HODGE-THEORETIC OBSTRUCTION TO THE EXISTENCE OF QUATERNION ALGEBRAS

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Abstract. This paper gives a necessary criterion in terms of Hodge theory for representability by quaternion algebras of certain 2-torsion classes in the unramified Brauer group of a complex function field. This criterion is used to give examples of threefolds with unramified Brauer group elements which are the classes of biquaternion division algebras.

1. Introduction

In this paper we exhibit complex function fields F and division algebras B over F which are of index strictly larger than the exponent in the Brauer group and which additionally are unramified; that is, they are the restrictions of classes in the Brauer groups of complete nonsingular models of F. These examples come from a general Hodge-theoretic criterion (Corollary 1) for certain 2-torsion elements of the unramified Brauer group of a complex function field to fail to be the classes of quaternion algebras. The proof uses a cycle map to equivariant cohomology [4]. At the heart of this result is a computation (5) of Chow rings of certain 'twisted' classifying stacks for the group $\mathbf{Z}/2$.

We now turn to the statement of the main theorem.

Theorem 1. *Let X be a nonsingular complex projective variety. Let $\alpha \in H^2(X, \mathbf{Z}/2)$ be an element such that* (i) *the image of α in the Brauer group of the function field $\mathrm{Br}(\mathbf{C}(X))$ is representable by a quaternion algebra over $\mathbf{C}(X)$, and* (ii) *there exists a preimage $\alpha_0 \in H^2(X, \mathbf{Z})$ under the natural map $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z}/2)$. Then there exists a codimension 2 algebraic cycle on X whose class in cohomology is equal to $4(\alpha_0^2 + 2\varepsilon)$, for some $\varepsilon \in H^4(X, \mathbf{Z})$.*

The promised Hodge-theoretic criterion is an immediate corollary, using the fact that the class of a codimension 2 algebraic cycle on X must be of Hodge type $(2, 2)$.

Corollary 1. *Let X be a nonsingular complex projective variety. Let N denote the Néron–Severi group of X and $i\colon (H^2(X, \mathbf{Z})/N) \otimes \mathbf{Z}/2 \to \mathrm{Br}(\mathbf{C}(X))$ the injective homomorphism to the Brauer group induced by $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z}/2)$ followed by the natural map to the Brauer group. Let $M = H^4(X, \mathbf{Z}) \cap H^{2,2}(X)$; then for any $\beta \in (H^2(X, \mathbf{Z})/N) \otimes \mathbf{Z}/2$, the image of β under the homomorphism*

$$(H^2(X, \mathbf{Z})/N) \otimes \mathbf{Z}/2 \to (H^4(X, \mathbf{Z})/M) \otimes \mathbf{Z}/2 \tag{1}$$

given by $\beta \mapsto \beta^2$ is an obstruction to representing $i(\beta)$ by a quaternion algebra over $\mathbf{C}(X)$.

Date: 31 October 2001.
2000 *Mathematics Subject Classification.* 16K50 (primary); 14C17, 14F22 (secondary).
The author was partially supported by an NSF Postdoctoral Research Fellowship.

Finally, the examples referred to above come about by computing the obstruction map (1) for some particular complex threefolds. The resulting statement is as follows.

Theorem 2. *Let X be a nonsingular complex projective surface. Assume that $h^{2,0}(X) \neq 0$, so there exists $\beta \in H^2(X, \mathbf{Z})$ whose reduction to $H^2(X, \mathbf{Z}/2)$ and image λ in the Brauer group of X is nonzero. Let $V \to X$ be a smooth conic which is a Brauer–Severi variety for λ, and let $F = \mathbf{C}(V)$, the function field of V. Then there exists a biquaternion division algebra B over F, such that B is the restriction of a globally defined sheaf of Azumaya algebras on V.*

By way of background, we quickly review some facts about Brauer groups and give the context for the present examples. The Brauer group of a field F is the group of finite dimensional central simple algebras over F with the tensor product operation, modulo the equivalence relation generated by $D \sim M_k(D)$ for every central division algebra D over F and natural number k, with $M_k(D)$ the $k \times k$ matrices with entries in D. By Wedderburn's Theorem, every central simple algebra is isomorphic to some $M_k(D)$, and the dimension of D as a vector space over F must be a perfect square n^2. In such a case we call n the *index* of the corresponding class in $\mathrm{Br}(F)$. The *exponent* of $\alpha \in \mathrm{Br}(F)$ is the smallest natural number k such that $k\alpha = 0$ in $\mathrm{Br}(F)$. Some classically known facts about exponent and index are: (i) we always have $k \mid n$; (ii) k and n have the same set of prime factors; and (iii) all pairs (k, n) satisfying (i) and (ii) occur as exponent and index of a Brauer group element of some field.

Assume $\operatorname{char} F \neq 2$. For fields of transcendence degree 2 over an algebraically closed field, or more generally for C_2-fields, every element in the Brauer group of exponent 2 must have index 2 (that is, it must be the class of a quaternion algebra); see the proof by Artin and Harris in [1] or the argument by Platonov given in [13]. A deep result of Merkurev [9] asserts that if F is any field of characteristic different from 2, then the 2-torsion subgroup of $\mathrm{Br}(F)$ is generated by quaternion algebras. By Albert's criterion, the tensor product of two quaternion algebras over F is a division algebra if and only if a quadratic form of type

$$ax^2 + by^2 - abz^2 - cu^2 - dv^2 + cdw^2 \tag{2}$$

has no nontrivial zeroes over F. So the condition that every element of exponent 2 in $\mathrm{Br}(F)$ should have index 2 is equivalent to isotropicity of the form (2) for every $a, b, c, d \in F^*$. One says F is *linked* if these equivalent conditions are satisfied. For instance, $F = \mathbf{C}((a))((b))((c))((d))$ is not linked, since (2) over F is anisotropic. One easily has examples of non-linkage for function fields (say, over $\mathbf{C}$) of transcendence degree greater than or equal to 3; see the discussion in [2] for examples, including some from early papers on Brauer groups.

While examples in Brauer groups where the exponent is not equal to the index are known classically, most examples over function fields do not come from anything global on a complete model. Here we use the cohomological interpretation of the Brauer group as H^2 with values in the étale sheaf $\mathbb{G}_m$. For instance, when $F = \mathbf{C}(x_1, \ldots, x_n)$ we have $\mathrm{Br}(X) = 0$ for any smooth complete model X. Indeed, $\mathrm{Br}(\mathbf{P}^n) = 0$, and it follows from cohomological purity and the valuative criterion for properness that for any smooth complete complex variety X, the image of the (injective) homomorphism $\mathrm{Br}(X) \to \mathrm{Br}(\mathbf{C}(X))$ can be identified with the unramified Brauer group [12] of $\mathbf{C}(X)$, and hence the cohomological Brauer group $\mathrm{Br}(X) = H^2(X, \mathbf{G}_m)$ is a birational invariant of a smooth complete complex variety. The novelty in the present work is that our examples are of elements with exponent different from index, which lie in the unramified Brauer group. Taking the present examples as inspiration, Colliot-Thélène has reproduced our Theorem 2 using different methods and has exhibited similar unramified elements of Brauer groups taking other values for exponent and index [2].

2. Preliminaries

All schemes are of finite type over a field. We use the language of algebraic stacks [3, 8]; all stacks are algebraic and of finite type over a field. For X a stack, A_*X denotes the cycle groups of [7], with $A^*X := A_{n-*}X$ when X is smooth of pure dimension n. The stacks we encounter are approximated closely enough by quotients of algebraic group actions on varieties, so that the relevant Chow groups turn out to be equivariant Chow groups in the sense of Edidin–Graham–Totaro [4, 14]. For a topological space X, $H^*(X)$ denotes $H^*(X, \mathbf{Z})$. When X is a complex variety, $H^*(X)$ denotes the cohomology of the underlying analytic space, and when X is a reduced algebraic stack of finite type over $\operatorname{Spec} \mathbf{C}$, $H^*(X)$ denotes the cohomology of a topological realization of the underlying analytic stack. $\mathrm{Br}(X)$ denotes the cohomological Brauer group of the scheme X; for basic facts on Brauer groups, see [6, 10].

Proposition 1. *Let X be a regular pure-dimensional algebraic stack with finite geometric stabilizers. Then the first Chern class homomorphism $\mathrm{Pic}(X) \to A^1X$ is an isomorphism.*

Proof. A finite cover by a scheme $Y \to X$ exists [5, Theorem 2.8]. We may suppose Y normal, hence finite flat over X away from a closed substack $Z \subset X$, empty or of codimension greater than or equal to 3. So, $U := X \smallsetminus Z$ is isomorphic to the stack quotient of a linear algebraic group acting on an algebraic space (see [7, Proposition 3.5.7]). By the isomorphisms $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ and $A^1X \to A^1U$, we are reduced to the corresponding statement for equivariant Chow groups, and this is [4, Theorem 1]. □

The results that follow use the fact that gerbes over a scheme X banded by the nth roots of unity μ_n are classified by $H^2(X, \mu_n)$; see [10, IV.2]. (Strictly speaking, this is cohomology for the flat topology, but this equals étale cohomology when n is invertible in the base field [6, III.11].) The reader is referred to [5] for a discussion of the relation between Brauer group elements being represented by sheaves of Azumaya algebras and the corresponding gerbes having vector bundles with faithful stabilizer actions.

Proposition 2. *Let X be an irreducible regular scheme, and let n be a positive integer. Let $\mathscr{G}$ be a gerbe over X, banded by μ_n. Then the following are equivalent.*

(i) *The gerbe $\mathscr{G} \to X$ is Zariski locally trivial.*
(ii) *The restriction of $\mathscr{G}$ to the generic point of X is trivial.*
(iii) *There exists a line bundle on $\mathscr{G}$ on which the action of stabilizer groups at geometric points of $\mathscr{G}$ is faithful.*

Proof. Clearly, (i) implies (ii). If $k(X)$ denotes the function field of X, we have $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ injective, so (ii) implies that the classifying element $\beta \in H^2(X, \mu_n)$ lies in the image of the boundary homomorphism

$$\delta\colon H^1(X, \mathbf{G}_m) \to H^2(X, \mu_n)$$

of the Kummer sequence. If $\beta = \delta(\alpha)$, then $\mathscr{G}$ is isomorphic to a $\mathbf{G}_m$-quotient of the principal bundle on X associated with α. So (ii) implies (iii). Given (iii), we can identify $\mathscr{G}$ with a $\mathbf{G}_m$-quotient of a principal bundle on X, say classified by $\alpha \in H^1(X, \mathbf{G}_m)$. Now $\delta(j\alpha) = \beta$ for some integer j prime to n, and (i) holds. □

Since a line bundle on $\mathscr{G}$ is a pullback from X if and only if the action of the stabilizer group at geometric points is trivial, we have the following corollary.

Corollary 2. *Let p be a prime. Let X be an irreducible regular scheme, and let $f\colon \mathscr{G} \to X$ be a gerbe banded by μ_p. Then*

$$f^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(\mathscr{G})$$

is an isomorphism if and only if (i)–(iii) *of Proposition 2 fail for $\mathscr{G}$.*

In the following two statements, we show that in the case that the equivalent conditions of Proposition 2 hold, the Chow groups of the gerbe can be explicitly related to the Chow groups of the base.

Proposition 3. *Let n be a positive integer. Let X be a scheme, and let $\mathscr{G}$ be a gerbe over X, banded by μ_n, classified by $\beta \in H^2(X, \mu_n)$. Suppose β is in the image of the boundary homomorphism of the Kummer sequence. If F is a line bundle on X whose class in* $\operatorname{Pic}(X)$ *is sent to β, then for each integer k there is an exact sequence*

$$\bigoplus_{s \geqslant 1} A_{k+s}X \to \bigoplus_{s \geqslant 0} A_{k+s}X \to A_k\mathscr{G} \to 0,$$

where leftmost map is $(\alpha_1, \alpha_2, \ldots) \mapsto (c_1(F) \cap \alpha_1, c_1(F) \cap \alpha_2 - n\alpha_1, \ldots)$.

Proof. We follow the program of equivariant intersection theory [4] as follows: $\mathscr{G}$ is a $\mathbf{G}_m$-quotient of F minus the zero section, so scheme approximations to $\mathscr{G}$ are complements of zero sections of line bundles over $X \times \mathbf{P}^r$:

$$A_k\mathscr{G} = A_{k+r+1}((F \otimes \mathscr{O}_{\mathbf{P}^r}(-n) \smallsetminus s(X \times \mathbf{P}^r)) \qquad (\text{any } r \gg 0).$$

Applying the standard excision sequence gives the result. □

Corollary 3. *If the conditions of Proposition 3 are met, and if additionally X is smooth, then there is a natural map*

$$A^*(X)[z]/(nz - c_1(F)) \to A^*\mathscr{G},$$

which is an isomorphism of rings.

An analogous result to Corollary 3 in the topological category is Proposition 4, stated below after a topological lemma.

Lemma 1. *Let Y be a topological space, and let E be a topological complex line bundle on Y, with zero section s. Assume E is numerable (that is, it trivializes on some numerable open cover; this is automatic if Y is paracompact). Then there is a natural injective homomorphism*

$$H^k(Y)/(c_1(E) \cup H^{k-2}(Y)) \to H^k(E \smallsetminus s(Y)),$$

for every k, which is surjective precisely when $c_1(E) \cup -\colon H^{k-1}(Y) \to H^{k+1}(Y)$ is injective.

Proof. We compactify E by setting $P = \mathbf{P}(E \oplus 1)$, the (complex) projectivization of the Whitney sum of E and a trivial complex line bundle. Now we have E and $F := P \smallsetminus s(Y)$, line bundles over Y, with $E \cap F = E \smallsetminus s(Y)$. The Mayer-Vietoris sequence in cohomology gives

$$H^k(P) \to H^k(Y)^2 \to H^k(E \smallsetminus s(Y)) \to H^{k+1}(P) \to H^{k+1}(Y)^2. \tag{3}$$

Let L denote the tautological complex line bundle on P; then the projective bundle theorem dictates that $H^k(Y) \oplus H^{k-2}(Y) \simeq H^k(P)$ via $(\alpha, \beta) \mapsto \alpha + (c_1(L) \cup \beta)$. Since L is trivial on E and $L|_{P \smallsetminus E} \simeq E$, the leftmost map in (3) is

$$(\alpha, \beta) \mapsto \big(\alpha, \alpha + (c_1(E) \cup \beta)\big).$$

The result now follows. □

Fix a realization of the classifying space $B(\mathbf{Z}/n)$ as a topological abelian group [11]; then $H^2(-, \mathbf{Z}/n)$ classifies principal $B(\mathbf{Z}/n)$-bundles.

Proposition 4. *Let X be a topological space, and assume X is homotopy-equivalent to a CW complex. Suppose that $\beta \in H^2(X, \mathbf{Z}/n)$ is the image of $\beta_0 \in H^2(X)$. Let $G \to X$ be a principal $B(\mathbf{Z}/n)$-bundle classified by β. Then there is an injective ring homomorphism*

$$H^*(X)[u]/(nu - \beta_0) \to H^*(G),$$

which is an isomorphism in degrees less than or equal to 2.

Proof. Let $\mathscr{O}_{\mathbf{CP}^\infty}(-n)$ denote the nth twist of the tautological line bundle on $\mathbf{CP}^\infty$. The complement of the zero section in $\mathscr{O}_{\mathbf{CP}^\infty}(-n)$ is homotopy-equivalent to $B(\mathbf{Z}/n)$, so by standard arguments, if L is a complex line bundle on X with $c_1(L) = \beta_0$, then the complement, over $X \times \mathbf{CP}^\infty$, of the zero section of $L \otimes \mathscr{O}_{\mathbf{CP}^\infty}(-n)$ is homotopy-equivalent to G. Now the result follows from Lemma 1 (applied to $Y = X \times \mathbf{CP}^\infty$) and the general fact that $H^0(X)$ and $H^1(X)$ are torsion free. □

Proposition 5. *Let X be a scheme of pure dimension k, let n be a positive integer, and let $f \colon \mathscr{G} \to X$ be a gerbe banded by μ_n. Then for any $\alpha \in A_{k-1}\mathscr{G}$, we have $n\alpha = f^*\beta$ for some $\beta \in A_{k-1}X$.*

Proof. We may assume X is reduced. Consider the components of the regular locus $X^{\mathrm{reg}} = \coprod X_i$. For each i, let f_i denote the restriction of f over X_i; we claim the image of $f_i^* \colon A_{k-1}X_i \to A_{k-1}\mathscr{G}_i$ contains $nA_{k-1}\mathscr{G}_i$. Indeed, this follows by Proposition 3 when f_i is Zariski locally trivial; otherwise by Corollary 2 combined with Proposition 1, f_i induces an isomorphism of Chow groups in dimension $k-1$. The result now follows by comparing the excision sequences for $X^{\mathrm{reg}} \subset X$ and $\mathscr{G}^{\mathrm{reg}} \subset \mathscr{G}$. □

3. Proof of the main theorem

Let X and $\alpha \in H^2(X, \mathbf{Z}/2)$ be as in the statement of Theorem 1. Let $\mathscr{G}$ be the associated $\mathbf{Z}/2$-gerbe. By condition (i), there exists a quaternion algebra over $\mathbf{C}(X)$ whose class in $\mathrm{Br}(\mathbf{C}(X))$ is the image of α. This spreads out as a sheaf of Azumaya algebras over an open subset of X, and to this we can associate a rank-2 vector bundle on an open subset of $\mathscr{G}$. Since X is regular, the bundle (and hence also the associated Brauer–Severi variety) extends to $X \smallsetminus Z$ for some closed $Z \subset X$, empty or of codimension greater than or equal to 3. So we get a vector bundle E on $\mathscr{G}_{X \smallsetminus Z} := \mathscr{G} \times_X (X \smallsetminus Z)$ and a Brauer–Severi variety $V \to X \smallsetminus Z$, with $\mathbf{P}(E) \simeq V \times_X \mathscr{G}$. As $\mathscr{G}_{X \smallsetminus Z}$ is a quotient stack of a smooth variety by a linear algebraic group, we can identify $A^*\mathscr{G}_{X\smallsetminus Z}$ and $H^*(\mathscr{G}_{X\smallsetminus Z})$, respectively, with equivariant Chow and cohomology groups.

The class in $\mathrm{Br}(X)$ of a Brauer–Severi variety over X is the obstruction to identification with the projectivization of an algebraic vector bundle. Condition (ii) is the vanishing of the analogous topological obstruction. Hence V is topologically isomorphic to a projectivized vector bundle. We let B denote a complex rank-2 topological vector bundle over $X \smallsetminus Z$, such that $V(\mathbf{C}) \simeq \mathbf{P}(B)$.

Let us denote the restrictions of $\mathscr{G}$, V, and E over the generic point $\operatorname{Spec}\mathbf{C}(X)$ by $\mathscr{G}_{\mathrm{gen}}$, V_{gen}, and E_{gen}, respectively. Since the conic V_{gen} is not rational, the Chow ring of V_{gen} is $\mathbf{Z}[y]/(y^2)$, with y the class of a degree-2 point on V_{gen}. The Chow ring of $\mathbf{P}(E_{\mathrm{gen}})$ is related to $A^*\mathscr{G}_{\mathrm{gen}}$ by the projective bundle formula and is related to A^*V_{gen} by Corollary 3, and hence

$$A^*\mathscr{G}_{\mathrm{gen}}[w]/(w^2 - c_1(E_{\mathrm{gen}})w + c_2(E_{\mathrm{gen}})) \simeq \mathbf{Z}[y,z]/(y^2, 2z - ky),$$

for some integer k. We have $A^1\mathscr{G}_{\mathrm{gen}} = 0$ by Corollary 2. Hence $A^1\mathbf{P}(E_{\mathrm{gen}}) \simeq \mathbf{Z}$, so k must be odd, and without loss of generality we can suppose that $k = 1$. Then w and z both generate $A^1\mathbf{P}(E_{\mathrm{gen}})$, so $w = \pm z$, and now

$$A^2\mathscr{G}_{\mathrm{gen}} \cong \mathbf{Z}/4, \tag{4}$$

with $c_2(E_{\mathrm{gen}})$ as a generator. For comparison, we record

$$\begin{aligned} A^*B(\mathbf{Z}/2) &\cong \mathbf{Z} \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \cdots, \\ A^*\mathscr{G}_{\mathrm{quatern}} &\cong \mathbf{Z} \oplus \quad 0 \quad \oplus \mathbf{Z}/4 \oplus \quad 0 \quad \oplus \mathbf{Z}/4 \oplus \cdots, \end{aligned} \tag{5}$$

for any base field k with $\operatorname{char} k \neq 2$, where $\mathscr{G}_{\mathrm{quatern}}$ is any $\mathbf{Z}/2$-banded gerbe over $\operatorname{Spec} k$ whose classifying element has image in $\mathrm{Br}(k)$ equal to the class of a quaternion algebra over k.

Topologically, Proposition 4 gives an injective ring homomorphism

$$H^*(X)[u]/(2u - \alpha_0) \to H^*(\mathscr{G}),$$

which is an isomorphism in degrees less than or equal to 2. Let f denote the projection $\mathscr{G}_{X\smallsetminus Z} \to X \smallsetminus Z$. Recall the topological vector bundle B on $X \smallsetminus Z$; we have $\mathbf{P}(f^*B) \simeq \mathbf{P}(E)$, and hence

$$f^*B \simeq E \otimes L$$

for some topological complex line bundle L on $\mathscr{G}_{X\smallsetminus Z}$. Comparing the restrictions over a point of X, we see that L must be nontrivial on fibers of f. So,

$$c_1(L) = u + f^*\delta$$

for some $\delta \in H^2(X \smallsetminus Z) \simeq H^2(X)$. There is a cycle class map to equivariant cohomology which respects Chern classes [4]; now, using $A^1\mathscr{G} \simeq A^1X$ (Corollary 2), we see that

$$c_2(E) = -c_1(L)^2 - c_1(E)c_1(L) + f^*c_2(B) = -u^2 + uf^*\beta + f^*\varepsilon$$

for some $\varepsilon \in H^4(X \smallsetminus Z) \simeq H^4(X)$ and $\beta \in H^2(X)$. Hence $2c_2(E) = -2u^2 + f^*\beta'$ in $H^4(\mathscr{G}_{X\smallsetminus Z})$, for some $\beta' \in H^4(X)$.

Let $\gamma = c_2(E) \in A^2\mathscr{G}_{X\smallsetminus Z} \simeq A^2\mathscr{G}$. By (4), $4c_2(E_{\mathrm{gen}}) = 0$ in $A^2\mathscr{G}_{\mathrm{gen}}$, and hence 4γ vanishes in $A^2\mathscr{G}_U$ for some nonempty open $U \subset X$. Let $Y = X \smallsetminus U$; we may assume Y has pure codimension 1. Let $n = \dim X$, and let i denote the inclusion $\mathscr{G}_Y := \mathscr{G} \times_X Y \to \mathscr{G}$. By excision, 4γ lies in the image of $i_*\colon A_{n-2}\mathscr{G}_Y \to A_{n-2}\mathscr{G}$. By Proposition 5, now, we have

$$8\gamma = f^*\eta$$

for some $\eta \in A^2X$. So the cycle class of η satisfies $f^* \operatorname{cl}(\eta) = -2f^*\alpha_0^2 + 4f^*\beta'$. The kernel of $f^*\colon H^4(X) \simeq H^4(X \smallsetminus Z) \to H^4\mathscr{G}_{X\smallsetminus Z}$ is 2-torsion, so $4\alpha_0^2 - 8\beta'$ is the class of an algebraic cycle on X, as required.

4. Brauer–Severi varieties over surfaces

Here we prove Theorem 2. Since X, here, is a complex surface, every (nonzero) 2-torsion element $\lambda \in \operatorname{Br}(\mathbf{C}(X))$ is represented by a quaternion algebra; moreover, if λ is unramified, then the quaternion algebra over $\mathbf{C}(X)$ is the restriction of a sheaf of Azumaya algebras defined on all of X. Associated, then, to $\beta \in H^2(X)$ as in the statement of Theorem 2 is a Brauer–Severi variety $V \to X$ for λ, of relative dimension 1.

Let $N = \operatorname{NS}(X)$, the Néron–Severi group. Without loss of generality, we may suppose the image of β in $H^2(X)/N$ is a primitive lattice element, and now since N contains the torsion in $H^2(X)$ we have a direct sum decomposition $H^2(X) = N \oplus \langle\beta\rangle \oplus T$, for some torsion-free T.

By the hypotheses, V is topologically the projectivization of a rank-2 complex vector bundle on X. We use the notation of the proof of Theorem 1 applied to X and β. In particular, B is a topological complex rank-2 vector bundle such that $V(\mathbf{C}) \simeq \mathbf{P}(B)$. So

$$H^*(V) = H^*(X)[x]/(x^2 - c_1(B)x + c_2(B)).$$

We claim that $\operatorname{NS}(V) = N \oplus \langle 2x + \gamma\rangle$ for some $\gamma \in H^2(X)$, in fact, with $\gamma \in \beta + 2H^2(X)$. This is clear. Now, in

$$H^4(V) = H^4(X) \oplus xN \oplus \langle x\beta\rangle \oplus xT,$$

we have

$$H^4(V) \cap H^{2,2}(V) = H^4(X) \oplus xN.$$

Now $f^*B \simeq E \otimes L$ gives $x^2 = xc_1(E) + 2xc_1(L) - c_2(B)$. Since $2c_1(L) \in \beta + 2H^2(X)$, the image of x under the obstruction map (1) is nonzero. The obstruction map for V is thus completely determined by its vanishing on $H^2(X)$ and its nonvanishing on x.

We conclude by showing that the obstructed Brauer group elements in $\mathbf{C}(V)$ are division algebras of index 4 (and hence biquaternion algebras by a result of Albert), and that these division algebras are restrictions of sheaves of Azumaya algebras on V. Every obstructed 2-torsion element of $\operatorname{Br}(\mathbf{C}(V))$ is the image of $x + \delta$ for some $\delta \in H^2(X)$. Such an element is the pullback of some 4-torsion element $\tau \in \operatorname{Br}(\mathbf{C}(X))$, with τ also unramified (take τ to be the image via $H^2(X) \to H^2(X, \mathbf{Z}/4) \cong H^2(X, \mu_4)$ of $2\delta - \gamma$). Again, because X is a complex surface, τ is represented by a division algebra of index 4 which is the restriction of a sheaf of Azumaya algebras on X, and this pulls back to V.

Acknowledgements. The author thanks J.-L. Colliot-Thélène, B. Hassett, R. Hoobler, and T. Pantev for helpful correspondence and advice. The anonymous referee of an earlier version of this work is credited with observing the connection with 4-torsion Brauer group elements which appears at the conclusion of the proof of Theorem 2.

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