look at complete flag varieties

$$F(n) = \{\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \mathbb{C}^n\} = B \backslash \mathrm{GL}(n)$$

where B is the Borel group of lower triangular matrices.

We can also consider partial flag varieties

$$F(a_1, \dots, a_j; n) = \{\Sigma_1 \subset \cdots \subset \Sigma_j \subset \mathbb{C}^n \mid \dim \Sigma_i = a_i \text{ for } i = 1, \dots, j\} = P \backslash \mathrm{GL}(n)$$

where P is a lower parabolic matrix corresponding to $a_1, \dots, a_j$.

Example 4.1. For $j = 2$ and $F(3, 5; 7)$, the Schubert cell / variety corresponding to $\sigma_{136,27} \in H^*(F(3,5;7))$ is described by the matrix

$$\left(\begin{array}{ccccccc} * & 0 & * & * & 0 & * & 1 \\ * & 0 & * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & * & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

where the first three rows span Σ_1, and all five rows span Σ_2.

In general,

$$H^*(F) = \bigoplus_{s \in S} \mathbb{Z} \cdot \sigma_s$$

for a suitable set S. In case of the complete flag variety $F(n)$, we have $S = S_n$, the permutation group.

In the case of a Grassmannian, i.e. $j = 1, k = a_1$, we have $S = \{1 \le i_1 < \cdots < i_k \le n\}$. The permutation corresponding to $\sigma_{i_1,\dots,i_k}$ can be described in the following way: We

define $1 \leq j_1 < \cdots < j_{n-k} \leq n$ so that $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}$. Then the corresponding Grassmannian permutation is

$$\begin{pmatrix} 1 & \ldots & k & k+1 & \ldots & n \\ i_1 & \ldots & i_k & j_1 & \ldots & j_{n-k} \end{pmatrix} \in S_n$$

In a 2-step flag, for example $\sigma_{136,27}$ corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 6 & 2 & 7 & 4 & 5 \end{pmatrix} \in S_7$$

consisting of three blocks of increasing numbers.

Remark 4.2. Certain minimum-length coset representatives of the Weyl group S_n (in this case) index the Schubert varieties – this is the situation for general homogeneous spaces $P\backslash G$.

Returning to the case $G(k,n)$, let $d \leq \min(k, n-k)$ and consider the diagram

$$\begin{array}{ccc} F(k-d,k,k+d;n) & \xrightarrow{\psi} & F(k-d,k+d;n) \\ \downarrow{\scriptstyle \pi} & & \\ G(k,n) & & \end{array}$$

and set $\sigma_\lambda^{(d)} = \psi_* \pi^* \sigma_\lambda \in H^*(F(k-d,k+d;n))$. Looking at the permutation corresponding to σ_λ, this means sorting the $2d$ entries on positions $k-d+1, \ldots, k+d$ to get three blocks of increasing numbers.

Theorem 4.3. *For $\lambda, \mu, \nu \subset (n-k)^k$, $|\lambda| + |\mu| + |\nu| = k(n-k) + dn$, we have*

$$\begin{aligned} c_{\lambda,\mu,\nu}^{(d)} &:= \int_{F(k-d,k+d;n)} \sigma_\lambda^{(d)} \cup \sigma_\mu^{(d)} \cup \sigma_\nu^{(d)} \\ &= \#\{\textit{degree } d \textit{ rational curves on } G(k,n) \textit{ incident to } X_\lambda(E_\bullet), X_\mu(F_\bullet), X_\nu(G_\bullet)\} \end{aligned}$$

for general triples $(E_\bullet, F_\bullet, G_\bullet)$ of flags.

Remark 4.4. The setting for this result is the following: $H^*G(k,n)$ admits a "quantum" deformation

$$QH^*G(k,n) = \bigoplus_{\lambda \subset (n-k)^k} \mathbb{Z}[q] \cdot \sigma_\lambda$$

(as $\mathbb{Z}[q]$-modules), with

$$\sigma_\lambda * \sigma_\mu = \sum_{\nu, d \geq 0} c_{\lambda,\mu,\nu^\vee}^{(d)} q^d \sigma_\nu.$$

It is an amazing fact that this multiplication is associative! It comes from the general theory of *quantum cohomology*, which associates QH^*X to H^*X for general complex projective manifolds.

The multiplication in QH^*X encodes enumerative information about rational curves in X.

Question 4.5. Getting back to $H^*G(k,n)$, we are interested in the constants $c_{\lambda,\mu,\nu^\vee}$ in the general product

$$\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda,\mu,\nu^\vee} \sigma_\nu$$

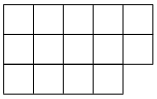
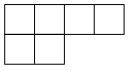
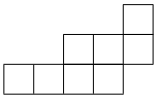
and similarly in the values of $c_{\lambda,\mu,\nu^\vee}^{(d)}$ for quantum extensions.

More generally, what are the structure constants in the product of Schubert classes for partial flag varieties?

The classical Littlewood–Richardson rule gives a combinatorial interpretation for $c_{\lambda,\mu,\nu}$, which has been extended recently to partial flag varieties.

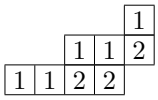
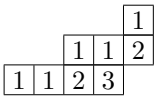
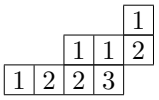
The classical rule for $G(k,n)$ is: $c_{\lambda,\mu,\nu}$ is equal to the number of *semistandard tableaux* of content ν on the skew diagram $\mu^\vee/\lambda$ (i.e. $\mu^\vee$ with λ removed as introduced in Proposition 3.2; the result is 0 if $\lambda \not\subset \mu^\vee$) satisfying a further combinatorial condition which is best illustrated by the following example:

Consider $G(3,8)$, $\lambda = 42$, $\mu = 1$. We want to calculate $\sigma_{42} \cup \sigma_1$. We have $\mu^\vee = 554$.

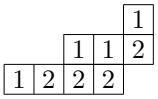
$\mu =$  $\mu^\vee =$  $\lambda =$  $\mu^\vee/\lambda =$ 

The resulting $\mu^\vee/\lambda$ can be filled with numbers which must satisfy the following conditions: in rows, the numbers must increase weakly from left to right; in columns, the numbers must increase strictly from top to bottom; and as the further "lattice word property", reading the numbers row by row from right to left must give a sequence such that for any initial subwords, $\#1's \geq \#2's \geq \dots$ must hold.

These conditions can be fulfilled in three different ways

while the diagram



gives the sequence 12112221 which does not satisfy the lattice word property since the initial subword of length 7 contains more $2's$ than $1's$.

These diagrams with numbers are translated into the partition $\nu = (\#1's, \#2's, \dots)$ of weight $|\mu^\vee| - |\lambda|$, and each of them contributes 1 to $c_{\lambda,\mu,\nu}$. The three examples give $\nu_1 = (53)$, $\nu_2 = (521)$, $\nu_3 = (431)$, with respective dual partitions (52), (43), (421). Therefore,

$$\sigma_{42} \cup \sigma_1 = \sigma_{52} + \sigma_{43} + \sigma_{421}.$$

Most proofs of the Littlewood–Richardson rule are combinatorial. Recently, there have been two geometric proofs by R. Vakil ([Vak06]) and I. Coskun ([Cos04]). These approaches are based on the technique of degeneration.

Example 4.6. To calculate $\sigma_1^2 \in H^*G(2,4)$, consider

$$\left\{ \begin{pmatrix} 0 & 0 & * & * \\ * & * & 0 & 0 \end{pmatrix} \right\}$$

whose closure is 2-dimensional in $G(2,4)$. This describes the space of all Σ meeting $\operatorname{span}(e_3, e_4)$ and $\operatorname{span}(e_1, e_2)$ non-trivially. We degenerate the latter to $\operatorname{span}(e_2, e_3)$. In the limit, any Σ not meeting the new intersection must be contained in the new span, i.e. must lie in the following translate of σ_{11} or of σ_2:

$$\sigma_{11} = \left\{ \begin{pmatrix} 0 & 0 & * & * \\ 0 & * & * & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} \qquad \sigma_2 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & 0 \end{pmatrix} \right\}$$

One can apply this repeatedly and get an algorithm, i.e., a combinatorial formula, for $\sigma_\lambda \cup \sigma_\mu$.

Example 4.7. $\sigma_1 \cup \sigma_{42} \in H^*(G(3,8))$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & * & 0 \\ * & * & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

Degenerating $\operatorname{span}(e_1,\dots,e_5)$ to $\operatorname{span}(e_2,\dots,e_6)$ gives in the case that Σ intersects the intersection $\operatorname{span}(e_4,e_5,e_6)$:

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & 0 & 0 \end{pmatrix}$$

and in the case that Σ is contained in the new spans:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & * & 0 \\ 0 & * & * & * & * & * & 0 & 0 \end{pmatrix}$$

Coskun has extended this to a Littlewood–Richardson rule for two-step flag varieties ([Cos04]) and general flag varieties (research announcement). This gives a complete solution for the problem of multiplying Schubert classes in $\mathrm{GL}(n)$-homogeneous spaces.

Example 4.8. $\sigma_{2,15} \cup \sigma_{3,16} \in H^*F(1,3;8)$:

$$\left(\begin{array}{cccccccc} 0 & 0 & * & * & * & * & * & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & * & * & * \\ * & * & * & * & 0 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} 0 & * & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & * & * & * & * & * & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \end{pmatrix}$$

This can be transformed into

$$\begin{pmatrix} 0 & 0 & * \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * \\ * & * & * & * & * & * & * & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

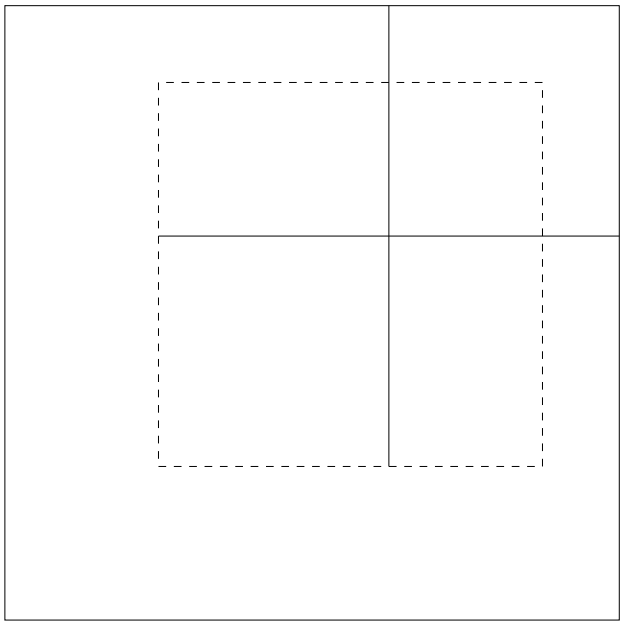
and

$$\begin{pmatrix} 0 & * & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & * & * & * & * & * & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

and, as a new type of move which did not occur in the previous example:

$$\begin{pmatrix} 0 & * & * \end{pmatrix} \cdot \begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

Coskun also introduced the new notation of "Mondrian tableaux". For the last variety, it is given by the following picture:



5. Bibliographic remarks

General references on Schubert varieties are for example an article by Kleiman and Laksov [KL72] and Fulton's book [Ful97].

On enumerative geometry, we mention the original book by Schubert [Sch79] and Kleiman's modern treatment [Kle76].

Modern work in this area can be found in articles by Buch, Kresch, and Tamvakis [BKT03], Vakil [Vak06], and Coskun [Cos04].

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