ASSOCIATIVITY RELATIONS IN QUANTUM COHOMOLOGY

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1. Introduction

The geometry of moduli spaces of stable maps of genus 0 curves into a complex projective manifold X leads to a system of quadratic equations in the tree-level (genus 0) *Gromov-Witten numbers* of X. In elementary examples, these equations solve for all such numbers, uniquely and consistently, from starting data. The beautiful paper of Di Francesco and Itzykson [1] presents a number of examples in this context.

One of the foundational papers in the area of quantum cohomology, [4], explains this phenomenon, at least in some cases, by proving the *first reconstruction theorem*. This theorem applies to manifolds X such that $H^*(X, \mathbb{Q})$ is generated by $H^2(X)$. This result gives an effective procedure for solving for genus 0 Gromov-Witten numbers from starting data using the quadratic relations (since this entire paper is concerned only with the genus 0 invariants, we omit explicit mention of genus from now on).

In all but the simplest cases there will be more than one way of using the relations to solve for the numbers. In other words, the system of equations is *overdetermined*. In the same paper the authors ask whether the seemingly redundant equations follow algebraically from the useful ones. Consistency of this overdetermined system of equations, as an algebraic (or combinatorial) statement rather than a geometric statement, has only been noted in the literature in isolated instances, cf. [2].

The equations were predicted on the basis of physical theories and later confirmed by rigorous study of the moduli spaces. The view taken by the physicists is quite useful: the numbers are combined into a generating function and the relations are represented by differential equations. The survey paper [6] documents these early studies.

This paper continues in the spirit of these early investigations into the structure of the relations. The main result is a generalization of the first reconstruction theorem. Keeping the hypothesis that the cohomology ring of X is generated by divisors, we show that an initial collection of numbers and relations determines, purely algebraically, the entire system of relations (*strong reconstruction*) as well as the Gromov-Witten numbers. Examples, in the last section, illustrate the existence of *non-geometric* solutions.

For a manifold X with $H = H^*(X, \mathbb{C})$, the associativity relations can be expressed geometrically by saying that an associated connection on T_H is flat, i.e., that the quantum potential function dictates on H the structure of a Frobenius manifold [2, 4]. Near a semisimple solution, the equations for flatness can be proved equivalent (with suitable assumptions) to the well-studied *Schlesinger equations*, cf. [5]. Strong

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reconstruction captures quantum deformations of H near the zero point, which is never semisimple.

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2. The basic problem

To set up the system of associativity relations, we start with the cohomology ring $A^* = H^{2*}(X,\mathbb{Q})$ of a complex projective manifold X which has cohomology only in even dimensions. If we fix an isomorphism $\int: A^n \simeq \mathbb{Q}$ then by duality the induced pairing $A^k \otimes A^{n-k} \to A^n \to \mathbb{Q}$ is nondegenerate for each k. Let us denote by $\{T_i \,|\, i \in I\}$ a basis for A as a $\mathbb{Q}$-module, consisting of homogeneous elements, and define $g_{ij} = \int T_i \cdot T_j$. We know that (g_{ij}) must be an invertible matrix; let us denote by (g^{ij}) the inverse matrix.

In order to get a well-defined system of equations, we need two more pieces of data: a maximal-rank integral lattice Λ in $(A^1)^*$ (we would ordinarily take Λ to be the dual lattice to $H^{1,1}(X) \cap H^2(X,\mathbb{Z})$), and a strongly convex polyhedral cone $\Theta \subset (A^1)^* \otimes \mathbb{R}$ (if we wish to capture geometry then Θ should contain the dual to the ample cone). We also assume that the chosen basis $\{T_i\}$ contains the identity element $T_1 = 1$, as well as the elements T_{σ_i} dual to some $\mathbb{Z}$-basis of Λ. We label the rest of the basis elements by T_{τ_j}, so we have $B = \{T_1, T_{\sigma_1}, \ldots, T_{\sigma_r}, T_{\tau_1}, \ldots, T_{\tau_s}\}$. We may assume the basis ordered so that $\operatorname{codim} T_{\tau_j} \le \operatorname{codim} T_{\tau_k}$ whenever $j \le k$. By our notation we have integers r and s defined to be the ranks of the first and the higher graded pieces of A, respectively.

We have taken $A^* = H^{2*}(X)$ for some manifold X since our main interest is in the systems of associativity equations coming from geometry. However, we may just as well take X to be a projective orbifold (we only require $\mathbb{Q}$-duality). Even more generally, there need not be any X at all: A may be any graded Gorenstein artinian $\mathbb{Q}$-algebra with socle in degree $n \ge 2$ such that $A^0 \cong \mathbb{Q}$ and $A^1 \ne 0$.

If A is a Gorenstein ring as above, with the additional data consisting of $\int$, Λ, and Θ, and if B be a basis for A as above, then we set $C = \Theta \cap \Lambda \setminus \{0\}$ (in the geometric situation, C corresponds to the set of effective curve classes). For any ω in the interior of the dual cone to Θ, the set $\{\beta \in \mathbb{Z}\langle \sigma_1^*, \ldots, \sigma_r^* \rangle \,|\, \langle \beta, \omega \rangle < k\}$ is finite for any k. The set of unknown numbers is defined to be the collection of all $N(\beta; d_1, \ldots, d_s)$ with $\beta \in C$ and $d_j \ge 0$ for all j. (We remark that there is nothing resembling a canonical class in the set-up). There is a system of equations in these unknowns, one for each 4-tuple (i,j,k,l) of elements of the basis indexing set $I = \{1, \sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s\}$ and each *degree* $(\beta; d_1, \ldots, d_s)$.

We introduce formal variables $\{y_i \,|\, i \in I\}$ and define the *potential function*

$$\Phi = \Phi^{\mathrm{cl}} + \Gamma$$

to be the sum of the *classical part*

$$\Phi^{\mathrm{cl}} = \frac{1}{6} \sum_{i,j,k \in I} \left(\int T_i \cdot T_j \cdot T_k \right) y_i y_j y_k$$

and the *quantum correction*

$$\Gamma = \sum_{\substack{\beta\in C\\ d_1,\ldots,d_s\geq 0}} N(\beta;d_1,\ldots,d_s)e^{c_1 y_{\sigma_1}}\cdots e^{c_r y_{\sigma_r}}\frac{y_{\tau_1}^{d_1}}{d_1!}\cdots\frac{y_{\tau_s}^{d_s}}{d_s!},$$

where for $\beta\in C$, we denote by c_i the pairing of β with T_{σ_i}. We see that Γ is an infinite sum of polynomials in $\{e^{\pm y_{\sigma_1}},\ldots,e^{\pm y_{\sigma_r}}\}$ times formal power series in $\{y_{\tau_1},\ldots,y_{\tau_s}\}$, with N's as coefficients. Since Θ is assumed strongly convex, it is clear that Γ lives in a formal power series ring.

Any 4-tuple (i,j,k,l) of elements of I determines a differential equation

$$\sum_{e,f}\Phi_{ije}g^{ef}\Phi_{fkl} = \sum_{e,f}\Phi_{jke}g^{ef}\Phi_{fil}. \tag{1}$$

We use a subscript $i\in I$ to denote partial differentiation with respect to y_i. Isolating the coefficient of $e^{c_1 y_{\sigma_1}}\cdots e^{c_r y_{\sigma_r}}y_{\tau_1}^{d_1}\cdots y_{\tau_s}^{d_s}$ on each side produces a quadratic equation in N's, which we call an *associativity relation* (they imply associativity of the so-called quantum product; see [3]). Following Dubrovin [2] we call the system of equations (1) the *WDVV equations* (after E. Witten, R. Dijkgraaf, H. Verlinde and E. Verlinde). A particular WDVV equation is represented symbolically by an equivalence of Feynman diagrams

$$\begin{pmatrix} i & & l \\ & \succ\!\prec & \\ j & & k\end{pmatrix} \sim \begin{pmatrix} i & \curlyvee & l \\ j & \curlywedge & k\end{pmatrix}$$

We adopt the notation $\begin{pmatrix} i & & l \\ & \succ\!\prec & \\ j & & k\end{pmatrix}$ to refer to the WDVV equation (1) and $\begin{pmatrix} i & & l \\ & \succ\!\prec & \\ j & & k\end{pmatrix}^{(\beta;d_1,\ldots,d_s)}$ to refer to a particular associativity relation. More generally, for $\xi,\pi,\rho,\sigma\in A$, we use the notation $\begin{pmatrix} \xi & & \sigma \\ & \succ\!\prec & \\ \pi & & \rho\end{pmatrix}$ to refer to the equation obtained by writing each element in terms of the basis and summing in a multilinear fashion. For subsets $\Xi,\Pi,\mathrm{P},\Sigma$ of A, we let $\begin{pmatrix} \Xi & & \Sigma \\ & \succ\!\prec & \\ \Pi & & \mathrm{P}\end{pmatrix}$ refer to the collection of equations $\begin{pmatrix} \xi & & \sigma \\ & \succ\!\prec & \\ \pi & & \rho\end{pmatrix}$ with ξ in Ξ, etc. As a special case, for integers w,x,y,z, $\begin{pmatrix} w & & z \\ & \succ\!\prec & \\ x & & y\end{pmatrix}$ refers to $\begin{pmatrix} A^w & & A^z \\ & \succ\!\prec & \\ A^x & & A^y\end{pmatrix}$. Also, if $i,j,k,l,m\in I$, we write $\begin{pmatrix} ij & & m \\ & \succ\!\prec & \\ k & & l\end{pmatrix}$ as shorthand for $\begin{pmatrix} T_i\cdot T_j & & T_m \\ & \succ\!\prec & \\ T_k & & T_l\end{pmatrix}$.

We can rewrite (1) by splitting Φ into its classical and quantum parts. If $T_i\cdot T_j = \sum_q t_q T_q$, we denote $\sum_q t_q\Gamma_{qkl}$ by $\Gamma_{(ij)kl}$. Then (1) becomes

$$\Gamma_{ij(kl)} + \Gamma_{(ij)kl} - \Gamma_{jk(il)} - \Gamma_{(jk)il} = \sum_{e,f}\Gamma_{jke}g^{ef}\Gamma_{fil} - \sum_{e,f}\Gamma_{ije}g^{ef}\Gamma_{fkl}. \tag{2}$$

We have thus split the WDVV equation into the *linear contribution* (left-hand side) and *quadratic contribution* (right-hand side).

Main Problem. Given A, $\int$, Λ, Θ as above, find solutions in rational numbers $N(\beta;d_1,\ldots,d_s)$ to the full set of WDVV equations (1).

The formal identity $\begin{pmatrix} i & & l \\ & \succ\!\prec & \\ j & & k\end{pmatrix} + \begin{pmatrix} j & & l \\ & \succ\!\prec & \\ k & & i\end{pmatrix} + \begin{pmatrix} k & & l \\ & \succ\!\prec & \\ i & & j\end{pmatrix} = 0$ tells us that if a collection of numbers $N(\beta;d_1,\ldots,d_s)$ satisfies two of the three indicated WDVV equations, then they also satisfies the third. We call refer to this fact as the *two-out-of-three* implication.

3. The three symbols relation

Let $i,j,k,l,m \in I$. Let $\Phi = \Phi^{\mathrm{cl}} + \Gamma$ be the potential function. The following algebraic identity, called the *three symbols identity*, holds:

$$
\begin{aligned}
&\frac{\partial}{\partial y_m}\Big(\sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} - \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{fil}\Big) \\
&\quad + \frac{\partial}{\partial y_j}\Big(\sum_{e,f} \Phi_{ile} g^{ef} \Phi_{fkm} - \sum_{e,f} \Phi_{lke} g^{ef} \Phi_{fim}\Big) \\
&\quad + \frac{\partial}{\partial y_l}\Big(\sum_{e,f} \Phi_{ime} g^{ef} \Phi_{fkj} - \sum_{e,f} \Phi_{mke} g^{ef} \Phi_{fij}\Big) = 0. \qquad (3)
\end{aligned}
$$

Let (β,d) be a degree. Define $e_{\sigma_i} = 0$ and $e_{\tau_j} = (0,\ldots,1,\ldots,0)$ with the 1 in the j^{th} place. Then (3) gives us

Proposition 1. *Suppose $i,j,k,l,m \in I$ with $\operatorname{codim} T_m \geq 2$, and let (β,d) be a degree with $d_m \geq 1$. Then the relations $\left(\begin{smallmatrix} i & & m \\ & \rangle\!\!-\!\!\langle & \\ l & & k \end{smallmatrix}\right)^{(\beta;d+e_j-e_m)}$ and $\left(\begin{smallmatrix} i & & j \\ & \rangle\!\!-\!\!\langle & \\ m & & k \end{smallmatrix}\right)^{(\beta;d+e_l-e_m)}$ together imply $\left(\begin{smallmatrix} i & & l \\ & \rangle\!\!-\!\!\langle & \\ j & & k \end{smallmatrix}\right)^{(\beta;d)}$.*

This we call the *three symbols relation* (3sr), and denote by the diagram $\left(\begin{smallmatrix} i & & j,l;m \\ & \succ\!\!\prec & \\ & & k \end{smallmatrix}\right)^{(\beta;d)}$. We now record one application of 3sr.

Lemma 1. *With the notation of the Main Problem, suppose $(\beta;d)$ is a degree with $d \neq 0$. Then the collection of all relations in degrees $(\beta;d')$ with $\sum_i d'_i = (\sum_i d_i) - 1$ implies $\left(\begin{smallmatrix} A & & A^1 \\ & \rangle\!\!-\!\!\langle & \\ A^1 & & A \end{smallmatrix}\right)^{(\beta;d)}$.*

Indeed, if $d_m \geq 1$ then $\left(\begin{smallmatrix} i & & 1,1;m \\ & \succ\!\!\prec & \\ & & j \end{smallmatrix}\right)$ yields $\left(\begin{smallmatrix} i & & 1 \\ & \rangle\!\!-\!\!\langle & \\ 1 & & j \end{smallmatrix}\right)^{(\beta;d)}$ for any $i,j \in I$.

4. The five symbols relation

Given $i,j,k,l,m \in I$, the following algebraic identity holds:

$$\sum_{e,f} \Gamma_{ij(me)} g^{ef} \Gamma_{fkl} = \sum_{e,f} \Gamma_{kl(me)} g^{ef} \Gamma_{fij}. \qquad (4)$$

Recall, if $T_m \cdot T_e = \sum_q t_q T_q$ then by $\Gamma_{ij(me)}$ we mean $\sum_q t_q \Gamma_{ijq}$. Now (4) follows by observing that with $g_{abc} = \int T_a \cdot T_b \cdot T_c$ we have $t_q = \sum_p g_{mep} g^{pq}$, and now the coefficient $\sum_{e,p} g^{ef} g_{mep} g^{pq}$ of $\Gamma_{ijq}\Gamma_{fkl}$ on the left-hand side is symmetric in f and q.

We write the expression $\sum \Gamma_{ij(me)} g^{ef} \Gamma_{fkl} - \sum \Gamma_{kl(me)} g^{ef} \Gamma_{fij}$ and add to it the four additional expressions obtained by permuting the variables i,j,k,l,m cyclically. We use identity (2) coming from the associativity relation $\left(\begin{smallmatrix} e & & j \\ & \rangle\!\!-\!\!\langle & \\ m & & i \end{smallmatrix}\right)$ and its cyclic translates to obtain

$$
\begin{aligned}
0 &= \Gamma_{ij(me)}\Gamma_{fkl} + \Gamma_{jk(ie)}\Gamma_{flm} + \Gamma_{kl(je)}\Gamma_{fmi} + \Gamma_{lm(ke)}\Gamma_{fij} + \Gamma_{mi(le)}\Gamma_{fjk} \\
&\quad - \Gamma_{mi(je)}\Gamma_{fkl} - \Gamma_{ij(ke)}\Gamma_{flm} - \Gamma_{jk(le)}\Gamma_{fmi} - \Gamma_{kl(me)}\Gamma_{fij} - \Gamma_{lm(ie)}\Gamma_{fjk} \\
&= \Gamma_{(mi)je}\Gamma_{fkl} + \Gamma_{(ij)ke}\Gamma_{flm} + \Gamma_{(jk)le}\Gamma_{fmi} + \Gamma_{(kl)me}\Gamma_{fij} + \Gamma_{(lm)ie}\Gamma_{fjk} \\
&\quad - \Gamma_{(ij)me}\Gamma_{fkl} - \Gamma_{(jk)ie}\Gamma_{flm} - \Gamma_{(kl)je}\Gamma_{fmi} - \Gamma_{(lm)ke}\Gamma_{fij} - \Gamma_{(mi)le}\Gamma_{fjk}.
\end{aligned}
$$

We have omitted summations symbols and g^{ef}'s to save space. We have also omitted the (cubic) terms obtained by substituting the quadratic contributions of the associativity relations, but the key point is that these cancel.

The final expression above is the quadratic contribution of a sum of associativity relations, conveniently written

$$\left({}^{mi}_{\ j}\succ\!\!\prec{}^{l}_{k}\right)-\left({}^{m}_{ij}\succ\!\!\prec{}^{l}_{k}\right)+\left({}^{m}_{i}\succ\!\!\prec{}^{l}_{jk}\right)-\left({}^{m}_{i}\succ\!\!\prec{}^{kl}_{j}\right)+\left({}^{lm}_{\ i}\succ\!\!\prec{}^{k}_{j}\right). \tag{5}$$

The linear contribution of (5) vanishes (as may be checked), so the indicated associativity relations imply vanishing of (5), at least formally. To get a precise statement, we grade the terms in Γ by degree. Using the notation of the Main Problem, we rewrite the above, isolate the coefficient of some degree $(\beta;d)$, and note that then every quadratic term is a sum over $\beta_1+\beta_2=\beta$ with $\langle\beta_i,\omega\rangle<\langle\beta,\omega\rangle$ for $i=1,2$. This establishes

Proposition 2. *Suppose $i,j,k,l,m\in I$, and let $(\beta;d)$ be a degree. Let ω be an element of the interior to the dual cone to Θ. The collection of relations consisting of $\left({}^{i}_{j}\succ\!\!\prec{}^{e}_{k}\right)^{(\beta';d')}$ and its cyclic translates through $\{i,j,k,l,m\}$, for all $e\in I$ and all degrees (β',d') with $\langle\beta',\omega\rangle<\langle\beta,\omega\rangle$ and $d'\le d$ (componentwise), implies the relation*

$$\begin{aligned}\left({}^{mi}_{\ j}\succ\!\!\prec{}^{l}_{k}\right)^{(\beta;d)}-\left({}^{m}_{ij}\succ\!\!\prec{}^{l}_{k}\right)^{(\beta;d)}+\left({}^{m}_{i}\succ\!\!\prec{}^{l}_{jk}\right)^{(\beta;d)}&\\ -\left({}^{m}_{i}\succ\!\!\prec{}^{kl}_{j}\right)^{(\beta;d)}+\left({}^{lm}_{\ i}\succ\!\!\prec{}^{k}_{j}\right)^{(\beta;d)}&.\end{aligned} \tag{6}$$

We call this the *five symbols relation* (5SR). We employ the notation $\left({}^{m\swarrow i}_{\quad j}\bigotimes{}^{l}_{k}\right)^{(\beta;d)}$ to describe the above relation.

5. Strong reconstruction for $n=2$

As an illustration, we work out a strong reconstruction theorem for $n=2$. For simplicity let us suppose (as expected from geometry) the $N(\beta;d)$ to be identically zero except when $d=\langle\beta,-K\rangle-1$ (for some $K\in A^1$). There is only one τ, and the associativity relations are of the following forms:

$$\text{(i)}\ \left({}^{\tau}_{\tau}\succ\!\!\prec{}^{\sigma_i}_{\sigma_j}\right)\qquad\text{(ii)}\ \left({}^{\tau}_{\sigma_i}\succ\!\!\prec{}^{\sigma_k}_{\sigma_j}\right)\qquad\text{(iii)}\ \left({}^{\sigma_i}_{\sigma_j}\succ\!\!\prec{}^{\sigma_l}_{\sigma_k}\right).$$

The potential function is composed of (we write g_{ef} for $g_{\sigma_e\sigma_f}$)

$$\Phi^{\mathrm{cl}}=\frac{1}{2}y_1^2y_\tau+\frac{1}{2}\sum_{e,f=1}^{r}g_{ef}y_1y_{\sigma_e}y_{\sigma_f},$$

$$\Gamma=\sum_{\substack{\langle\beta,-K\rangle\ge 1\\ \beta=\sum c_i\sigma_i^*\in C\\ d=\langle\beta,-K\rangle-1}}N(\beta;d)e^{c_1y_{\sigma_1}}\cdots e^{c_ry_{\sigma_r}}\frac{y_\tau^d}{d!}.$$

Suppose we are given all $N(\beta;d)$ with $d\le 2$, and suppose these satisfy

$$\left({}^{\tau}_{\sigma_i}\succ\!\!\prec{}^{\sigma_k}_{\sigma_j}\right)^{(\beta;0)}\ \text{and}\ \left({}^{\sigma_i}_{\sigma_j}\succ\!\!\prec{}^{\sigma_l}_{\sigma_k}\right)^{(\beta;0)}$$

for all $i, j, k, l \in \{1, \ldots, r\}$ and all β. We claim that relations of type (i) allow us to solve for all further $N(\beta; d)$ (*reconstruction*) and that the numbers thus obtained satisfy the full system of WDVV equations (*strong reconstruction*).

Indeed, by the three symbols relation,

$$\begin{pmatrix}\tau & & \sigma_l \\ & \rangle\!\!\!\langle & \\ \sigma_j & & \sigma_i\end{pmatrix}^{(\beta;0)} \text{ and } \begin{pmatrix}\tau & & \sigma_l \\ & \rangle\!\!\!\langle & \\ \sigma_j & & \sigma_k\end{pmatrix}^{(\beta;0)} \Rightarrow \begin{pmatrix}\sigma_i & & \sigma_l \\ & \rangle\!\!\!\langle & \\ \sigma_j & & \sigma_k\end{pmatrix}^{(\beta;1)},$$

and thus the hypothesis implies relations (ii) $(\beta; 1)$. Similarly, relations (i) $(\beta; d)$ imply (ii) $(\beta; d + 1)$ and (iii) $(\beta; d + 2)$.

Inductively on d, assume all $N(\beta; d')$ known and all relations satisfied for $\langle \beta, -K\rangle < d + 4$. Relation $\begin{pmatrix}\tau & & \sigma_i \\ & \rangle\!\!\!\langle & \\ \tau & & \sigma_j\end{pmatrix}^{(\beta;d)}$ reads $g_{ij}\Gamma^{(\beta;d)}_{zzz} = Q^{(\beta;d)}_{ij}$, where $Q^{(\beta;d)}_{ij}$ is a quadratic expression in known quantities. Now $\begin{pmatrix} & \tau & & \sigma_i \\ & & \bigotimes & \\ \sigma_k & \sigma_l & & \sigma_j\end{pmatrix}$ tells us $g_{kl}Q^{(\beta;d)}_{ij} = g_{ij}Q^{(\beta;d)}_{kl}$, which says we can solve for $\Gamma^{(\beta;d)}_{zzz}$ (that is, $N(\beta; d + 3)$) satisfying (i), and we have just seen that the relations of type (i) imply all the relations in degree β.

6. Strong reconstruction theorem

Theorem 1. *With the notation of the Main Problem, suppose A is generated by A^1. Then the collection of $N(\beta; d)$ with $\sum_{i=1}^{s} d_i \le 2$ extends to a solution to WDVV if and only if* $\begin{pmatrix}A & & A^1 \\ & \rangle\!\!\!\langle & \\ A^1 & & A\end{pmatrix}^{(\beta;0)}$ *is satisfied for all β.*

We begin by organizing notation. By hypothesis, we may assume the basis B chosen such that for each j, $1 \le j \le s$, there exists $i_j \in \{1, \ldots, r\}$ and $\mu_j \in I$ such that $T_{\tau_j} = T_{\sigma_{i_j}} \cdot T_{\mu_j}$.

We wish to impose a partial order on the collection of degrees $d = (d_1, \ldots, d_s)$ with fixed $|d| := \sum_{i=1}^{s} d_s$, such that $(d_1, \ldots, d_s)$ precedes $(d_1, \ldots, d_i + 1, \ldots, d_j - 1, \ldots, d_s)$ for any $i < j$. A convenient way is to order by $\sum id_i$.

Let us give an outline of the proof of the theorem. Let ω be an element of the interior to the dual cone to Θ. Inducting on $\langle\beta, \omega\rangle$, then on $|d|$, then downwards on $\sum id_i$, we verify all associativity relations in degree $(\beta; d)$, showing that those of the form

$$\begin{pmatrix}\mu_j & & \tau_l \\ & \rangle\!\!\!\langle & \\ \sigma_{i_j} & & \tau_k\end{pmatrix}^{(\beta;d)}$$

with $\operatorname{codim} T_{\tau_j} \le \operatorname{codim} T_{\tau_k} \le \operatorname{codim} T_{\tau_l}$ and $\max(j, k, l) \le \min\{m \,|\, d_m \ne 0\}$ determine the numbers $N(\beta; d + e_j + e_k + e_l)$ (here $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the i^{th} position).

7. Proof of strong reconstruction theorem

The induction breaks up into an outer induction on degrees and an inner induction within each degree. The outer induction proceeds with respect to the partial order: $(\beta', d') \prec (\beta, d)$ if $\langle\beta', \omega\rangle < \langle\beta, \omega\rangle$ and $|d'| \le |d|$; $\beta' = \beta$ and $|d'| < |d|$; or $\beta' = \beta$, $|d'| = |d|$, and $\sum id'_i > \sum id_i$. The inner induction is on (u, c, a, b) with u (corresponding to $\operatorname{codim} T_{\mu_j}$ above) up from 1, c ($= \operatorname{codim} T_{\tau_k} + \operatorname{codim} T_{\tau_l}$) up from $2(u + 1)$, and a ($= \operatorname{codim} T_{\tau_k}$) up from $u + 1$ to $[c/2]$. Define $b = c - a$; then we always have $a \le b$.

The induction hypothesis, at a given step (β, d, u, c, a, b), consists of all relations in previous degrees plus all numbers they refer to (i.e., all $N(\beta'; d' + e')$ with $(\beta', d') \prec$

(β, d), $|e'| \leq 3)$, plus, in the current degree, all $\begin{pmatrix} z & & y \\ & \succ\!\prec & \\ 1 & & x \end{pmatrix}$ with $\min(x, y, z) < u$, all $\begin{pmatrix} u & & y \\ & \succ\!\prec & \\ 1 & & x \end{pmatrix}$ and $\begin{pmatrix} x & & u \\ & \succ\!\prec & \\ 1 & & y \end{pmatrix}$ with $x \geq u+1$, $y \geq u+1$, and either $x + y < c$ or $x + y = c$ with $\min(x, y) < a$, together with all numbers these relations refer to.

In any degree, for any integers x and y, $\begin{pmatrix} x & & 1 \\ & \succ\!\prec & \\ 1 & & y \end{pmatrix}$ follows either by hypothesis ($d = 0$) or by the induction hypothesis and Lemma 1 ($|d| \geq 1$). When $u \geq 2$ we obtain $\begin{pmatrix} x & & u \\ & \succ\!\prec & \\ 1 & & y \end{pmatrix}$ for $x \geq u$ and $y \geq u$ from $\begin{pmatrix} x & \nwarrow & u-1\,1 \\ & \bigotimes & \\ 1 & & y \end{pmatrix}$, and now $\begin{pmatrix} u & & x \\ & \succ\!\prec & \\ 1 & & u \end{pmatrix}$ for $x > u$ from $\begin{pmatrix} u-1\,1 & \nearrow & x \\ & \bigotimes & \\ 1 & & u \end{pmatrix}$.

The main step is to deduce $\begin{pmatrix} u & & b \\ & \succ\!\prec & \\ 1 & & a \end{pmatrix}$. Here the linear terms coming from the associativity relation possibly involve new N's. We divide this into two steps.

First, we show it suffices to prove a distinguished set of $\begin{pmatrix} u & & b \\ & \succ\!\prec & \\ 1 & & a \end{pmatrix}$. Let S be the set of relations $\begin{pmatrix} \mu_j & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_k \end{pmatrix}$ with $\operatorname{codim} T_{\tau_j} = u+1$, $\operatorname{codim} T_{\tau_k} = a$, and $\operatorname{codim} T_{\tau_l} = b$. We claim that S (and the new N's referred to) implies $\begin{pmatrix} u & & b \\ & \succ\!\prec & \\ 1 & & a \end{pmatrix}$. Indeed, if $\operatorname{codim} T_\mu = u$ and $\operatorname{codim} T_\sigma = 1$ with $T_\sigma \cdot T_\mu = \sum \lambda_j T_{\tau_j}$, then comparing $\begin{pmatrix} \mu & & \tau_l \\ & \bigotimes & \\ \sigma & & \sigma_{i_k} \nearrow \mu_k \end{pmatrix}$ with $\sum \lambda_j \begin{pmatrix} \mu_j & & \tau_l \\ & \bigotimes & \\ \sigma_{i_j} & & \sigma_{i_k} \nearrow \mu_k \end{pmatrix}$ establishes $\begin{pmatrix} \mu & & \tau_l \\ & \succ\!\prec & \\ \sigma & & \tau_k \end{pmatrix}$ from the relations in S.

For the second step, we establish all relations in S. Each $\begin{pmatrix} \mu_j & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_k \end{pmatrix}$ in S involves the variable $N(\beta; d + e_j + e_k + e_l)$. For a, b, $u+1$ distinct, there is a one-to-one correspondence between elements of S and such variables. In other cases, we shall need symmetrizing arguments to show any two relations in S sharing a common such variable are equivalent. In case $a = b$, the two-out-of-three implication gives $\begin{pmatrix} \mu_j & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_k \end{pmatrix} \Leftrightarrow \begin{pmatrix} \mu_j & & \tau_k \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_l \end{pmatrix}$. In case $a = u+1$, we get $\begin{pmatrix} \mu_j & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_k \end{pmatrix} \Leftrightarrow \begin{pmatrix} \sigma_{i_j} & & \tau_l \\ & \succ\!\prec & \\ \mu_j & & \tau_k \end{pmatrix} \Leftrightarrow \begin{pmatrix} \mu_k & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_k} & & \tau_j \end{pmatrix}$ by two-out-of-three and $\begin{pmatrix} \sigma_{i_j} & & \tau_l \\ & \bigotimes & \\ \mu_j & & \sigma_{i_k}\, \mu_k \nearrow \end{pmatrix}$.

Thus, it suffices to establish only those $\begin{pmatrix} \mu_j & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_k \end{pmatrix} \in S$ such that $j \leq k \leq l$. In case $d_m \geq 1$ for some $m < l$, $\begin{pmatrix} \mu_j & \times & \sigma_{i_j}, \tau_l; \tau_m \\ & & \tau_k \end{pmatrix}$ establishes $\begin{pmatrix} \mu_j & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_k \end{pmatrix}$. Otherwise, $N(\beta; d + e_j + e_k + e_l)$ is actually an unknown, so solving $\begin{pmatrix} \mu_j & & \tau_l \\ & \succ\!\prec & \\ \sigma_{i_j} & & \tau_k \end{pmatrix}$ establishes simultaneously the number and the relation. Finally, two-out-of-three establishes $\begin{pmatrix} u & & a \\ & \succ\!\prec & \\ 1 & & b \end{pmatrix}$ from $\begin{pmatrix} u & & b \\ & \succ\!\prec & \\ 1 & & a \end{pmatrix}$.

Having finished the inner induction, to establish general $\begin{pmatrix} w & & z \\ & \succ\!\prec & \\ x & & y \end{pmatrix}$ is an easy induction on $\min(w, x, y, z)$, using 5SR by decomposing the entry of lowest codimension.

8. Examples

For several manifolds/orbifolds X we give a description of the solution space to the WDVV equations corresponding to X. Since our focus in on the equations coming from geometry, we impose the standard dimension restriction on N's: $N(\beta;d)=0$ whenever $\sum_j d_j(\operatorname{codim} T_{\tau_j}-1)\neq\langle\beta,c_1(X)\rangle+n-3$ ($n=\dim X$). Geometry dictates one solution to WDVV, together with a family of rescalings with as many degrees of freedom as the Picard number of X. The common theme to these examples is the existence of solutions besides the geometric solution and its rescalings.

Example 1. Let Q be a smooth quadric threefold, and let $X=Q\times Q$. Let $A=A^*_{\mathbb{Q}}X$. Strong reconstruction dictates 52 equations in 40 unknowns. We shall see that the solution set has two degenerate irreducible components (i.e., with $N(\beta;d)$ nontrivial only for β contained in a ray of the effective cone) and two irreducible components having nontrivial $N(\beta;d)$ for β spanning $(A^1)^*$. So, if one carries out a program in the spirit of [1] of computing Gromov Witten invariants starting from 2 basic numbers (one counting curves of type $(1,0)$ with some simple incidence conditions, and the other, curves of type $(0,1)$), then one encounters two remarkable phenomena. First, one must supply an additional basic number in order to produce numbers ad infinitum (eventually, the algorithm of the first reconstruction theorem becomes applicable). Second, there are two valid choices for this additional number: zero, which yields the geometric solution, or a unique nonzero integer, which yields a consistent non-geometric solution.

The cohomology of Q has one generator in each codimension: 1, h (hyperplane class), ℓ (class of a line), and p (point class). With $A\cong A^*_{\mathbb{Q}}Q\otimes A^*_{\mathbb{Q}}Q$ we have the basis consisting of identity, divisor classes $h\otimes 1$ and $1\otimes h$, and classes in codimension ≥ 2 which we order as follows:

$$\begin{gathered}\ell\otimes 1, h\otimes h, 1\otimes\ell, p\otimes 1, \ell\otimes h, h\otimes\ell, 1\otimes p,\\ p\otimes h, \ell\otimes\ell, h\otimes p, p\otimes\ell, \ell\otimes p, p\otimes p\end{gathered}$$

Denote by r and s the homology classes corresponding to line $\times$ point and point $\times$ line, respectively. So, for instance, $N(r;\,1{,}0{,}0,\,0{,}0{,}0{,}0,\,0{,}0{,}0,\,0{,}0,\,1)$ corresponds to the number of lines on the first copy of Q incident to a line and a point.

The set of basic relations dictated by the strong reconstruction theorem includes 20 equations in degree r, namely $\left(\begin{matrix}\varphi & & 1\otimes h\\ & \asymp & \\ h\otimes 1 & & \psi\end{matrix}\right)^{(r;0)}$ for all basis elements φ and ψ whose codimensions add up to 7. The reader may check that of the 17 $N(r;d)$'s with $\sum d_i\leq 2$, the relations in degree r determine that 11 of these vanish and that the remaining 6 are linearly related:

$$\begin{aligned}&N(r;\,0{,}0{,}0,\,0{,}1{,}0{,}0,\,0{,}0{,}0,\,1{,}0,\,0)\\&\qquad=N(r;\,1{,}0{,}0,\,0{,}0{,}0{,}0,\,0{,}0{,}0,\,0{,}0,\,1)\\&N(r;\,0{,}0{,}0,\,0{,}0{,}0{,}0,\,1{,}1{,}0,\,0{,}0,\,0)\\&\qquad=N(r;\,1{,}0{,}0,\,0{,}0{,}0{,}0,\,0{,}0{,}0,\,0{,}0,\,1)\\&N(r;\,0{,}0{,}0,\,1{,}0{,}0{,}0,\,0{,}0{,}0,\,0{,}1,\,0)\\&\qquad=N(r;\,1{,}0{,}0,\,0{,}0{,}0{,}0,\,0{,}0{,}0,\,0{,}0,\,1)\\&N(r;\,0{,}0{,}0,\,0{,}0{,}0{,}0,\,2{,}0{,}0,\,0{,}0,\,0)\\&\qquad=2N(r;\,0{,}0{,}0,\,1{,}0{,}0{,}0,\,0{,}0{,}0,\,1{,}0,\,0)\end{aligned}$$

The basic relations in degree s determine, by symmetry, analogous constraints on the basic $N(s;d)$.

Now there are 6 more basic numbers, 2 each in degrees $2r$, $2s$, and $r+s$. These are constrained by basic relations $\left(\begin{smallmatrix} p\otimes h \\ h\otimes 1\end{smallmatrix} \succ\!\!\prec \begin{smallmatrix} 1\otimes h \\ p\otimes p\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \ell\otimes \ell \\ h\otimes 1\end{smallmatrix} \succ\!\!\prec \begin{smallmatrix} 1\otimes h \\ p\otimes p\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} h\otimes p \\ h\otimes 1\end{smallmatrix} \succ\!\!\prec \begin{smallmatrix} 1\otimes h \\ p\otimes p\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} p\otimes \ell \\ h\otimes 1\end{smallmatrix} \succ\!\!\prec \begin{smallmatrix} 1\otimes h \\ \ell\otimes p\end{smallmatrix}\right)$ in each degree.

In degrees $2r$ and $2s$ the relations determine that the basic $N(2r;d)$ and $N(2s;d)$ vanish. In degree $(r+s)$ we get interesting equations. The interested reader may write out the 4 equations and make the linear substitutions indicated above to see that the entire system of equations reduces to:

$$XW = 0 \qquad\qquad YZ = 0$$

with

$$\begin{aligned}
X &= N(r;\ 0{,}0{,}0,\ 1{,}0{,}0{,}0,\ 0{,}0{,}0,\ 1{,}0,\ 0)\\
Y &= N(s;\ 0{,}0{,}0,\ 0{,}0{,}0{,}1,\ 0{,}0{,}0,\ 0{,}1,\ 0)\\
Z &= 2N(r;\ 1{,}0{,}0,\ 0{,}0{,}0{,}0,\ 0{,}0{,}0,\ 0{,}0,\ 1)\\
&\quad - N(r;\ 0{,}0{,}0,\ 1{,}0{,}0{,}0,\ 0{,}0{,}0,\ 1{,}0,\ 0)\\
W &= 2N(s;\ 0{,}0{,}1,\ 0{,}0{,}0{,}0,\ 0{,}0{,}0,\ 0{,}0,\ 1)\\
&\quad - N(s;\ 0{,}0{,}0,\ 0{,}0{,}0{,}1,\ 0{,}0{,}0,\ 0{,}1,\ 0)
\end{aligned}$$

Given the constraints (from geometry) $N(r;\ 1{,}0{,}0,\ 0{,}0{,}0{,}0,\ 0{,}0{,}0,\ 0{,}0,\ 1) = N(s;\ 0{,}0{,}1,\ 0{,}0{,}0{,}0,\ 0{,}0{,}0,\ 0{,}0,\ 1) = 1$, the system of equations above dictates two solutions: the geometric solution,

$$\begin{aligned}
&N(r;\ 0{,}0{,}0,\ 1{,}0{,}0{,}0,\ 0{,}0{,}0,\ 1{,}0,\ 0)\\
&\qquad = N(s;\ 0{,}0{,}0,\ 0{,}0{,}0{,}1,\ 0{,}0{,}0,\ 0{,}1,\ 0) = 0
\end{aligned}$$

and the non-geometric solution,

$$\begin{aligned}
&N(r;\ 0{,}0{,}0,\ 1{,}0{,}0{,}0,\ 0{,}0{,}0,\ 1{,}0,\ 0)\\
&\qquad = N(s;\ 0{,}0{,}0,\ 0{,}0{,}0{,}1,\ 0{,}0{,}0,\ 0{,}1,\ 0) = 2.
\end{aligned}$$

By the strong reconstruction theorem, both of these extend uniquely to solutions to the full set of WDVV equations for X.

Example 2. $X = G(2,4)$ and $X = \operatorname{Sym}^2 \mathbb{P}^2$. The cohomology rings (which are isomorphic, up to scale) are not generated by divisors, so we are outside the scope of the strong reconstruction theorem. However, the method of proof still applies, and with a little extra work (and a genericity hypothesis, namely that some starting number is nonzero) we can deduce strong reconstruction from the expected data (one number for $G(2,4)$ and three numbers for $(\operatorname{Sym}^2 \mathbb{P}^2)$, with a vacuous set of relations).

We take as cohomology basis the powers of the ample generator h of A^1X, plus an extra codimension 2 element, chosen orthogonal to h. So $B = \{1, h, c, h^2, h^3, h^4\}$ with $c \cdot h = 0$, $\int h^4 \neq 0$, $\int c^2 \neq 0$. We have $K = -4h$, $K = -3h$ in the cases of the two respective varieties; set $\kappa = 4$, $\kappa = 3$ accordingly. It will be helpful to recall the

(a)	$N(\beta;\kappa\beta-5,0,0,2)$	by	$\left(\begin{smallmatrix} h^4 & & c \\ & \succ\!\prec & \\ h & & h^3 \end{smallmatrix}\right)^{(\beta;\kappa\beta-6,0,0,0)}$	$(\beta\geq 2)$
(b)	$N(\beta-1;$ $\kappa\beta-\kappa+1,0,0,0)$	by	$\left(\begin{smallmatrix} c & & c \\ & \succ\!\prec & \\ h & & h^4 \end{smallmatrix}\right)^{(\beta;\kappa\beta-5,0,0,0)}$	$(\beta\geq 2)$
(c)	$N(\beta;t,u,v,w)$ with $u+v+w\geq 3$	by	$\left(\begin{smallmatrix} \langle h\rangle & & \langle h\rangle \\ & \succ\!\prec & \\ \langle h\rangle & & \langle h\rangle \end{smallmatrix}\right)^{(\beta;d)}$	
(d)	$N(\beta;\kappa\beta-4,0,1,1)$	by	$\left(\begin{smallmatrix} h^4 & & c \\ & \succ\!\prec & \\ h & & h^2 \end{smallmatrix}\right)^{(\beta;\kappa\beta-5,0,0,0)}$	$(\beta\geq 2)$
(e)	$N(\beta;\kappa\beta-3,1,0,1)$	by	$\left(\begin{smallmatrix} h^4 & & c \\ & \succ\!\prec & \\ h & & h \end{smallmatrix}\right)^{(\beta;\kappa\beta-4,0,0,0)}$	$(\kappa+\beta\geq 5)$
(f)	$N(\beta;\kappa\beta-3,0,2,0)$	by	$\left(\begin{smallmatrix} h^3 & & c \\ & \succ\!\prec & \\ h & & h^2 \end{smallmatrix}\right)^{(\beta;\kappa\beta-4,0,0,0)}$	$(\kappa+\beta\geq 5)$
(g)	$N(\beta;\kappa\beta-2,0,0,1)$	by	$\left(\begin{smallmatrix} h^3 & & c \\ & \succ\!\prec & \\ h & & c \end{smallmatrix}\right)^{(\beta;\kappa\beta-4,0,0,0)}$	$(\kappa+\beta\geq 5)$
(h)	$N(\beta;\kappa\beta-2,1,1,0)$	by	$\left(\begin{smallmatrix} h^3 & & c \\ & \succ\!\prec & \\ h & & h \end{smallmatrix}\right)^{(\beta;\kappa\beta-3,0,0,0)}$	
(i)	$N(\beta;\kappa\beta-1,0,1,0)$	by	$\left(\begin{smallmatrix} h^2 & & c \\ & \succ\!\prec & \\ h & & c \end{smallmatrix}\right)^{(\beta;\kappa\beta-3,0,0,0)}$	
(j)	$N(\beta;\kappa\beta-1,2,0,0)$	by	$\left(\begin{smallmatrix} h^2 & & c \\ & \succ\!\prec & \\ h & & h \end{smallmatrix}\right)^{(\beta;\kappa\beta-2,0,0,0)}$	
(k)	$N(\beta;\kappa\beta,1,0,0)$	by	$\left(\begin{smallmatrix} h & & c \\ & \succ\!\prec & \\ h & & c \end{smallmatrix}\right)^{(\beta;\kappa\beta-2,0,0,0)}$	

TABLE 1. Order, within the Outer Induction, for the Proof of Strong Reconstruction for $G(2,4)/\mathrm{Sym}^2\,\mathbb{P}^2$

dimension condition on relations. For $\left(\begin{smallmatrix} \xi & & \sigma \\ & \succ\!\prec & \\ \pi & & \rho \end{smallmatrix}\right)^{(\beta;d)}$ to be nontrivial requires

$$\begin{aligned} &\langle\beta,-K\rangle-\sum_{j=1}^{s}d_j(\operatorname{codim}T_{\tau_j}-1)\\ &\qquad=\operatorname{codim}\xi+\operatorname{codim}\pi+\operatorname{codim}\rho+\operatorname{codim}\sigma-n. \end{aligned}\tag{7}$$

The relations in curve class β never involve the number $N(\beta;\kappa\beta+1,0,0,0)$. We recover this exceptional number from a particular degree $\beta+1$ relation in which it appears in a quadratic term, for which, to be able to solve, we must add the hypothesis $N(1;0,0,1,1)\neq 0$ (resp. $N(1;1,0,0,1)\neq 0$) when $\kappa=4$ (resp. $\kappa=3$).

For each β, there are 10 numbers $N(\beta;d)$ which are not of the form $N(\beta;t,u,v,w)$ with $u+v+w\geq 3$; these are the numbers unreachable by the proof of strong reconstruction applied to the subring of A generated by h. We must show how to solve for 9 of these (all except $N(\beta;\kappa\beta+1,0,0,0)$) as well as the leftover degree $\beta-1$ number. Table 1 outlines how to do this.

We induct first on curve class β. The induction hypothesis consists of all relations and all numbers in degrees less than $\beta-1$, and all relations and all numbers except $N(\beta-1;\kappa\beta-\kappa+1,0,0,0)$ in degree $\beta-1$. The relations indicated in entries (a), (b) of Table 1 give us two new numbers, including $N(\beta-1;\kappa\beta-\kappa+1,0,0,0)$. Thus from now on we assume all relations and all numbers in degrees less than or equal to $\beta-1$.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
t_1	t_2	t_4	t_5	t_6	$(1/3)t_7$	$5t_8$	0	t_9
t_2	t_4	t_6	$(1/3)t_7$	$5t_8$	0	$5t_9$	0	0
t_3	t_5	$(1/3)t_7$	$t_6 - (11/3)t_7$	0	$5t_8$	0	$15t_9$	0
t_4	t_6	$5t_8$	0	$5t_9$	0	0	0	0
t_5	$(1/3)t_7$	0	$5t_8$	0	$5t_9$	0	0	0
t_6	$5t_8$	$5t_9$	0	0	0	0	0	0
t_7	0	0	$15t_9$	0	0	0	0	0
t_8	t_9	0	0	0	0	0	0	0

TABLE 2. Multiplication Table for $A^*_{\mathbb{Q}}G(2,5)$

Now, inductively on d via the partial ordering $d' = (t', u', v', w') \prec d = (t, u, v, w) \Leftrightarrow |d'| < |d|$ or $|d'| = |d|$, $t' > t$ or $|d'| = |d|$, $t' = t$, $u' + 2v' + 3w' > u + 2v + 3w$, we establish the relations indicated in (c)–(k) of the table, as applicable to the current degree. Step (c) is the inner induction of the proof of the strong reconstruction theorem, applied to the subring of A generated by h.

Finally, it follows from 5SR (this takes a bit of checking) that in any degree, the (31 out of 55 total) relations indicated in Table 1 imply all the relations. The 21 encoded by step (c) follow by the proof of strong reconstruction, so we are reduced to establishing the remaining 10. When $u = v = w = 0$, we are done by Table 1 (each remaining relation solves for an unknown number). Otherwise, since each of the Feynman diagrams indicated in (a), (b), (d)–(k) of the table has a diagonal containing c and h, an application of 3SR reduces us to relations obtained in previous degrees (via the partial ordering above).

Example 3. $X = G(2,5)$. We find a two-dimensional family of solutions, the generic one non-geometric (X has Picard number 1). Thinking of X as the space of rank 2 quotients of $\mathbb{C}^5$, let Q be the universal quotient bundle and $c_i = c_i(Q)$. We use the following basis for $A^*_{\mathbb{Q}}X$, suggested by T. Graber (a multiplication table is given in Table 2):

$$
\begin{aligned}
&(L1) \quad N(\beta;0,0,0,0,0,5\beta,0,1) - 3N(\beta;0,0,0,0,0,5\beta-1,2,0)\\
&(L2) \quad N(\beta;1,0,0,0,0,5\beta,1,0) - N(\beta;0,1,0,0,0,5\beta+1,0,0)\\
&\qquad\qquad + \beta N(\beta;0,0,0,0,0,5\beta-1,2,0)\\
&(L3) \quad N(\beta;1,0,0,0,0,5\beta,1,0) - 2\beta N(\beta;0,0,0,0,0,5\beta-1,2,0)\\
&(L4) \quad N(\beta;1,0,0,0,0,5\beta+2,0,0) - 2\beta N(\beta;0,0,0,0,0,5\beta+1,1,0)\\
&\qquad\qquad - 11\beta^2 N(\beta;0,0,0,0,0,5\beta-1,2,0)\\
&(L5) \quad N(\beta;2,0,0,0,0,5\beta+1,0,0) - 4\beta^2 N(\beta;0,0,0,0,0,5\beta-1,2,0)
\end{aligned}
$$

TABLE 3. The Linear Expressions Obtained by Degree β Relations

(a)	$N(\beta;0,0,1,0,1,5\beta-5,0,0)$	by	$\left({}^{t_9}_{t_1}\rangle\!\!-\!\!\langle{}^{t_3}_{t_4}\right)^{(\beta;0,0,0,0,0,5\beta-6,0,0)}$
(b)	$N(\beta;0,0,0,2,0,5\beta-5,0,0)$	by	$\left({}^{t_8}_{t_1}\rangle\!\!-\!\!\langle{}^{t_3}_{t_6}\right)^{(\beta;0,0,0,0,0,5\beta-6,0,0)}$
(c)	$N(\beta;0,0,0,1,1,5\beta-6,0,0)$	by	$\left({}^{t_9}_{t_1}\rangle\!\!-\!\!\langle{}^{t_3}_{t_6}\right)^{(\beta;0,0,0,0,0,5\beta-7,0,0)}$
(d)	$N(\beta;0,0,0,0,1,5\beta-5,0,1)$	by	$\left({}^{t_8}_{t_1}\rangle\!\!-\!\!\langle{}^{t_3}_{t_7}\right)^{(\beta;0,0,0,0,0,5\beta-6,0,0)}$
(e)	$N(\beta;0,0,0,1,0,5\beta-4,0,1)$	by	$\left({}^{t_6}_{t_1}\rangle\!\!-\!\!\langle{}^{t_3}_{t_7}\right)^{(\beta;0,0,0,0,0,5\beta-5,0,0)}$
(f)	$N(\beta;0,0,0,0,1,5\beta-4,1,0)$	by	$\left({}^{t_8}_{t_1}\rangle\!\!-\!\!\langle{}^{t_3}_{t_5}\right)^{(\beta;0,0,0,0,0,5\beta-5,0,0)}$
(g)	$N(\beta-1;0,0,0,0,0,5\beta-2,0,0)$	by	$\left({}^{t_3}_{t_1}\rangle\!\!-\!\!\langle{}^{t_3}_{t_9}\right)^{(\beta;0,0,0,0,0,5\beta-5,0,0)}$

TABLE 4. Path to the Remaining Degree $(\beta-1)$ Numbers for $G(2,5)$

codim 0:	$t_0 = 1$	
codim 1:	$t_1 = c_1$	
codim 2:	$t_2 = c_1^2$	$t_3 = 2c_1^2 - 5c_2$
codim 3:	$t_4 = c_1^3$	$t_5 = 2c_1^3 - 5c_1c_2$
codim 4:	$t_6 = c_1^4$	$t_7 = c_1^4 - 5c_2^2$
codim 5:	$t_8 = c_1c_2^2$	
codim 6:	$t_9 = c_2^3$ (point class)	

We denote a typical unknown by $N(\beta;d_2,d_4,d_6,d_8,d_9,d_3,d_5,d_7)$ (note special order). Inductively on degree β, we show that degree β relations solve consistently for all but 8 degree β numbers, plus the 5 linear expressions shown in Table 3. The 8 exceptions are the 7 numbers appearing in Table 3 as well as $N(\beta;0,0,0,0,0,5\beta+3,0,0)$.

The genericity assumption is $N(1;0,0,0,0,1,0,0,1) \neq 0$. Given the induction hypothesis, we solve for the remaining degree $(\beta-1)$ numbers according to Table 4, first by the path shown with $(0,0,0,0,0,-4,2,0)$ added to all degrees, then by the path shown with $(0,0,0,0,0,-2,1,0)$ added to all degrees, and then by the path as shown. Only for $\beta=2$, during the first pass, we must substitute $\left({}^{t_9}_{t_1}\rangle\!\!-\!\!\langle{}^{t_5}_{t_6}\right)^{(\beta;0,0,0,0,0,5\beta-10,1,0)}$ for step (c). Once we have all the numbers in degree $\beta-1$, we then induct on d with respect to the partial ordering $d' \prec d \Leftrightarrow$

(i) $|d'| < |d|$, or
(ii) $|d'| = |d|$ and $d'_2+d'_4+d'_6+d'_8+d'_9 < d_2+d_4+d_6+d_8+d_9$, or
(iii) $|d'| = |d|$ and $d'_2+d'_4+d'_6+d'_8+d'_9 = d_2+d_4+d_6+d_8+d_9$, but $d'_2+2d'_4+3d'_6+4d'_8+5d'_9 > d_2+2d_4+3d_6+4d_8+5d_9$, or
(iv) $|d'| = |d|$, $(d'_2,d'_4,d'_6,d'_8,d'_9) = (d_2,d_4,d_6,d_8,d_9)$, and $d'_3+2d'_5+3d'_7 < d_3+2d_5+3d_7$.

For each d, we use the inner induction of the proof of strong reconstruction to obtain all relations involving only powers of t_1. Next, we obtain all of

$$\begin{array}{rll}
N(\beta;0,0,0,0,2,u,v,w) & & \\
u+2v+3w=5\beta-7 & \text{by} & \left({}^{t_9}_{t_1}{\succ\!\prec}{}^{t_7}_{t_8}\right)^{(\beta;0,0,0,0,0,u,v,w-1)} \\
 & \text{or} & \left({}^{t_9}_{t_1}{\succ\!\prec}{}^{t_5}_{t_8}\right)^{(\beta;0,0,0,0,0,u,v-1,w)} \\
 & \text{or} & \left({}^{t_9}_{t_1}{\succ\!\prec}{}^{t_3}_{t_8}\right)^{(\beta;0,0,0,0,0,u-1,v,w)} \\
N(\beta;0,0,0,1,1,u,v,w) & & \\
u+2v+3w=5\beta-6 & \text{by} & \left({}^{t_9}_{t_1}{\succ\!\prec}{}^{t_7}_{t_6}\right)^{(\beta;0,0,0,0,0,u,v,w-1)} \\
 & \text{or} & \text{etc.} \\
\ldots & & \\
N(\beta;2,0,0,0,0,u,v,w) & & \\
u+2v+3w=5\beta+1 & \text{by} & \left({}^{t_2}_{t_1}{\succ\!\prec}{}^{t_7}_{t_1}\right)^{(\beta;0,0,0,0,0,u,v,w-1)} \quad \text{etc.}
\end{array}$$

coming from relations in degree $(\beta;d)$. The exception to be noted occurs in attempting to solve for $N(\beta;2,0,0,0,0,5\beta+1,0,0)$: we get a value for $(L5)$ of Table 3 rather than a single N.

Still in a particular degree, we obtain

$$\begin{array}{rll}
N(\beta;0,0,0,0,1,u,v,w) & & \\
u+2v+3w=5\beta-2 & \text{by} & \left({}^{t_8}_{t_1}{\succ\!\prec}{}^{t_7}_{t_7}\right)^{(\beta;0,0,0,0,0,u,v,w-2)} \\
 & \text{or} & \left({}^{t_8}_{t_1}{\succ\!\prec}{}^{t_5}_{t_7}\right)^{(\beta;0,0,0,0,0,u,v-1,w-1)} \\
 & \text{etc.} &
\end{array}$$

with exceptions noted below:

$$\begin{array}{ll}
(L2) \quad \text{by} & \left({}^{t_2}_{t_1}{\succ\!\prec}{}^{t_3}_{t_3}\right)^{(\beta;0,0,0,0,0,5\beta-1,0,0)} \\
(L3) \quad \text{by} & \left({}^{t_1}_{t_1}{\succ\!\prec}{}^{t_3}_{t_5}\right)^{(\beta;0,0,0,0,0,5\beta-1,0,0)} \\
(L4) \quad \text{by} & \left({}^{t_1}_{t_1}{\succ\!\prec}{}^{t_3}_{t_3}\right)^{(\beta;0,0,0,0,0,5\beta,0,0)}
\end{array}$$

Lastly, we have numbers of the form $N(\beta;0,0,0,0,0,u,v,w)$ and the relations that produce these:

relation	for cases			relation	for cases		
$\left({}^{t_5}_{t_1}{\succ\!\prec}{}^{t_7}_{t_7}\right)$			$w \geq 3$	$\left({}^{t_3}_{t_1}{\succ\!\prec}{}^{t_3}_{t_7}\right)$	$u \geq 1$	$v \geq 1$	$w \geq 1$
$\left({}^{t_3}_{t_1}{\succ\!\prec}{}^{t_7}_{t_7}\right)$		$v \geq 1$	$w \geq 2$	$\left({}^{t_5}_{t_1}{\succ\!\prec}{}^{t_5}_{t_3}\right)$		$v \geq 3$	
$\left({}^{t_5}_{t_1}{\succ\!\prec}{}^{t_3}_{t_7}\right)$	$u \geq 1$		$w \geq 2$	$\left({}^{t_3}_{t_1}{\succ\!\prec}{}^{t_3}_{t_5}\right)$	(only to get $(L1)$)		
$\left({}^{t_3}_{t_1}{\succ\!\prec}{}^{t_5}_{t_7}\right)$		$v \geq 2$	$w \geq 1$				

Finally, we obtain $\left({}^{t_3}_{t_1}{\succ\!\prec}{}^{t_3}_{t_9}\right)$, $\left({}^{t_5}_{t_1}{\succ\!\prec}{}^{t_3}_{t_9}\right)$, and $\left({}^{t_7}_{t_1}{\succ\!\prec}{}^{t_3}_{t_9}\right)$. First we consider the case $(d_2,d_4,d_6,d_8,d_9)=(0,0,0,0,0)$ and $d_5=d_7=0$. Starting with $\left({}^{t_3}_{t_1}{\succ\!\prec}{}^{t_3}_{t_9}\right)^{(\beta;0,0,0,0,0,5\beta-9,2,0)}$

Assumptions	Implications
$\left({}^{t_8}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_7}\right)^{(5\beta-8,1,0)}$	$\left({}^{t_3}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(5\beta-9,2,0)} + 3\textsc{sr} \;\Rightarrow\; \left({}^{t_3}_{t_1}\!\succ\!\!\prec\!{}^{t_5}_{t_9}\right)^{(5\beta-8,1,0)}$
$\left({}^{t_8}_{t_1}\!\succ\!\!\prec\!{}^{t_5}_{t_5}\right)^{(5\beta-8,1,0)}$	$\left(\nwarrow\ {}^{t_1}_{t_6\,t_1}\!\bigotimes\!{}^{t_3}_{t_7}\right) \;\Rightarrow\; \left({}^{t_8}_{t_1}\!\succ\!\!\prec\!{}^{t_7}_{t_3}\right)^{(5\beta-8,1,0)}$
$\left({}^{t_6}_{t_1}\!\succ\!\!\prec\!{}^{t_5}_{t_7}\right)^{(5\beta-8,1,0)}$	$\left({}^{t_5}_{t_1}\!\bigotimes\!{}^{t_3}_{t_8\,t_1}\ \nearrow\right) \;\Rightarrow\; \left({}^{t_5}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(5\beta-8,1,0)}$
$\left({}^{t_8}_{t_1}\!\succ\!\!\prec\!{}^{t_5}_{t_7}\right)^{(5\beta-7,0,0)}$	$3\textsc{sr} \;\Rightarrow\; \left({}^{t_5}_{t_1}\!\succ\!\!\prec\!{}^{t_5}_{t_9}\right)^{(5\beta-7,0,0)}$
	$\left({}^{t_1}_{t_7}\!\bigotimes\!{}^{t_3}_{t_1\,t_8}\ \nearrow\right) \;\Rightarrow\; \left({}^{t_1}_{t_7}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(5\beta-7,0,0)}$
	$\left(\nwarrow\ {}^{t_1}_{t_5\,t_1}\!\bigotimes\!{}^{t_3}_{t_9}\right) \;\Rightarrow\; \left({}^{t_7}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(5\beta-7,0,0)}$

Table 5. Steps for Deducing $\left({}^{t_3}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(5\beta-9,2,0)} \Rightarrow \left({}^{t_7}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(5\beta-7,0,0)}$ Using 3sr and 5sr (Notation (u,v,w) Is Shorthand for $(\beta;0,0,0,0,0,u,v,w)$)

(assuming $\beta \geq 2$) — which, we recall, we used to solve for one of the remaining degree-$(\beta-1)$ unknowns — we apply the sequence of implications in Table 5 to deduce $\left({}^{t_7}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(\beta;0,0,0,0,0,5\beta-7,0,0)}$. Starting with $\left({}^{t_3}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(\beta;0,0,0,0,0,5\beta-7,1,0)}$ $(\beta \geq 2)$ the first three steps of Table 5 (with degrees suitably adjusted) give us $\left({}^{t_5}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(\beta;0,0,0,0,0,5\beta-6,0,0)}$. Lastly, we consider $\left({}^{t_3}_{t_1}\!\succ\!\!\prec\!{}^{t_3}_{t_9}\right)^{(\beta;0,0,0,0,0,5\beta-5,0,0)}$. This comes about in solving for the last of the degree-$(\beta-1)$ unknowns when $\beta \geq 2$. When $\beta = 1$ this relation imposes an actual constraint on the starting data. If one writes out (e)–(g) of Table 4 then one finds the constraint

$$\begin{aligned} 11N(1;0,0,0,0,1,0,0,1) = {} & 6N(1;0,0,1,0,1,0,0,0) \\ & + 15N(1;0,0,0,2,0,0,0,0). \end{aligned} \tag{8}$$

Next, in case $(d_2,d_4,d_6,d_8,d_9) = (0,0,0,0,0)$ but $d_5 \neq 0$ or $d_7 \neq 0$, then because of the induction order, some of the (85 total) relations listed as determining numbers will determine numbers that have already been solved for. But in each such case, 3sr allows us to deduce the relation in question. Finally, when $(d_2,d_4,d_6,d_8,d_9) \neq (0,0,0,0,0)$ then all 85 relations follow by 3sr just as in Example 2.

As in Example 2, we must now verify that the relations indicated in the above lists (85 in number) plus the 120 relations which involve only powers of t_1 imply the remaining 461 relations by 5sr. This takes a bit of checking. The seemingly daunting task is made tractable by associating to each relation its "weight" (the quantity appearing on the right-hand side of (7)), and then noting that the terms appearing in each 5sr expression (6) all carry the same weight. Now, there is just a bit of checking to do within each weight class. Also, once we deduce all the relations which involve t_1, the relations which do not involve t_1 follow immediately (find an entry other than t_3, factor it into t_1 times something else, and invoke 5sr).

Summarizing, any choice of starting data satisfying the genericity condition $N(1; 0, 0, 0, 0, 1, 0, 0, 1) \neq 0$ and the constraint (8) extends uniquely to a full solution to WDVV for $G(2, 5)$. These solutions form a two-dimensional family.

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