

# ON COVERINGS OF DELIGNE–MUMFORD STACKS AND SURJECTIVITY OF THE BRAUER MAP

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ABSTRACT. This paper proves a result on the existence of finite flat scheme covers of Deligne–Mumford stacks. This result is used to prove that a large class of smooth Deligne–Mumford stacks with affine moduli space are quotient stacks, and in the case of quasi-projective moduli space, to reduce the question to a classical question on Brauer groups of schemes.

## 1. INTRODUCTION

This paper is concerned with finite covers of Deligne–Mumford stacks by schemes, in connection with the theory of Brauer group. The reader is referred to [6] for basic references on algebraic stacks and Brauer groups. We are primarily concerned with Deligne–Mumford stacks; every Deligne–Mumford stack, separated and of finite type over a field, possesses a coarse moduli algebraic space ([10]).

Mumford, in [13], constructed intersection theory on the Deligne–Mumford stack  $\overline{\mathcal{M}}_g$  of stable curves of genus  $g$ . One tool that he used was the existence of a finite flat morphism  $Z \rightarrow \overline{\mathcal{M}}_g$ , where  $Z$  is a Cohen–Macaulay scheme of abelian level structures; his construction would have been made much simpler by the existence of such a morphism with  $Z$  smooth.

Afterwards Looijenga, in [11], showed the existence of a finite morphism  $Z \rightarrow \overline{\mathcal{M}}_g$ , where  $Z$  is a smooth projective scheme, over the field of complex numbers. A more algebraic construction, working over more general bases, was given by Pikaart and De Jong in [14], and another in [2].

In this paper we show that in fact the existence of such a  $Z$  is a very general phenomenon. We prove that a separated Deligne–Mumford stack with quasi-projective moduli space which is a quotient stack always possesses a finite flat cover by a scheme (Theorem 1); if the stack is smooth, then the scheme cover can also be chosen to be smooth. The hypothesis of being a global quotient stack with quasi-projective moduli space is often verified in practice (for example, for  $\overline{\mathcal{M}}_g$ , or for stacks of stable maps into a projective variety).

We also apply this result to the question of the surjectivity of the map from the usual Brauer group to the cohomological Brauer group of a scheme. We recall briefly some essential facts from [9]. The (Azumaya) Brauer group of a scheme  $X$  is the group of classes of sheaves of Azumaya algebras on  $X$ , and this group maps, via the Brauer map, into the cohomological Brauer group, that is, the torsion subgroup of

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the étale cohomology group  $H^2(X, \mathbb{G}_m)$ . An algebraic stack is called a quotient stack if it is isomorphic to the stack quotient of an algebraic space by a linear algebraic group scheme.

The connection between stacks and Brauer groups is one of the central themes of [6]: the Brauer map is surjective (hence an isomorphism) if and only if some associated algebraic stacks are all quotient stacks. In op. cit. it was shown that knowing the Brauer map to be an isomorphism implies the quotient stack property for tame Deligne–Mumford stacks that are gerbes, that is, have nonvarying stabilizer group, of order prime to the characteristic of the ground field. Our Theorem 1 is used to show (Theorem 2) that the surjectivity of the Brauer map implies the quotient stack property for smooth Deligne–Mumford stacks of finite type over a field, with some hypothesis on the characteristic. A final, unconditional, result (Theorem 3) guarantees the existence of Zariski coverings by quotient stacks for quite general classes of smooth Deligne–Mumford stacks.

## 2. RESULTS

**Theorem 1.** *Let  $X$  be a Deligne–Mumford stack, separated and of finite type over a field  $k$ , with quasi-projective coarse moduli space. Assume  $X$  is a quotient stack. Then there exists a quasi-projective scheme  $Z$  and a finite flat local complete intersection morphism  $Z \rightarrow X$ , such that the singular locus of  $Z$  is the inverse image of the singular locus of  $X$ .*

Recall that a Deligne–Mumford stack  $X$  over a field  $k$  is called *tame* if the order of the stabilizer group at any geometric point of  $X$  is relatively prime to the characteristic of  $k$  (or if  $\text{char } k = 0$ ). We say that a Deligne–Mumford stack is *generically tame* if it has a tame dense open substack.

**Theorem 2.** *Given a field  $k$  and a positive integer  $n$ , the following two conditions are equivalent.*

- (a) *Every smooth separated generically tame Deligne–Mumford stack over  $k$  of dimension  $n$  with quasi-projective moduli space is a quotient stack.*
- (b) *The Azumaya Brauer group of any smooth quasi-projective scheme over  $k$  of dimension  $n$  coincides with the cohomological Brauer group.*

*Remark.* In any discussion of quotient stacks, it is worth drawing a comparison with a related condition for algebraic stacks, known as the resolution property, which asserts that every coherent sheaf admits a surjection from a locally free coherent sheaf. The resolution property is discussed in some detail in a recent paper by Totaro [15]. For the stacks in Theorem 2, it is known (cf. [6], Remarks 2.15 and 4.3) that the condition to be a quotient stack is equivalent to the resolution property.

For  $n \leq 2$  the assertions of Theorem 2 have long been known to be true. For general  $n$ , statement (b) is given as an open question in [12], and it is widely conjectured that the statement is true. Techniques developed by Gabber may soon settle this question; already an announcement of proof by Gabber has been made (see, e.g., [3, p. 19]). Gabber has settled the case of affine schemes and separated unions of two affines [7]. So, we have:

**Theorem 3.** *Every smooth separated generically tame Deligne–Mumford stack over a field, whose moduli space is either affine or the union of two affine schemes, is a quotient stack.*

## 3. PROOFS OF RESULTS

The proof of Theorem 1 relies upon the following lemma.

**Lemma 1.** *Let  $f: U \rightarrow V$  be a proper morphism of quasi-projective schemes over an infinite base field  $k$ , with constant fiber dimension  $r > 0$ . Choose a projective embedding  $U \rightarrow \mathbb{P}^N$  for some  $N$ . Then for sufficiently large  $d$  the intersection of  $U$  with a generic hypersurface of degree  $d$  in  $\mathbb{P}^N$  is a Cartier divisor in  $U$ , of constant fiber dimension  $(r - 1)$  over  $V$  and whose singular locus is the intersection of the hypersurface with the singular locus of  $U$ .*

*Proof.* The irreducible components  $E \subset f^{-1}(v)$  of fibers of  $f$  over geometric points  $v: \text{Spec } \bar{k} \rightarrow V$  vary over a bounded family, hence only finite many Hilbert polynomials occur. Choosing  $d$  sufficiently large, we may assume the sheaf of ideals  $\mathcal{I}_E$  of any component  $E$  of a geometric fiber of  $f$  satisfies  $H^i(\mathbb{P}^N, \mathcal{I}_E(d)) = 0$  for  $i > 0$ . Then the codimension of the space of polynomials of degree  $d$  vanishing along some component of a fiber is bounded below by a polynomial of degree  $r$  in  $d$ . Since  $r > 0$ , this codimension is positive for sufficiently large  $d$ . Hence a generic hypersurface of degree  $d$  will not contain any component of any geometric fiber of  $f$ . This establishes the assertion concerning fiber dimension of the intersection. The generic hypersurface avoids all the associated points of  $U$ , meaning the intersection is a Cartier divisor in  $U$ . A standard Bertini-type argument establishes the assertion concerning the singular locus.  $\square$

*Proof of Theorem 1.* We are easily reduced to the case of an infinite base field  $k$ . Since  $X$  is a quotient stack, there exists a projectivized vector bundle  $P_0 \rightarrow X$  and a representable dense open substack  $Q_0 \subset P_0$  with  $Q_0 \rightarrow X$  surjective (see [5]). Call  $d$  the fiber dimension of  $P_0 \rightarrow X$ . Let  $S_0 \subseteq P_0$  be  $P_0 \setminus Q_0$  (with the reduced closed substack structure). Call  $P$  the fiber product  $P_0^t = P_0 \times_X \cdots \times_X P_0$  of  $t$  copies of  $P_0$ . Then the projections  $P \rightarrow P_0$  are representable, hence the complement of  $S_0^t = S_0 \times_X \cdots \times_X S_0$  is an algebraic space. The dimension of  $S_0^t$  is at most equal to  $t(d - 1) + \dim X$ , while the fiber dimension of  $P \rightarrow X$  is  $td$ ; so for large  $t$  we get a smooth projective morphism  $P \rightarrow X$  with a representable open substack  $Q \subseteq P$  such that the complement of  $Q$  in  $P$  has dimension less than the fiber dimension of  $\pi$ .

Let  $U$  be the moduli space of  $P$  and let  $V$  be the moduli space of  $X$ . We have an induced morphism  $f: U \rightarrow V$  which is proper. By hypothesis,  $V$  is quasi-projective. Let us show that  $U$  is quasi-projective as well. This can be shown using Geometric Invariant Theory, or with the following more elementary method.

**Lemma 2.** *Let  $\mathcal{L}$  be an invertible sheaf on a Deligne–Mumford stack  $T$ , separated and of finite type over a noetherian base scheme, and let  $U$  be the moduli space of  $T$ . Then some power  $\mathcal{L}^{\otimes d}$  (with  $d > 0$ ) is the pullback of an invertible sheaf  $\mathcal{M}$  on  $U$ .*

*Proof.* First of all, the statement is equivalent to saying that, if we call  $\pi: T \rightarrow U$  the canonical homomorphism,  $\pi_*(\mathcal{L}^d)$  is an invertible sheaf on  $U$ , and the adjunction map  $\pi^*\pi_*(\mathcal{L}^d) \rightarrow \mathcal{L}^d$  is an isomorphism. This is a local question in the étale topology, and by [10, Proposition 4.2] we may assume that  $T$  is of the form  $[S/G]$ , where  $G$  is a finite group acting on an affine scheme  $S$ . There is a spectral sequence

$$E_2^{pq} = H^p(G, H^q(S, \mathbb{G}_m)) \implies H^{p+q}([S/G], \mathbb{G}_m)$$

showing that the kernel of the pullback on Picard groups  $\text{Pic}[S/G] \rightarrow \text{Pic} S$  is  $H^1(G, \mathcal{O}^*(S))$ . By shrinking  $S$  we may assume that the pullback of  $\mathcal{L}$  to  $S$  is trivial; this means that  $\mathcal{L}$  comes from  $H^1(G, \mathcal{O}^*(S))$ , so some tensor power is trivial. But  $\pi_* \mathcal{O}_T = \mathcal{O}_U$ , by definition of moduli space, so this proves the lemma.  $\square$

By Lemma 2, there exists an invertible sheaf  $\mathcal{M}$  on  $U$  whose pullback to  $P$  is ample relative to  $P \rightarrow X$ . We claim that  $\mathcal{M}$  is ample relative to  $U \rightarrow V$ . Again, this is a local question in the étale topology on  $V$ , so we may assume that  $X$  is of the form  $[S/G]$ , where  $G$  is a finite group and  $S$  an affine scheme. We set  $T = S \times_X P$ , so that  $P = [T/G]$ . The pullback of  $\mathcal{M}$  to  $T$  is ample, and the projection  $T \rightarrow T/G = U$  is finite and surjective; hence  $\mathcal{M}$  is ample on  $U$ , and  $U$  is quasiprojective.

By repeated applications of Lemma 1, there is a complete intersection subscheme  $Z \subset U$  with  $Z \rightarrow V$  surjective and finite; by dimension reasoning we may take  $Z$  to be disjoint from the image in  $U$  of  $P \setminus Q$ . Since  $Q$  is representable, the morphism  $P \rightarrow U$  restricts to an isomorphism of  $Q$  to its image, hence  $Z$  lifts to a (representable) substack of  $P$ , also a complete intersection. Since  $P \rightarrow X$  is smooth, by the local criterion for flatness it follows that  $Z$  is flat over  $X$ . Also, the singular locus of  $Z$  is the pre-image of the singular locus of  $X$ .  $\square$

*Proof of Theorem 2.* First we show that (a) implies (b). Suppose  $\beta \in H^2(X, \mathbf{G}_m)$  is  $n$ -torsion, and we want to show  $\beta$  is in the image of the Brauer map. It suffices to show this after finite flat pullback, so in the case  $\text{char } k = p$  with  $p$  dividing  $n$  we take  $\varphi: X \rightarrow X$  to be a suitable iteration of the Frobenius map and by considering  $\varphi^* \beta$  we are reduced to the case  $n$  is relatively prime to  $p$ .

Now the associated gerbe banded by the  $n$ th roots of unity is a tame Deligne–Mumford stack with moduli space  $X$ , hence is a quotient stack by (a). Then by [6, Theorem 3.6],  $\beta$  lies in the image of the Brauer map.

Now we show (b) implies (a). Let  $X$  be as in statement (a). Then, first,  $X$  is a gerbe over a smooth separated stack  $Y$  such that  $Y$  has trivial generic stabilizer and thus by [6, Theorem 2.18],  $Y$  is a quotient stack. More precisely, if  $I \rightarrow X$  denotes the inertia stack and  $U$  is the open substack of  $X$  where the morphism  $I \rightarrow X$  is flat, with inverse image  $I_U$ , then the closure  $J$  of  $I_U$  in  $I$  is étale over  $F$ , and  $Y$  is the rigidification of  $X$  along  $J$  as in [2, Section 5.1]. The stacks  $X$  and  $Y$  have the same moduli space.

By Theorem 1, there is a smooth scheme  $Z$  and a finite flat surjective morphism  $Z \rightarrow Y$ , with  $Z$  smooth. The fiber product  $W := X \times_Y Z$  is a tame gerbe over  $Z$ . We claim that, after replacing  $Z$  by a finite étale cover, we can produce a finite representable étale cover  $W' \rightarrow W$ , so that  $W'$  is a gerbe banded by a product of groups of roots of unity over  $Z$ .

This can be checked as follows. First of all, we may assume that  $Z$  is connected; then the group of automorphisms of any two geometric points are isomorphic. Let  $G$  be such a group;  $X$  is a  $G$ -gerbe. Then ([6, Proposition 3.5]) we may replace  $Z$  by a finite étale cover and assume that  $W \rightarrow Z$  is banded by  $G$ . After a further cover, we also assume that the center  $H$  of  $G$  is isomorphic to a product of groups of roots of unity over  $Z$ . Take  $W'$  to be the stack of equivalences of the trivial  $G$ -gerbe with  $W$  which induce the identity map on bands; by [4, Section 1.2.5] (see also [8]) the stack  $W'$  is banded by  $H$ . There is a tautological evaluation map  $W' \rightarrow W$ , and this is finite, étale, and representable.

This  $W'$  is a fiber product  $W_1 \times_Z \cdots \times_Z W_r$ , where each  $W_i$  is a gerbe over  $Z$  banded by a group of roots of unity. Then, by the hypothesis (b) and the main result of [6], each  $W_i$  is a quotient stack, hence so is  $W'$ ; by [6, Lemma 2.13],  $X$  is a quotient stack as well.  $\square$

*Proof of Theorem 3.* The argument is just as in the proof that (b) implies (a) in Theorem 2, except that we invoke surjectivity of the Brauer map for affine schemes, or schemes that are unions of two affines ([7]) in place of the hypothesis (b).  $\square$

*Note added in proof* (September 2003) A. J. de Jong recently announced a new proof of assertion (b) of Theorem 2. The authors are grateful to him for sending them a preliminary copy of his manuscript.

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