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Q-Valued Functions Revisited

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Contents

Abstract

In this note we revisit Almgren's theory of Q-valued functions, that are functions taking values in the space $\mathcal{A}_Q(\mathbb{R}^n)$ of unordered Q-tuples of points in $\mathbb{R}^n$. In particular:

- we give shorter versions of Almgren's proofs of the existence of Dir-minimizing Q-valued functions, of their Hölder regularity and of the dimension estimate of their singular set;
- we propose an alternative, intrinsic approach to these results, not relying on Almgren's biLipschitz embedding $\xi : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^{N(Q,n)}$;
- we improve upon the estimate of the singular set of planar D-minimizing functions by showing that it consists of isolated points.

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Introduction

The aim of this paper is to provide a simple, complete and self-contained reference for Almgren's theory of Dir-minimizing Q-valued functions, so to make it an easy step for the understanding of the remaining parts of the Big regularity paper [**Alm00**]. We propose simpler and shorter proofs of the central results on Q-valued functions contained there, suggesting new points of view on many of them. In addition, parallel to Almgren's theory, we elaborate an intrinsic one which reaches his main results avoiding the extrinsic mappings $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ (see Section 2.1 and compare with 1.2 of [**Alm00**]). This "metric" point of view is clearly an original contribution of this paper. The second new contribution is Theorem 0.12 where we improve Almgren's estimate of the singular set in the planar case, relying heavily on computations of White [**Whi83**] and Chang [**Cha88**].

Simplified and intrinsic proofs of parts of Almgren's big regularity paper have already been established in [**Gob06a**] and [**Gob06b**]. In fact our proof of the Lipschitz extension property for Q-valued functions is essentially the one given in [**Gob06a**] (see Section 1.2). Just to compare this simplified approach to Almgren's, note that the existence of the retraction $\boldsymbol{\rho}$ is actually an easy corollary of the existence of $\boldsymbol{\xi}$ and of the Lipschitz extension theorem. In Almgren's paper, instead, the Lipschitz extension theorem is a corollary of the existence of $\boldsymbol{\rho}$, which is constructed explicitly (see 1.3 in [**Alm00**]) . However, even where our proofs differ most from his, we have been clearly influenced by his ideas and we cannot exclude the existence of hints to our strategies in [**Alm00**] or in his other papers [**Alm83**] and [**Alm86**]: the amount of material is very large and we have not explored it in all the details.

Almgren asserts that some of the proofs in the first chapters of [**Alm00**] are more involved than apparently needed because of applications contained in the other chapters, where he proves his celebrated partial regularity theorem for area-minimizing currents. We instead avoid any complication which looked unnecessary for the theory of Dir-minimizing Q-functions. For instance, we do not show the existence of Almgren's improved Lipschitz retraction $\boldsymbol{\rho}^*$ (see 1.3 of [**Alm00**]), since it is not needed in the theory of Dir-minimizing Q-valued functions. This retraction is instead used in the approximation of area-minimizing currents (see Chapter 3 of [**Alm00**]) and will be addressed in the forthcoming paper [**DLS**].

In our opinion the portion of Almgren's Big regularity paper regarding the theory of Q-valued functions is simply a combination of clean ideas from the theory of elliptic partial differential equations with elementary observations of combinatorial nature, the latter being much less complicated than what they look at a first sight. In addition our new "metric" point of view reduces further the combinatorial part, at the expense of introducing other arguments of more analytic flavor.

The metric space $\mathcal{A}_Q(\mathbb{R}^n)$. Roughly speaking, our intuition of Q-valued functions is that of mappings taking their values in the unordered sets of Q points of $\mathbb{R}^n$, with the understanding that multiplicity can occur. We formalize this idea by identifying the space of Q unordered points in $\mathbb{R}^n$ with the set of positive atomic measures of mass Q.

Definition 0.1 (Unordered Q-tuples). We denote by $[\![P_i]\!]$ the Dirac mass in $P_i \in \mathbb{R}^n$ and we define the space of Q-points as

$$\mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] \,:\, P_i \in \mathbb{R}^n \text{ for every } i = 1, \ldots, Q \right\}.$$

In order to simplify the notation, we use $\mathcal{A}_Q$ in place of $\mathcal{A}_Q(\mathbb{R}^n)$ and we write $\sum_i [\![P_i]\!]$ when n and Q are clear from the context. Clearly, the points P_i do not have to be distinct: for instance $Q\,[\![P]\!]$ is an element of $\mathcal{A}_Q(\mathbb{R}^n)$. We endow $\mathcal{A}_Q(\mathbb{R}^n)$ with a metric which makes it a complete metric space (the completeness is an elementary exercise left to the reader).

Definition 0.2. For every $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n)$, with $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!]$, we define

$$\mathcal{G}(T_1, T_2) := \min_{\sigma \in \mathscr{P}_Q} \sqrt{\sum_i \left|P_i - S_{\sigma(i)}\right|^2},$$

where $\mathscr{P}_Q$ denotes the group of permutations of $\{1, \ldots, Q\}$.

Remark 0.3. $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ is a closed subset of a "convex" complete metric space. Indeed, $\mathcal{G}$ coincides with the L^2-Wasserstein distance on the space of positive measures with finite second moment (see for instance [**AGS05**] and [**Vil03**]). In Section 4.1 we will also use the fact that $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ can be embedded isometrically in a separable Banach space.

The metric theory of Q-valued functions starts from this remark. It avoids the Euclidean embedding and retraction theorems of Almgren but is anyway powerful enough to prove the main results on Q-valued functions addressed in this note. We develop it fully in Chapter 4 after presenting (in Chapters 1, 2 and 3) Almgren's theory with easier proofs. However, since the metric point of view allows a quick, intrinsic definition of Sobolev mappings and of the Dirichlet energy, we use it already here to state immediately the main theorems.

Q-valued functions and the Dirichlet energy. For the rest of the paper Ω will be a bounded open subset of the Euclidean space $\mathbb{R}^m$. If not specified, we will assume that the regularity of $\partial\Omega$ is Lipschitz. Continuous, Lipschitz, Hölder and (Lebesgue) measurable functions from Ω into $\mathcal{A}_Q$ are defined in the usual way. As for the spaces $L^p(\Omega, \mathcal{A}_Q)$, they consist of those measurable maps $u : \Omega \to \mathcal{A}_Q$ such that $\|\mathcal{G}(u, Q\,[\![0]\!])\|_{L^p}$ is finite. Observe that, since Ω is bounded, this is equivalent to ask that $\|\mathcal{G}(u, T)\|_{L^p}$ is finite for every $T \in \mathcal{A}_Q$.

It is a general fact (and we show it in Section 1.1) that any measurable Q-valued function can be written as the "sum" of Q measurable functions.

Proposition 0.4 (Measurable selection). *Let $B \subset \mathbb{R}^m$ be a measurable set and let $f : B \to \mathcal{A}_Q$ be a measurable function. Then, there exist $f_1, \ldots, f_Q$ measurable*

$\mathbb{R}^n$-valued functions such that

$$f(x) = \sum_i [\![f_i(x)]\!] \quad \textit{for a.e. } x \in B. \tag{0.1}$$

Obviously, such a choice is far from being unique, but, in using notation (0.1), we will always think of a measurable Q-valued function as coming together with such a selection.

We now introduce the Sobolev spaces of functions taking values in the metric space of Q-points, as defined independently by Ambrosio in [**Amb90**] and Reshetnyak in [**Res04**].

Definition 0.5 (Sobolev Q-valued functions). A measurable function $f : \Omega \to \mathcal{A}_Q$ is in the Sobolev class $W^{1,p}$ $(1 \leq p \leq \infty)$ if there exist m functions $\varphi_j \in L^p(\Omega, \mathbb{R}^+)$ such that

(i) $x \mapsto \mathcal{G}(f(x), T) \in W^{1,p}(\Omega)$ for all $T \in \mathcal{A}_Q$;
(ii) $|\partial_j \mathcal{G}(f, T)| \leq \varphi_j$ almost everywhere in Ω for all $T \in \mathcal{A}_Q$ and for all $j \in \{1, \ldots, m\}$.

Definition 0.5 can be easily generalized when the domain is a Riemannian manifold M. In this case we simply ask that $f \circ x^{-1}$ is a Sobolev Q-function for every open set $U \subset M$ and every chart $x : U \to \mathbb{R}^n$. It is not difficult to show the existence of minimal functions $\tilde{\varphi}_j$ fulfilling (ii), i.e. such that

$$\tilde{\varphi}_j \leq \varphi_j \text{ a.e. } \text{ for any other } \varphi_j \text{ satisfying (ii)},$$

(see Proposition 4.2). We denote them by $|\partial_j f|$. We will later characterize $|\partial_j f|$ by the following property (cp. with Proposition 4.2): for every countable dense subset $\{T_i\}_{i \in \mathbb{N}}$ of $\mathcal{A}_Q$ and for every $j = 1, \ldots, m$,

$$|\partial_j f| = \sup_{i \in \mathbb{N}} |\partial_j \mathcal{G}(f, T_i)| \quad \text{almost everywhere in } \Omega. \tag{0.2}$$

In the same way, given a vector field X, we can define intrinsically $|\partial_X f|$ and prove the formula corresponding to (0.2). For functions $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$, we set

$$|Df|^2 := \sum_{j=1}^m |\partial_j f|^2 . \tag{0.3}$$

For functions on a general Riemannian manifold M, we choose an orthonormal frame $X_1, \ldots X_m$ and set $|Df|^2 = \sum |\partial_{X_i} f|^2$. This definition is independent of the choice of coordinates (resp. of frames), as it can be seen from Proposition 2.17.

Definition 0.6. The Dirichlet energy of $f \in W^{1,2}(U, \mathcal{A}_Q)$, where U is an open subset of a Riemannian manifold, is given by $\mathrm{Dir}(f, U) := \int_U |Df|^2$.

It is not difficult to see that, when f can be decomposed into finitely many regular single-valued functions, i.e. $f(x) = \sum_i [\![f_i(x)]\!]$ for some differentiable functions f_i, then

$$\mathrm{Dir}(f, U) = \sum_i \int_U |Df_i|^2 = \sum_i \mathrm{Dir}(f_i, U).$$

The usual notion of trace at the boundary can be easily generalized to this setting.

Definition 0.7 (Trace of Sobolev Q-functions). Let $\Omega \subset \mathbb{R}^m$ be a Lipschitz bounded open set and $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. A function g belonging to $L^p(\partial\Omega, \mathcal{A}_Q)$ is said to be the trace of f at $\partial\Omega$ (and we denote it by $f|_{\partial\Omega}$) if, for every $T \in \mathcal{A}_Q$, the trace of the real-valued Sobolev function $\mathcal{G}(f, T)$ coincides with $\mathcal{G}(g, T)$.

It is straightforward to check that this notion of trace coincides with the restriction of f to the boundary when f is a continuous function which extends continuously to $\overline{\Omega}$. In Section 4.2, we show the existence and uniqueness of the trace for every $f \in W^{1,p}$. Hence, we can formulate a Dirichlet problem for Q-valued functions: $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is said to be Dir-minimizing if

$$\mathrm{Dir}(f, \Omega) \leq \mathrm{Dir}(g, \Omega) \quad \text{for all } g \in W^{1,2}(\Omega, \mathcal{A}_Q) \text{ with } f|_{\partial\Omega} = g|_{\partial\Omega}.$$

The main results proved in this paper. We are now ready to state the main theorems of Almgren reproved in this note: an existence theorem and two regularity results.

Theorem 0.8 (Existence for the Dirichlet Problem). *Let $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$. Then, there exists a* Dir*-minimizing function $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ such that $f|_{\partial\Omega} = g|_{\partial\Omega}$.*

Theorem 0.9 (Hölder regularity). *There exists a positive constant $\alpha = \alpha(m, Q) > 0$ with the following property. If $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is* Dir*-minimizing, then $f \in C^{0,\alpha}(\Omega')$ for every $\Omega' \subset\subset \Omega \subset \mathbb{R}^m$. For two-dimensional domains, we have the explicit constant $\alpha(2, Q) = 1/Q$.*

For the second regularity theorem we need the definition of singular set of f.

Definition 0.10 (Regular and singular points). A Q-valued function f is regular at a point $x \in \Omega$ if there exists a neighborhood B of x and Q analytic functions $f_i : B \to \mathbb{R}^n$ such that

$$f(y) = \sum_i [\![f_i(y)]\!] \quad \text{for almost every } y \in B$$

and either $f_i(x) \neq f_j(x)$ for every $x \in B$ or $f_i \equiv f_j$. The singular set Σ_f of f is the complement of the set of regular points.

Theorem 0.11 (Estimate of the singular set). *Let f be a* Dir*-minimizing function. Then, the singular set Σ_f of f is relatively closed in Ω. Moreover, if $m = 2$, then Σ_f is at most countable, and if $m \geq 3$, then the Hausdorff dimension of Σ_f is at most $m - 2$.*

Following in part ideas of [**Cha88**], we improve this last theorem in the following way.

Theorem 0.12 (Improved estimate of the singular set). *Let f be* Dir*-minimizing and $m = 2$. Then, the singular set Σ_f of f consists of isolated points.*

This note is divided into five parts. Chapter 1 gives the "elementary theory" of Q-valued functions, while Chapter 2 focuses on the "combinatorial results" of Almgren's theory. In particular we give there very simple proofs of the existence of Almgren's biLipschitz embedding $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^{N(Q,n)}$ and of a Lipschitz retraction $\boldsymbol{\rho}$ of $\mathbb{R}^{N(Q,n)}$ onto $\boldsymbol{\xi}(\mathbb{R}^{N(Q,n)})$. Following Almgren's approach, $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ are then used to generalize the classical Sobolev theory to Q-valued functions. In Chapter 4 we develop the intrinsic theory and show how the results of Chapter 2 can be recovered independently of the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. Chapter 3 gives simplified proofs of Almgren's regularity theorems for Q-valued functions and Chapter 5 contains the improved estimate of Theorem 0.12. Therefore, to get a proof of the four main Theorems listed above, the reader can choose to follow Chapters 1, 2, 3 and 5, or to follow Chapters 1, 4, 3 and 5.

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CHAPTER 1

The elementary theory of Q-valued functions

This chapter consists of three sections. The first one introduces a recurrent theme: decomposing Q-valued functions in simpler pieces. We will often build on this and prove our statements inductively on Q, relying ultimately on well-known properties of single-valued functions. Section 1.2 contains an elementary proof of the following fact: any Lipschitz map from a subset of $\mathbb{R}^m$ into $\mathcal{A}_Q$ can be extended to a Lipschitz map on the whole Euclidean space. This extension theorem, combined with suitable truncation techniques, is the basic tool of various approximation results. Section 1.3 introduces a notion of differentiability for Q-valued maps and contains some chain–rule formulas and a generalization of the classical theorem of Rademacher. These are the main ingredients of several computations in later sections.

1.1. Decomposition and selection for Q-valued functions

Given two elements $T \in \mathcal{A}_{Q_1}(\mathbb{R}^n)$ and $S \in \mathcal{A}_{Q_2}(\mathbb{R}^n)$, the sum $T + S$ of the two measures belongs to $\mathcal{A}_Q(\mathbb{R}^n) = \mathcal{A}_{Q_1+Q_2}(\mathbb{R}^n)$. This observation leads directly to the following definition.

Definition 1.1. Given finitely many Q_i-valued functions f_i, the map $f_1 + f_2 + \ldots + f_N$ defines a Q-valued function f, where $Q = Q_1 + Q_2 + \ldots + Q_N$. This will be called a *decomposition of f into N simpler functions*. We speak of Lebesgue measurable (Lipschitz, Hölder, etc.) decompositions, when the f_i's are measurable (Lipschitz, Hölder, etc.). In order to avoid confusions with the summation of vectors in $\mathbb{R}^n$, we will write, with a slight abuse of notation,

$$f = [\![f_1]\!] + \ldots + [\![f_N]\!] \,.$$

If $Q_1 = \ldots = Q_N = 1$, the decomposition is called a *selection*.

Proposition 0.4 ensures the existence of a measurable selection for any measurable Q-valued function. The only role of this proposition is to simplify our notation.

1.1.1. Proof of Proposition 0.4.

We prove the proposition by induction on Q. The case $Q = 1$ is of course trivial. For the general case, we will make use of the following elementary observation:

(D) if $\bigcup_{i\in\mathbb{N}} B_i$ is a covering of B by measurable sets, then it suffices to find a measurable selection of $f|_{B_i \cap B}$ for every i.

Let first $\mathcal{A}_0 \subset \mathcal{A}_Q$ be the closed set of points of type $Q\,[\![P]\!]$ and set $B_0 = f^{-1}(\mathcal{A}_0)$. Then, B_0 is measurable and $f|_{B_0}$ has trivially a measurable selection.

Next we fix a point $T \in \mathcal{A}_Q \setminus \mathcal{A}_0$, $T = \sum_i [\![P_i]\!]$. We can subdivide the set of indexes $\{1, \ldots, Q\} = I_L \cup I_K$ into two nonempty sets of cardinality L and K, with the property that

$$|P_k - P_l| > 0 \quad \text{for every } l \in I_L \text{ and } k \in I_K. \tag{1.1}$$

For every $S = \sum_i [\![Q_i]\!]$, let $\pi_S \in \mathscr{P}_Q$ be a permutation such that

$$\mathcal{G}(S,T)^2 = \sum_i |P_i - Q_{\pi_S(i)}|^2.$$

If U is a sufficiently small neighborhood of T in $\mathcal{A}_Q$, by (1.1), the maps

$$\tau : U \ni S \mapsto \sum_{l \in I_L} [\![Q_{\pi_S(l)}]\!] \in \mathcal{A}_L, \quad \sigma : U \ni S \mapsto \sum_{k \in I_K} [\![Q_{\pi_S(k)}]\!] \in \mathcal{A}_K$$

are continuous. Therefore, $C = f^{-1}(U)$ is measurable and $[\![\sigma \circ f|_C]\!] + [\![\tau \circ f|_C]\!]$ is a measurable decomposition of $f|_C$. Then, by inductive hypothesis, $f|_C$ has a measurable selection.

According to this argument, it is possible to cover $\mathcal{A}_Q \setminus \mathcal{A}_0$ with open sets U's such that, if $B = f^{-1}(U)$, then $f|_B$ has a measurable selection. Since $\mathcal{A}_Q \setminus \mathcal{A}_0$ is an open subset of a separable metric space, we can find a countable covering $\{U_i\}_{i\in\mathbb{N}}$ of this type. Since $\{B_0\} \cup \{f^{-1}(U_i)\}_{1=1}^{\infty}$ is a measurable covering of B, from (D) we conclude the proof.

1.1.2. One dimensional $W^{1,p}$-decomposition. A more serious problem is to find selections which are as regular as f itself. Essentially, this is always possible when the domain of f is 1-dimensional. For our purposes we just need the Sobolev case of this principle, which we prove in the next two propositions.

In this subsection $I = [a,b]$ is a closed bounded interval of $\mathbb{R}$ and the space of absolutely continuous functions $AC(I, \mathcal{A}_Q)$ is defined as the space of those continuous $f : I \to \mathcal{A}_Q$ such that, for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: for every $a \leq t_1 < t_2 < \ldots < t_{2N} \leq b$,

$$\sum_i (t_{2i} - t_{2i-1}) < \delta \quad \text{implies} \quad \sum_i \mathcal{G}\big(f(t_{2i}), f(t_{2i-1})\big) < \varepsilon.$$

Proposition 1.2. *Let $f \in W^{1,p}(I, \mathcal{A}_Q)$. Then,*

- (a) *$f \in AC(I, \mathcal{A}_Q)$ and, moreover, $f \in C^{0,1-\frac{1}{p}}(I, \mathcal{A}_Q)$ for $p > 1$;*
- (b) *there exists a selection $f_1, \ldots, f_Q \in W^{1,p}(I, \mathbb{R}^n)$ of f such that $|Df_i| \leq |Df|$ almost everywhere.*

Remark 1.3. A similar selection theorem holds for continuous Q-functions. This result needs a subtler combinatorial argument and is proved in Almgren's Big regularity paper [**Alm00**] (Proposition 1.10, p. 85). The proof of Almgren uses the Euclidean structure, whereas a more general argument has been proposed in [**DLGT04**].

Proposition 1.2 cannot be extended to maps $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$. For example, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and $\mathbb{S}^1$ with the set $\{z \in \mathbb{C} : |z| = 1\}$ and we consider the map $f : \mathbb{S}^1 \to \mathcal{A}_Q(\mathbb{R}^2)$ given by $f(z) = \sum_{\zeta^2 = z} [\![\zeta]\!]$. Then, f is Lipschitz (and hence belongs to $W^{1,p}$ for every p) but it does not have a continuous selection. Nonetheless, we can use Proposition 1.2 to decompose any $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$ into "irreducible pieces".

Definition 1.4. $f \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$ is called *irreducible* if there is no decomposition of f into 2 simpler $W^{1,p}$ functions.

PROPOSITION 1.5. *For every Q-function $g \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q(\mathbb{R}^n))$, there exists a decomposition $g = \sum_{j=1}^J [\![g_j]\!]$, where each g_j is an irreducible $W^{1,p}$ map. A function g is irreducible if and only if*

(i) *card* $(\mathrm{supp}\,(g(z))) = Q$ *for every* $z \in \mathbb{S}^1$ *and*
(ii) *there exists a $W^{1,p}$ map $h : \mathbb{S}^1 \to \mathbb{R}^n$ with the property that* $f(z) = \sum_{\zeta^Q = z} [\![h(\zeta)]\!]$.

Moreover, for every irreducible g, there are exactly Q maps h fulfilling (ii).

The existence of an irreducible decomposition in the sense above is an obvious consequence of the definition of irreducible maps. The interesting part of the proposition is the characterization of the irreducible pieces, a direct corollary of Proposition 1.2.

PROOF OF PROPOSITION 1.2. We start with (a). Fix a dense set $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{A}_Q$. Then, for every $i \in \mathbb{N}$, there is a negligible set $E_i \subset I$ such that, for every $x < y \in I \setminus E_i$,

$$\left|\mathcal{G}(f(x), T_i) - \mathcal{G}(f(y), T_i)\right| \le \left|\int_x^y \mathcal{G}(f, T_i)'\right| \le \int_x^y |Df|.$$

Fix $x < y \in I \setminus \cup_i E_i$ and choose a sequence $\{T_{i_l}\}$ converging to $f(x)$. Then,

$$\mathcal{G}(f(x), f(y)) = \lim_{l\to\infty} \left|\mathcal{G}(f(x), T_{i_l}) - \mathcal{G}(f(y), T_{i_l})\right| \le \int_x^y |Df|\,. \tag{1.2}$$

Clearly, (1.2) gives the absolute continuity of f outside $\cup_i E_i$. Moreover, f can be redefined in a unique way on the exceptional set so that the estimate (1.2) holds for every pair x, y. In the case $p > 1$, we improve (1.2) to $\mathcal{G}(f(x), f(y)) \le \|\,|Df|\,\|_{L^p} |x-y|^{(p-1)/p}$, thus concluding the Hölder continuity.

For (b), the strategy is to find $f_1, \ldots, f_Q$ as limit of approximating piecewise linear functions. To this aim, fix $k \in \mathbb{N}$ and set

$$\Delta_k := \frac{b-a}{k} \quad \text{and} \quad t_l := a + l\,\Delta_k, \quad \text{with} \quad l = 0, \ldots, k.$$

By (a), without loss of generality, we assume that f is continuous and we consider the points $f(t_l) = \sum_i [\![P_i^l]\!]$. Moreover, after possibly reordering each $\{P_i^l\}_{i\in\{1,\ldots,Q\}}$, we can assume that

$$\mathcal{G}(f(t_{l-1}), f(t_l))^2 = \sum_i \left|P_i^{l-1} - P_i^l\right|^2. \tag{1.3}$$

Hence, we define the functions f_i^k as the linear interpolations between the points (t_l, P_i^l), that is, for every $l = 1, \ldots, k$ and every $t \in [t_{l-1}, t_l]$, we set

$$f_i^k(t) = \frac{t_l - t}{\Delta_k} P_i^{l-1} + \frac{t - t_{l-1}}{\Delta_k} P_i^l.$$

It is immediate to see that the f_i^k's are $W^{1,1}$ functions; moreover, for every $t \in (t_{l-1}, t_l)$, thanks to (1.3), the following estimate holds,

$$\left|Df_i^k(t)\right| = \frac{\left|P_i^{l-1} - P_i^l\right|}{\Delta_k} \le \frac{\mathcal{G}(f(t_{l-1}), f(t_l))}{\Delta_k} \le ⨍_{t_{l-1}}^{t_l} |Df|\,(\tau)\,d\tau =: h^k(t). \tag{1.4}$$

Since the functions h^k converge in L^p to $|Df|$ for $k \to +\infty$, we conclude that the f_i^k's are equi-continuous and equi-bounded. Hence, up to passing to a subsequence, which we do not relabel, there exist functions $f_1, \ldots, f_Q$ such that $f_i^k \to f_i$ uniformly. Passing to the limit, (1.4) implies that $|Df_i| \leq |Df|$ and it is a very simple task to verify that $\sum_i [\![f_i]\!] = f$. $\square$

Proof of Proposition 1.5. The decomposition of g into irreducible maps is a trivial corollary of the definition of irreducibility. Moreover, it is easily seen that a map satisfying (i) and (ii) is necessarily irreducible.

Let now g be an irreducible $W^{1,p}$ Q-function. Consider g as a function on $[0, 2\pi]$ with the property that $g(0) = g(2\pi)$ and let $h_1, \ldots, h_Q$ in $W^{1,p}([0, 2\pi], \mathbb{R}^n)$ be a selection as in Proposition 1.2. Since we have $g(0) = g(2\pi)$, there exists a permutation σ such that $h_i(2\pi) = h_{\sigma(i)}(0)$. We claim that any such σ is necessarily a Q-cycle. If not, there is a partition of $\{1, \ldots, Q\}$ into two disjoint nonempty subsets I_L and I_K, with cardinality L and K respectively, such that $\sigma(I_L) = I_L$ and $\sigma(I_K) = I_K$. Then, the functions

$$g_L = \sum_{i \in I_L} [\![h_i]\!] \quad \text{and} \quad g_K = \sum_{i \in I_K} [\![h_i]\!]$$

would provide a decomposition of f into two simpler $W^{1,p}$ functions.

The claim concludes the proof. Indeed, for what concerns (i), we note that, if the support of $g(0)$ does not consist of Q distinct points, there is always a permutation σ such that $h_i(2\pi) = h_{\sigma(i)}(0)$ and which is not a Q-cycle. For (ii), without loss of generality, we can order the h_i in such a way that $\sigma(Q) = 1$ and $\sigma(i) = i+1$ for $i \leq Q-1$. Then, the map $h : [0, 2\pi] \to \mathbb{R}^n$ defined by

$$h(\theta) = h_i(Q\theta - 2(i-1)\pi), \quad \text{for } \theta \in [2(i-1)\pi/Q, 2i\pi/Q],$$

fulfils (ii). Finally, if a map $\tilde{h} \in W^{1,p}(\mathbb{S}^1, \mathbb{R}^n)$ satisfies

$$g(\theta) = \sum_i \left[\!\!\left[\tilde{h}((\theta + 2i\pi)/Q)\right]\!\!\right] \quad \text{for every } \theta, \tag{1.5}$$

then there is $j \in \{1, \ldots, Q\}$ such that $\tilde{h}(0) = h(2j\pi/Q)$. By (i) and the continuity of h and $\tilde{h}$, the identity $\tilde{h}(\theta) = h(\theta + 2j\pi/Q)$ holds for θ in a neighborhood of 0. Therefore, since $\mathbb{S}^1$ is connected, a simple continuation argument shows that $\tilde{h}(\theta) = h(\theta + 2j\pi/Q)$ for every θ. On the other hand, all the $\tilde{h}$ of this form are different (due to (i)) and enjoy (1.5): hence, there are exactly Q distinct $W^{1,p}$ functions with this property. $\square$

1.1.3. Lipschitz decomposition. For general domains of dimension $m \geq 2$, there are well-known obstructions to the existence of regular selections. However, it is clear that, when f is continuous and the support of $f(x)$ does not consist of a single point, in a neighborhood U of x, there is a decomposition of f into two continuous simpler functions. When f is Lipschitz, this decomposition holds in a sufficiently large ball, whose radius can be estimated from below with a simple combinatorial argument. This fact will play a key role in many subsequent arguments.

Proposition 1.6. *Let $f : B \subset \mathbb{R}^m \to \mathcal{A}_Q$ be a Lipschitz function, $f = \sum_{i=1}^Q [\![f_i]\!]$. Suppose that there exist $x_0 \in B$ and $i, j \in \{1, \ldots, Q\}$ such that*

$$|f_i(x_0) - f_j(x_0)| > 3\,(Q-1)\,Lip(f)\,diam(B). \tag{1.6}$$

Then, there is a decomposition of f into two simpler Lipschitz functions f_K and f_L with $Lip(f_K), Lip(f_L) \leq Lip(f)$ and $\mathrm{supp}\,(f_K(x)) \cap \mathrm{supp}\,(f_L(x)) = \emptyset$ *for every x.*

PROOF. Call a "squad" any subset of indices $I \subset \{1, \dots, Q\}$ such that

$$|f_l(x_0) - f_r(x_0)| \le 3\,(|I| - 1)\,\mathrm{Lip}(f)\,\mathrm{diam}(B) \quad \text{for all } l, r \in I,$$

where $|I|$ denotes the cardinality of I. Let I_L be a maximal squad containing 1, where L stands for its cardinality. By (1.6), $L < Q$. Set $I_K = \{1, \dots, Q\} \setminus I_L$. Note that, whenever $l \in I_L$ and $k \in I_K$,

$$|f_l(x_0) - f_k(x_0)| > 3\,\mathrm{Lip}(f)\,\mathrm{diam}(B), \tag{1.7}$$

otherwise I_L would not be maximal. For every x, $y \in B$, we let π_x, $\pi_{x,y} \in \mathscr{P}_Q$ be permutations such that

$$\mathcal{G}(f(x_0), f(x))^2 = \sum_i \left| f_i(x_0) - f_{\pi_x(i)}(x) \right|^2,$$
$$\mathcal{G}(f(x), f(y))^2 = \sum_i \left| f_i(x) - f_{\pi_{x,y}(i)}(y) \right|^2.$$

We define the functions f_L and f_K as

$$f_L(x) = \sum_{i \in I_L} [\![f_{\pi_x(i)}(x)]\!] \quad \text{and} \quad f_K(x) = \sum_{i \in I_K} [\![f_{\pi_x(i)}(x)]\!].$$

Observe that $f = [\![f_L]\!] + [\![f_K]\!]$: it remains to show the Lipschitz estimate. For this aim, we claim that $\pi_{x,y}(\pi_x(I_L)) = \pi_y(I_L)$ for every x and y. Assuming the claim, we conclude that, for every $x, y \in B$,

$$\mathcal{G}(f(x), f(y))^2 = \mathcal{G}(f_L(x), f_L(y))^2 + \mathcal{G}(f_K(x), f_K(y))^2,$$

and hence $\mathrm{Lip}(f_L), \mathrm{Lip}(f_K) \le \mathrm{Lip}(f)$.

To prove the claim, we argue by contradiction: if it is false, choose x, $y \in B$, $l \in I_L$ and $k \in I_K$ with $\pi_{x,y}(\pi_x(l)) = \pi_y(k)$. Then, $\left| f_{\pi_x(l)}(x) - f_{\pi_y(k)}(y) \right| \le \mathcal{G}(f(x), f(y))$, which in turn implies

$$\begin{aligned}
3\,\mathrm{Lip}(f)\,\mathrm{diam}(B) &\overset{(1.7)}{<} |f_l(x_0) - f_k(x_0)| \\
&\le \left| f_l(x_0) - f_{\pi_x(l)}(x) \right| + \left| f_{\pi_x(l)}(x) - f_{\pi_y(k)}(y) \right| + \left| f_{\pi_y(k)}(y) - f_k(x_0) \right| \\
&\le \mathcal{G}(f(x_0), f(x)) + \mathcal{G}(f(x), f(y)) + \mathcal{G}(f(y), f(x_0)) \\
&\le \mathrm{Lip}(f)\,(|x_0 - x| + |x - y| + |y - x_0|) \\
&\le 3\,\mathrm{Lip}(f)\,\mathrm{diam}(B).
\end{aligned}$$

This is a contradiction and, hence, the proof is complete. $\square$

1.2. Extension of Lipschitz Q-valued functions

This section is devoted to prove the following extension theorem.

THEOREM 1.7 (Lipschitz Extension). *Let $B \subset \mathbb{R}^m$ and $f : B \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz. Then, there exists an extension $\bar{f} : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ of f, with $Lip(\bar{f}) \le C(m, Q)\, Lip(f)$. Moreover, if f is bounded, then, for every $T \in Q\,[\![P]\!]$,*

$$\sup_{x \in \mathbb{R}^m} \mathcal{G}(\bar{f}(x), T) \le C(m, Q) \sup_{x \in B} \mathcal{G}(f(x), T). \tag{1.8}$$

Note that, in his Big regularity paper, Almgren deduces Theorem 1.7 from the existence of the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ of Section 2.1. We instead follow a sort of reverse path and conclude the existence of $\boldsymbol{\rho}$ from that of $\boldsymbol{\xi}$ invoking Theorem 1.7.

It has already been observed by Goblet in [**Gob06a**] that the Homotopy Lemma 1.8 below can be combined with a Whitney-type argument to yield an easy direct proof of the Lipschitz extension Theorem, avoiding Almgren's maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. In [**Gob06a**] the author refers to the general theory built in [**LS97**] to conclude Theorem 1.7 from Lemma 1.8. For the sake of completeness, we give here the complete argument.

1.2.1. Homotopy Lemma. Let C be a cube with sides parallel to the coordinate axes. As a first step, we show the existence of extensions to C of Lipschitz Q-valued functions defined on ∂C. This will be the key point in the Whitney type argument used in the proof of Theorem 1.7.

LEMMA 1.8 (Homotopy lemma). *There is a constant $c(Q)$ with the following property. For any closed cube with sides parallel to the coordinate axes and any Lipschitz Q-function $h : \partial C \to \mathcal{A}_Q(\mathbb{R}^n)$, there exists an extension $f : C \to \mathcal{A}_Q(\mathbb{R}^n)$ of h which is Lipschitz with $Lip(f) \leq c(Q) Lip(h)$. Moreover, for every $T = Q\,[\![P]\!]$,*

$$\max_{x\in C} \mathcal{G}(f(x), T) \leq 2\,Q \max_{x\in\partial C} \mathcal{G}(h(x), T). \tag{1.9}$$

PROOF. By rescaling and translating, it suffices to prove the lemma when $C = [0,1]^m$. Since C is biLipschitz equivalent to the closed unit ball $\overline{B_1}$ centered at 0, it suffices to prove the lemma with $\overline{B_1}$ in place of C. In order to prove this case, we proceed by induction on Q. For $Q = 1$, the statement is a well-known fact (it is very easy to find an extension $\bar f$ with $\mathrm{Lip}\,(\bar f) \leq \sqrt{n}\,\mathrm{Lip}(f)$; the existence of an extension with the same Lipschitz constant is a classical, but subtle, result of Kirszbraun, see 2.10.43 in [**Fed69**]). We now assume that the lemma is true for every $Q < Q^*$, and prove it for Q^*.

Fix any $x_0 \in \partial B_1$. We distinguish two cases: either (1.6) of Proposition 1.6 is satisfied with $B = \partial B_1$, or it is not. In the first case we can decompose h as $[\![h_L]\!] + [\![h_K]\!]$, where h_L and h_K are Lipschitz functions taking values in $\mathcal{A}_L$ and $\mathcal{A}_K$, and K and L are positive integers. By the induction hypothesis, we can find extensions of h_L and h_K satisfying the requirements of the lemma, and it is not difficult to verify that $f = [\![f_L]\!] + [\![f_K]\!]$ is the desired extension of h to $\overline{B_1}$.

In the second case, for any pair of indices i, j we have that

$$|h_i(x_0) - h_j(x_0)| \leq 6\,Q^*\,\mathrm{Lip}(h).$$

We use the following cone-like construction: set $P := h_1(x_0)$ and define

$$f(x) = \sum_i \left[\!\!\left[|x|\,h_i\left(\frac{x}{|x|}\right) + \big(1 - |x|\,\big)\,P \right]\!\!\right]. \tag{1.10}$$

Clearly f is an extension of h. For the Lipschitz regularity, note first that

$$\mathrm{Lip}(f|_{\partial B_r}) = \mathrm{Lip}(h), \text{ for every } 0 < r \leq 1.$$

Next, for any $x \in \partial B$, on the segment $\sigma_x = [0, x]$ we have

$$\mathrm{Lip} f|_{\sigma_x} \leq Q^* \max_i |h_i(x) - P| \leq 6\,(Q^*)^2\,\mathrm{Lip}(h).$$

So, we infer that $\mathrm{Lip}(f) \leq 12\,(Q^*)^2\,\mathrm{Lip}(h)$. Moreover, (1.9) follows easily from (1.10). □

1.2.2. Proof of Theorem 1.7. Without loss of generality, we can assume that B is closed. Consider a Whitney decomposition $\{C_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}^m \setminus B$ (see Figure 1). More precisely (cp. with Theorem 3, page 16 of [**Ste70**]):

(W1) each C_k is a closed dyadic cube, i.e. the length l_k of the side is 2^k for some $k \in \mathbb{Z}$ and the coordinates of the vertices are integer multiples of l_k;

(W2) distinct cubes have disjoint interiors and

$$c(m)^{-1}\mathrm{dist}(C_k, B) \le l_k \le c(m)\,\mathrm{dist}(C_k, B). \tag{1.11}$$

As usual, we call j-skeleton the union of the j-dimensional faces of C_k. We now construct the extension $\bar f$ by defining it recursively on the skeletons.

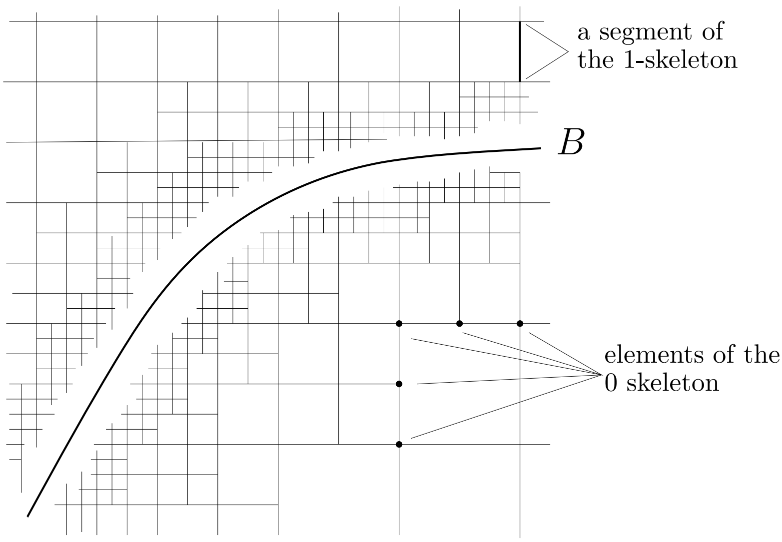

FIGURE 1. The Whitney decomposition of $\mathbb{R}^2 \setminus B$.

Consider the 0-skeleton, i.e. the set of the vertices of the cubes. For each vertex x, we choose $\tilde x \in B$ such that $|x - \tilde x| = \mathrm{dist}(x, B)$ and set $\bar f(x) = f(\tilde x)$. If x and y are two adjacent vertices of the same cube C_k, then

$$\max\big\{\,|x-\tilde x|\,,|y-\tilde y|\,\big\} \le \mathrm{dist}(C_k, B) \le c\,l_k = c\,|x-y|\,.$$

Hence, we have

$$\begin{aligned}\mathcal{G}\left(\bar f(x), \bar f(y)\right) = \mathcal{G}\left(f(\tilde x), f(\tilde y)\right) &\le \mathrm{Lip}(f)\,|\tilde x - \tilde y| \le \mathrm{Lip}(f)\big(|\tilde x - x| + |x-y| + |y-\tilde y|\big)\\ &\le c\,\mathrm{Lip}(f)\,|x-y|\,.\end{aligned}$$

Using the Homotopy Lemma 1.8, we extend f to $\bar f$ on each side of the 1-skeleton. On the boundary of any 2-face, $\bar f$ has Lipschitz constant smaller than $9\,C(m,Q)\,\mathrm{Lip}(f)$. Applying Lemma 1.8 recursively we find an extension of $\bar f$ to all $\mathbb{R}^m$ such that (1.8) holds and which is Lipschitz in each cube of the decomposition, with constant smaller than $C(m,Q)\,\mathrm{Lip}(f)$.

It remains to show that $\bar f$ is Lipschitz on the whole $\mathbb{R}^m$. Consider $x, y \in \mathbb{R}^m$, not lying in the same cube of the decomposition. Our aim is to show the inequality

$$\mathcal{G}\left(\bar f(x), \bar f(y)\right) \le C\,\mathrm{Lip}(f)\,|x-y|, \tag{1.12}$$

with some C depending only on m and Q. Without loss of generality, we can assume that $x \notin B$. We distinguish then two possibilities:

(a) $[x, y] \cap B \neq \emptyset$;
(b) $[x, y] \cap B = \emptyset$.

In order to deal with (a), assume first that $y \in B$. Let C_k be a cube of the decomposition containing x and let v be one of the nearest vertices of C_k to x. Recall, moreover, that $\bar{f}(v) = f(\tilde{v})$ for some $\tilde{v}$ with $|\tilde{v} - v| = \mathrm{dist}(v, B)$. We have then

$$
\begin{aligned}
\mathcal{G}\left(\bar{f}(x), \bar{f}(y)\right) &\leq \mathcal{G}\left(\bar{f}(x), \bar{f}(v)\right) + \mathcal{G}\left(\bar{f}(v), f(y)\right) = \mathcal{G}\left(\bar{f}(x), \bar{f}(v)\right) + \mathcal{G}\left(f(\tilde{v}), f(y)\right) \\
&\leq C\,\mathrm{Lip}(f)\,|x - v| + \mathrm{Lip}(f)\,|\tilde{v} - y| \\
&\leq C\,\mathrm{Lip}(f)\big(\,|x - v| + |\tilde{v} - v| + |v - x| + |x - y|\,\big) \\
&\leq C\,\mathrm{Lip}(f)\big(l_k + \mathrm{dist}(C_k, B) + \mathrm{diam}\,(C_k) + |x - y|\,\big) \\
&\stackrel{(1.11)}{\leq} C\,\mathrm{Lip}(f)\,|x - y|\,.
\end{aligned}
$$

If (a) holds but $y \notin B$, then let $z \in]a, b[\cap B$. From the previous argument we know $\mathcal{G}(\bar{f}(x), \bar{f}(z)) \leq C|x - z|$ and $\mathcal{G}\left(\bar{f}(y), \bar{f}(z)\right) \leq C|y - z|$, from which (1.12) follows easily.

If (b) holds, then $[x, y] = [x, P_1] \cup [P_1, P_2] \cup \ldots \cup [P_s, y]$ where each interval belongs to a cube of the decomposition. Therefore (1.12) follows trivially from the Lipschitz estimate for $\bar{f}$ in each cube of the decomposition.

1.3. Differentiability and Rademacher's Theorem

In this section we introduce the notion of differentiability for Q-valued functions and prove two related theorems. The first one gives chain-rule formulas for Q-valued functions and the second is the extension to the Q-valued setting of the classical result of Rademacher.

Definition 1.9. Let $f : \mathbb{R}^m \supset B \to \mathcal{A}_Q(\mathbb{R}^n)$ and $x_0 \in B$. We say that f is differentiable at x_0 if there exist Q matrices L_i satisfying:

(i) $\mathcal{G}(f(x), T_{x_0} f) = o(|x - x_0|)$, where

$$
T_{x_0} f(x) := \sum_i [\![L_i \cdot (x - x_0) + f_i(x_0)]\!]\,; \tag{1.13}
$$

(ii) $L_i = L_j$ if $f_i(x_0) = f_j(x_0)$.

The Q-valued map $T_{x_0} f$ will be called the *first-order approximation* of f at x_0. The point $\sum_i [\![L_i]\!] \in \mathcal{A}_Q(\mathbb{R}^{n\times m})$ will be called the differential of f at x_0 and is denoted by $Df(x_0)$.

Remark 1.10. What we call "differentiable" is called "strongly affine approximable" by Almgren.

Remark 1.11. The differential $Df(x_0)$ of a Q-function f does not determine univocally its first-order approximation $T_{x_0} f$. To overcome this ambiguity, we write Df_i for L_i in Definition 1.9, thus making evident which matrix has to be associated to $f_i(x_0)$ in (i). Note that (ii) implies that this notation is consistent: namely, if $g_1, \ldots, g_Q$ is a different selection for f, x_0 a point of differentiability and π a permutation such that $g_i(x_0) = f_{\pi(i)}(x_0)$ for all $i \in \{1, \ldots, Q\}$, then $Dg_i(x_0) = Df_{\pi(i)}(x_0)$. Even though the f_i's are not, in general, differentiable, observe that, when they are differentiable and f is differentiable, the Df_i's coincide with the classical differentials.

If D is the set of points of differentiability of f, the map $x \mapsto Df(x)$ is a measurable Q-valued map, which we denote by Df. In a similar fashion, we define the directional derivatives $\partial_\nu f(x) = \sum_i [\![Df_i(x) \cdot \nu]\!]$ and establish the notation $\partial_\nu f = \sum_i [\![\partial_\nu f_i]\!]$.

1.3.1. Chain rules. In what follows, we will deal with several natural operations defined on Q-valued functions. Consider a function $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$. For every $\Phi : \tilde{\Omega} \to \Omega$, the right composition $f \circ \Phi$ defines a Q-valued function on $\tilde{\Omega}$. On the other hand, given a map $\Psi : \Omega \times \mathbb{R}^n \to \mathbb{R}^k$, we can consider the left composition, $x \mapsto \sum_i [\![\Psi(x, f_i(x))]\!]$, which defines a Q-valued function denoted, with a slight abuse of notation, by $\Psi(x, f)$.

The third operation involves maps $F : (\mathbb{R}^n)^Q \to \mathbb{R}^k$ such that, for every Q points $(y_1, \ldots, y_Q) \in (\mathbb{R}^n)^Q$ and $\pi \in \mathscr{P}_Q$,

$$F(y_1, \ldots, y_Q) = F\left(y_{\pi(1)}, \ldots, y_{\pi(Q)}\right). \tag{1.14}$$

Then, $x \mapsto F(f_1(x), \ldots, f_Q(x))$ is a well defined map, denoted by $F \circ f$.

Proposition 1.12 (Chain rules). *Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be differentiable at x_0.*

(*i*) *Consider $\Phi : \tilde{\Omega} \to \Omega$ such that $\Phi(y_0) = x_0$ and assume that Φ is differentiable at y_0. Then, $f \circ \Phi$ is differentiable at y_0 and*

$$D(f \circ \Phi)(y_0) = \sum_i [\![Df_i(x_0) \cdot D\Phi(y_0)]\!]. \tag{1.15}$$

(*ii*) *Consider $\Psi : \Omega_x \times \mathbb{R}^n_u \to \mathbb{R}^k$ such that Ψ is differentiable at $(x_0, f_i(x_0))$ for every i. Then, $\Psi(x, f)$ fulfills (i) of Definition* 1.9. *Moreover, if (ii) holds, then*

$$D\Psi(x, f)(x_0) = \sum_i [\![D_u\Psi(x_0, f_i(x_0)) \cdot Df_i(x_0) + D_x\Psi(x_0, f_i(x_0))]\!]. \tag{1.16}$$

(*iii*) *Consider $F : (\mathbb{R}^n)^Q \to \mathbb{R}^k$ as in* (1.14) *and differentiable at $(f_1(x_0), \ldots, f_Q(x_0))$. Then, $F \circ f$ is differentiable at x_0 and*

$$D(F \circ f)(x_0) = \sum_i D_{y_i} F(f_1(x_0), \ldots, f_Q(x_0)) \cdot Df_i(x_0). \tag{1.17}$$

Proof. All the formulas are just routine modifications of the classical chain-rule. The proof of (i) follows easily from Definition 1.9. Since f is differentiable at x_0, we have

$$\begin{aligned}\mathcal{G}\left(f \circ \Phi(y), \sum_i [\![Df_i(x_0) \cdot (\Phi(y) - \Phi(y_0)) + f_i(\Phi(y_0))]\!]\right) &= o\left(|\Phi(y) - \Phi(y_0)|\right) \\ &= o\left(|y - y_0|\right),\end{aligned} \tag{1.18}$$

where the last equality follows from the differentiability of Φ at y_0. Moreover, again due to the differentiability of Φ, we infer that

$$Df_i(x_0) \cdot (\Phi(y) - \Phi(y_0)) = Df_i(x_0) \cdot D\Phi(y_0) \cdot (y - y_0) + o(|y - y_0|). \tag{1.19}$$

Therefore, (1.18) and (1.19) imply (1.15).

For what concerns (ii), we note that we can reduce to the case of $\mathrm{card}(f(x_0)) = 1$, i.e.

$$f(x_0) = Q\,[\![u_0]\!] \quad \text{and} \quad Df(x_0) = Q\,[\![L]\!]\,. \tag{1.20}$$

Indeed, since f is differentiable (hence, continuous) in x_0, in a neighborhood of x_0 we can decompose f as the sum of differentiable multi-valued functions g_k, $f = \sum_k [\![g_k]\!]$, such that $\mathrm{card}(g_k(x_0)) = 1$. Then, $\Psi(x, f) = \sum_k [\![\Psi(x, g_k)]\!]$ in a neighborhood of x_0, and the differentiability of $\Psi(x, f)$ follows from the differentiability of the $\Psi(x, g_k)$'s. So, assuming (1.20), without loss of generality, we have to show that

$$h(x) = Q\,[\![D_u\,\Psi(x_0, u_0)\cdot L\cdot(x-x_0) + D_x\,\Psi(x_0,u_0)\cdot(x-x_0) + \Psi(x_0,u_0)]\!]$$

is the first-order approximation of $\Psi(x, f)$ in x_0. Set

$$A_i(x) = D_u\,\Psi(x_0,u_0)\cdot(f_i(x)-u_0) + D_x\,\Psi(x_0,u_0)\cdot(x-x_0) + \Psi(x_0,u_0).$$

From the differentiability of Ψ, we deduce that

$$\mathcal{G}\left(\Psi(x,f), \sum_i [\![A_i(x)]\!]\right) = o\big(|x-x_0| + \mathcal{G}(f(x), f(x_0))\big) = o\,(|x-x_0|)\,, \tag{1.21}$$

where we used the differentiability of f in the last step. Hence, we can conclude (1.16), i.e.

$$\begin{aligned}
\mathcal{G}\,(\Psi(x,f), h(x)) &\le \mathcal{G}\left(\Psi(x,f), \sum_i [\![A_i(x)]\!]\right) + \mathcal{G}\left(\sum_i [\![A_i(x)]\!]\,, h(x)\right)\\
&\le o\,(|x-x_0|) + \|D_u\,\Psi(x_0,u_0)\|\mathcal{G}\left(\sum_i [\![f_i(x)]\!], Q\,[\![L\cdot(x-x_0)+u_0]\!]\right)\\
&= o\,(|x-x_0|)\,.
\end{aligned}$$

where $\|D_u\psi(x_0,u_0)\|$ denotes the Hilbert–Schmidt norm of the matrix $D_u\,\Psi(x_0,u_0)$.

Finally, to prove (iii), fix x and let π be such that

$$\mathcal{G}\big(f(x), f(x_0)\big)^2 = \sum_i |f_{\pi(i)}(x) - f_i(x_0)|^2.$$

By the continuity of f and (ii) of Definition 1.9, for $|x-x_0|$ small enough we have

$$\mathcal{G}\big(f(x), T_{x_0}f(x)\big)^2 = \sum_i |f_{\pi(i)}(x) - Df_i(x_0)\cdot(x-x_0) - z_i|^2. \tag{1.22}$$

Set $f_i(x_0) = z_i$ and $z = (z_1, \ldots, z_Q) \in (\mathbb{R}^n)^Q$. The differentiability of F implies

$$\begin{aligned}
\left|F\circ f(x) - F\circ f(x_0) - \sum_i D_{y_i}F(z)\cdot\big(f_{\pi(i)}(x) - z_i\big)\right| &= o\,(\mathcal{G}(f(x), f(x_0))\\
&= o(|x-x_0|).
\end{aligned} \tag{1.23}$$

Therefore, for $|x-x_0|$ small enough, we conclude

$$\begin{aligned}
\left|\sum_i D_{y_i}F(z)\cdot\big(f_{\pi(i)}(x) - z_i - Df_i(x_0)\cdot(x-x_0)\big)\right| \le\\
\le C\sum_i |f_{\pi(i)}(x) - Df_i(x_0)\cdot(x-x_0) - z_i| \overset{(1.22)}{=} o(|x-x_0|),
\end{aligned} \tag{1.24}$$

with $C = \sup_i \|D_{y_i}F(z)\|$. Therefore, using (1.23) and (1.24), we conclude (1.17). $\square$

1.3.2. Rademacher's Theorem. In this subsection we extend the classical theorem of Rademacher on the differentiability of Lipschitz functions to the Q-valued setting. Our proof is direct and elementary, whereas in Almgren's work the theorem is a corollary of the existence of the biLipschitz embedding $\boldsymbol{\xi}$ (see Section 2.1). An intrinsic proof has been already proposed in [**Gob06b**]. However our approach is considerably simpler.

THEOREM 1.13 (Rademacher). *Let $f : \Omega \to \mathcal{A}_Q$ be a Lipschitz function. Then, f is differentiable almost everywhere in Ω.*

PROOF. We proceed by induction on the number of values Q. The case $Q = 1$ is the classical Rademacher's theorem (see, for instance, 3.1.2 of [**EG92**]). We next assume that the theorem is true for every $Q < Q^*$ and we show its validity for Q^*.

We write $f = \sum_{i=1}^{Q^*} [\![f_i]\!]$, where the f_i's are a measurable selection. We let $\tilde{\Omega}$ be the set of points where f takes a single value with multiplicity Q:

$$\tilde{\Omega} = \big\{x \in \Omega \,:\, f_1(x) = f_i(x)\ \forall i\big\}.$$

Note that $\tilde{\Omega}$ is closed. In $\Omega \setminus \tilde{\Omega}$, f is differentiable almost everywhere by inductive hypothesis. Indeed, by Proposition 1.6, in a neighborhood of any point $x \in \Omega \setminus \tilde{\Omega}$, we can decompose f in the sum of two Lipschitz simpler multi-valued functions, $f = [\![f_L]\!] + [\![f_K]\!]$, with the property that $\mathrm{supp}\,(f_L(x)) \cap \mathrm{supp}\,(f_K(x)) = \emptyset$. By inductive hypothesis, f_L and f_K are differentiable, hence, also f is.

It remains to prove that f is differentiable a.e. in $\tilde{\Omega}$. Note that $f_1|_{\tilde{\Omega}}$ is a Lipschitz vector valued function and consider a Lipschitz extension of it to all Ω, denoted by g. We claim that f is differentiable in all the points x where

(a) $\tilde{\Omega}$ has density 1;
(b) g is differentiable.

Our claim would conclude the proof. In order to show it, let $x_0 \in \tilde{\Omega}$ be any given point fulfilling (a) and (b) and let $T_{x_0}g(y) = L \cdot (y - x_0) + f_1(x_0)$ be the first order Taylor expansion of g at x_0, that is

$$|g(y) - L \cdot (y - x_0) - f_1(x_0)| = o(|y - x_0|). \tag{1.25}$$

We will show that $T_{x_0}f(y) := Q\,[\![L \cdot (y - x_0) + f_1(x_0)]\!]$ is the first order expansion of f at x_0. Indeed, for every $y \in \mathbb{R}^m$, let $r = |y - x_0|$ and choose $y^* \in \tilde{\Omega} \cap \overline{B_{2r}(x_0)}$ such that

$$|y - y^*| = \mathrm{dist}\left(y, \tilde{\Omega} \cap \overline{B_{2r}(x_0)}\right).$$

Being f, g and Tg Lipschitz with constant at most $\mathrm{Lip}(f)$, using (1.25), we infer that

$$\begin{aligned}
\mathcal{G}\big(f(y), T_{x_0}f(y)\big) &\leq \mathcal{G}\big(f(y), f(y^*)\big) + \mathcal{G}\big(T_{x_0}f(y^*), T_{x_0}f(y)\big) + \mathcal{G}\big(f(y^*), T_{x_0}f(y^*)\big)\\
&\leq \mathrm{Lip}(f)\,|y - y^*| + Q\,\mathrm{Lip}(f)\,|y - y^*| +\\
&\quad + \mathcal{G}\big(Q\,[\![g(y^*)]\!]\,, Q\,[\![L \cdot (y^* - x_0) + f_1(x_0)]\!]\,\big)\\
&\leq (Q+1)\,\mathrm{Lip}(f)\,|y - y^*| + o\big(\,|y^* - x_0|\,\big).
\end{aligned} \tag{1.26}$$

Since $|y^* - x_0| \leq 2\,r = 2\,|y - x_0|$, it remains to estimate $\rho := |y - y^*|$. Note that the ball $B_\rho(y)$ is contained in $B_r(x_0)$ and does not intersect $\tilde{\Omega}$. Therefore

$$|y - y^*| = \rho \leq C\,\left|B_{2r}(x_0) \setminus \tilde{\Omega}\right|^{1/m} \leq C(m)\,r\left(\frac{|B_{2r}(x_0) \setminus \tilde{\Omega}|}{|B_{2r}(x_0)|}\right)^{\frac{1}{m}}. \tag{1.27}$$

Since x_0 is a point of density 1, we can conclude from (1.27) that $|y - y^*| = |y - x_0|\,o(1)$. Inserting this inequality in (1.26), we conclude that $\mathcal{G}(f(y), T_{x_0}f(y)) = o(|y - x_0|)$, which shows that $T_{x_0}f$ is the first order expansion of f at x_0. $\square$

CHAPTER 2

Almgren's extrinsic theory

Two "extrinsic maps" play a pivotal role in the theory of Q-functions developed in [**Alm00**]. The first one is a biLipschitz embedding $\boldsymbol{\xi}$ of $\mathcal{A}_Q(\mathbb{R}^n)$ into $\mathbb{R}^{N(Q,n)}$, where $N(Q,n)$ is a sufficiently large integer. Almgren uses this map to define Sobolev Q-functions as classical $\mathbb{R}^N$-valued Sobolev maps taking values in $\mathcal{Q} := \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$. Using $\boldsymbol{\xi}$, many standard facts of Sobolev maps can be extended to the Q-valued setting with little effort. The second map $\boldsymbol{\rho}$ is a Lipschitz retraction of $\mathbb{R}^{N(Q,n)}$ onto $\mathcal{Q}$, which is used in various approximation arguments.

The existence of the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ is proved in Section 2.1. In Section 2.2 we show that Sobolev Q-valued functions in the sense of Almgren coincide with those of Definition 0.5 and we use $\boldsymbol{\xi}$ to derive their basic properties. Finally, Section 2.3 shows that our definition of Dirichlet's energy coincides with Almgren's one and proves the Existence Theorem 0.8. Except for Section 2.2, no other portion of this paper makes direct use of $\boldsymbol{\xi}$ or of $\boldsymbol{\rho}$: the regularity theory of Chapters 3 and 5 needs only the propositions stated in Section 2.2, which we are going to prove again in Chapter 4 within the frame of an "intrinsic" approach, that is independent of $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$.

2.1. The biLipschitz embedding $\boldsymbol{\xi}$ and the retraction $\boldsymbol{\rho}$

Theorem 2.1. *There exist $N = N(Q,n)$ and an injective map $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ such that:*

(i) *$Lip(\boldsymbol{\xi}) \leq 1$;*
(ii) *if $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q)$, then $Lip(\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}) \leq C(n,Q)$.*

Moreover, there exists a Lipschitz map $\boldsymbol{\rho} : \mathbb{R}^N \to \mathcal{Q}$ which is the identity on $\mathcal{Q}$.

The existence of $\boldsymbol{\rho}$ is a trivial consequence of the Lipschitz regularity of $\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}$ and of the Extension Theorem 1.7.

Proof of the existence of $\boldsymbol{\rho}$ given $\boldsymbol{\xi}$. Consider $\boldsymbol{\xi}^{-1} : \mathcal{Q} \to \mathcal{A}_Q$. Since this map is Lipschitz, by Theorem 1.7 there exists a Lipschitz extension f of $\boldsymbol{\xi}^{-1}$ to the entire space. Therefore, $\boldsymbol{\rho} = \boldsymbol{\xi} \circ f$ is the desired retraction. $\square$

For the proof of the first part of Theorem 2.1, we follow instead the ideas of Almgren. A slight modification of these ideas, moreover, leads to the construction of a special biLipschitz embedding: this observation, due to B. White, was noticed in [**Cha88**].

Corollary 2.2. *There exist $M = M(Q,n)$ and an injective map $\boldsymbol{\xi}_{BW} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^M$ with the following properties: $\boldsymbol{\xi}_{BW}$ satisfies (i) and (ii) of Theorem 2.1 and, for every $T \in \mathcal{A}_Q(\mathbb{R}^n)$, there exists $\delta > 0$ such that*

$$|\boldsymbol{\xi}_{BW}(T) - \boldsymbol{\xi}_{BW}(S)| = \mathcal{G}(T,S) \quad \forall\, S \in B_\delta(T) \subset \mathcal{A}_Q(\mathbb{R}^n). \tag{2.1}$$

We point out that we will not make any use in the following of such special embedding $\boldsymbol{\xi}_{BW}$, since all the properties of Q-valued functions are independent of the embedding we choose. Nevertheless, we give a proof of Corollary 2.2 because it provides a better intuition on Q-valued functions (see Proposition 2.20) and can be used to give shorter proofs of several technical lemmas (see [**DLS**]).

2.1.1. A combinatorial Lemma. The key to the proof of Theorem 2.1 is the following combinatorial statement.

LEMMA 2.3 (Almgren's combinatorial Lemma). *There exist $\alpha = \alpha(Q,n) > 0$ and a set of $h = h(Q,n)$ unit vectors $\Lambda = \{e_1, \ldots e_h\} \subset \mathbb{S}^{n-1}$ with the following property: given any set of Q^2 vectors, $\{v_1, \ldots, v_{Q^2}\} \subset \mathbb{R}^n$, there exists $e_l \in \Lambda$ such that*

$$|v_k \cdot e_l| \geq \alpha\, |v_k| \quad \text{for all } k \in \{1, \ldots, Q^2\}. \tag{2.2}$$

PROOF. Choose a unit vector e_1 and let $\alpha(Q,n)$ be small enough in order to ensure that the set $E := \{x \in \mathbb{S}^{n-1} : |x \cdot e_1| < \alpha\}$ has sufficiently small measure, that is

$$\mathcal{H}^{n-1}(E) \leq \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{8 \cdot 5^{n-1}\, Q^2}. \tag{2.3}$$

Note that E is just the α-neighborhood of an equatorial $(n-2)$-sphere of $\mathbb{S}^{n-1}$. Next, we use Vitali's covering Lemma (see 1.5.1 of [**EG92**]) to find a finite set $\Lambda = \{e_1, \ldots, e_h\} \subset \mathbb{S}^{n-1}$ and a finite number of radii $0 < r_i < \alpha$ such that

(a) the balls $B_{r_i}(e_i)$ are disjoint;
(b) the balls $B_{5\, r_i}(e_i)$ cover the whole sphere.

We claim that Λ satisfies the requirements of the lemma. Let, indeed, $V = \{v_1, \ldots, v_{Q^2}\}$ be a set of vectors. We want to show the existence of $e_l \in \Lambda$ which satisfies (2.2). Without loss of generality, we assume that each v_i is nonzero. Moreover, we consider the sets $C_k = \{x \in \mathbb{S}^{n-1} : |x \cdot v_k| < \alpha\, |v_k|\}$ and we let C_V be the union of the C_k's. Each C_k is the α-neighborhood of the equatorial sphere given by the intersection of $\mathbb{S}^{n-1}$ with the hyperplane orthogonal to v_i. Thus, by (2.3),

$$\mathcal{H}^{n-1}\left(C_V\right) \leq \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{8 \cdot 5^{n-1}}. \tag{2.4}$$

Note that, due to the bound $r_i < \alpha$,

$$e_i \in C_V \quad \Rightarrow \quad \mathcal{H}^{n-1}\left(C_V \cap B_{r_i}(e_i)\right) \geq \frac{\mathcal{H}^{n-1}(B_{r_i}(e_i) \cap \mathbb{S}^{n-1})}{2}. \tag{2.5}$$

By our choices, there must be one e_l which does not belong to C_V, otherwise

$$\begin{aligned}
\frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{2 \cdot 5^{n-1}} &\overset{(a)\,\&\,(b)}{\leq} \sum_i \mathcal{H}^{n-1}\left(B_{r_i}(e_i) \cap \mathbb{S}^{n-1}\right) \overset{(2.5)}{\leq} 2 \sum_i \mathcal{H}^{n-1}\left(C_V \cap B_{r_i}(e_i)\right) \\
&\overset{(a)}{\leq} \quad 2\,\mathcal{H}^{n-1}\left(C_V\right) \overset{(2.4)}{\leq} \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{4 \cdot 5^{n-1}},
\end{aligned}$$

which is a contradiction (here we used the fact that, though the sphere is curved, for α sufficiently small the $(n-1)$-volume of $B_{r_i}(e_i) \cap \mathbb{S}^{n-1}$ is at least $2^{-1}5^{-n+1}$ times the volume of $B_{5\, r_i}(e_i) \cap \mathbb{S}^{n-1}$). Having chosen $e_l \notin C_V$, we have $e_l \notin C_k$ for every k, which in turn implies (2.2). $\square$

2.1.2. Proof of the existence of $\boldsymbol{\xi}$. Let $\Lambda = \{e_1, \dots e_h\}$ be a set satisfying the conclusion of Lemma 2.3 and set $N = Q\,h$. Fix $T \in \mathcal{A}_Q(\mathbb{R}^n)$, $T = \sum_i [\![P_i]\!]$. For any $e_l \in \Lambda$, we consider the Q projections of the points P_i on the e_l direction, that is $P_i \cdot e_l$. This gives an array of Q numbers, which we rearrange in increasing order, getting a Q-dimensional vector $\pi_l(T)$. The map $\boldsymbol{\xi} : \mathcal{A}_Q \to \mathbb{R}^N$ is, then, defined by $\boldsymbol{\xi}(T) = h^{-1/2}(\pi_1(T), \dots, \pi_h(T))$.

The Lipschitz regularity of $\boldsymbol{\xi}$ is a trivial corollary of the following rearrangement inequality:

(Re) if $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$, then, for every permutation σ of the indices,

$$(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2 \leq (a_1 - b_{\sigma(1)})^2 + \cdots + (a_n - b_{\sigma(n)})^2.$$

Indeed, fix two points $T = \sum_i [\![P_i]\!]$ and $S = \sum_i [\![R_i]\!]$ and assume, without loss of generality, that

$$\mathcal{G}(T,S)^2 = \sum_i |P_i - R_i|^2 . \tag{2.6}$$

Fix an l. Then, by (Re), $|\pi_l(T) - \pi_l(S)|^2 \leq \sum((P_i - R_i) \cdot \mathrm{e}_l)^2$. Hence, we get

$$\begin{aligned} |\boldsymbol{\xi}(T) - \boldsymbol{\xi}(S)|^2 &\leq \frac{1}{h}\sum_{l=1}^{h}\sum_{i=1}^{Q}((P_i - R_i)\cdot \mathrm{e}_l)^2 \leq \frac{1}{h}\sum_{l=1}^{h}\sum_{i=1}^{Q}|P_i - R_i|^2 \\ &\stackrel{(2.6)}{=} \frac{1}{h}\sum_{l=1}^{h}\mathcal{G}(T,S)^2 = \mathcal{G}(T,S)^2. \end{aligned}$$

Next, we conclude the proof by showing, for $T = \sum_i [\![P_i]\!]$ and $S = \sum_i [\![R_i]\!]$, the inequality

$$\mathcal{G}(T,S) \leq \frac{\sqrt{h}}{\alpha}\,|\boldsymbol{\xi}(T) - \boldsymbol{\xi}(S)|\,, \tag{2.7}$$

where α is the constant in Lemma 2.3. Consider, indeed, the Q^2 vectors $P_i - R_j$, for $i, j \in \{1, \dots, Q\}$. By Lemma 2.3, we can select a unit vector $e_l \in \Lambda$ such that

$$|(P_i - R_j)\cdot e_l| \geq \alpha\,|P_i - R_j|\,, \quad \text{for all } i, j \in \{1, \dots, Q\}. \tag{2.8}$$

Let τ and λ be permutations such that

$$\pi_l(T) = (P_{\tau(1)} \cdot e_l, \dots, P_{\tau(Q)} \cdot e_l) \quad \text{and} \quad \pi_l(S) = (R_{\lambda(1)} \cdot e_l, \dots, R_{\lambda(Q)} \cdot e_l).$$

Then, we conclude (2.7),

$$\begin{aligned} \mathcal{G}(T,S)^2 &\leq \sum_{i=1}^{Q}\left|P_{\tau(i)} - R_{\lambda(i)}\right|^2 \stackrel{(2.8)}{\leq} \alpha^{-2}\sum_{i=1}^{Q}\left((P_{\tau(i)} - R_{\lambda(i)})\cdot e_l\right)^2 \\ &= \alpha^{-2}\,|\pi_l(T) - \pi_l(S)|^2 \\ &\leq \alpha^{-2}\,h\,|\boldsymbol{\xi}(T) - \boldsymbol{\xi}(S)|^2 . \end{aligned}$$

2.1.3. Proof of Corollary 2.2. Let $\Lambda = \{e_1, \ldots e_h\}$ be the set of unit vectors in the proof of Theorem 2.1. We consider the enlarged set Γ of $n\,h$ vectors containing an orthonormal frame for each $e_l \in \Lambda$,

$$\Gamma = \left\{e_1^1, \ldots, e_1^n, \ldots, e_h^1, \ldots, e_h^n\right\},$$

where, for every $\alpha \in \{1, \ldots, h\}$, $e_\alpha^1 = e_\alpha$ and $\{e_\alpha^1, \ldots, e_\alpha^n\}$ is an orthonormal basis of $\mathbb{R}^n$. Note that, in principle, the vectors e_α^β may not be all distinct: this can happen, for example, if there exist two vectors e_j and e_l which are orthogonal. Nevertheless, we can assume, without loss of generality, that Γ is made of $n\,h$ distinct vectors (in passing, this is can always be reached by perturbing slightly Λ).

Then, we define the map $\boldsymbol{\xi}_{BW}$ in the same way as $\boldsymbol{\xi}$, with Γ replacing Λ: for $T = \sum_i [\![P_i]\!]$,

$$\boldsymbol{\xi}_{BW}(T) = h^{-1/2}\big(\pi_1^1(T), \ldots, \pi_1^n(T), \ldots, \pi_h^1(T) \ldots, \pi_h^n(T)\big),$$

where $\pi_\alpha^\beta(T)$ is the array of Q scalar products $P_i \cdot e_\alpha^\beta$ rearranged in increasing order.

Clearly, $\boldsymbol{\xi}_{BW}$ satisfies the conclusion of Theorem 2.1. We need only to show (2.1).

To this aim, we start noticing that, given $T = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$, there exists $\delta > 0$ with the following property: for every $S = \sum_i [\![R_i]\!] \in B_\delta(T)$ and every π_α^β, assuming that $\mathcal{G}(T,S)^2 = \sum_i |P_i - R_i|^2$, there exists a permutation $\sigma_\alpha^\beta \in \mathscr{P}_Q$ such that the arrays $(P_i \cdot e_\alpha^\beta)$ and $(R_i \cdot e_\alpha^\beta)$ are ordered increasingly by the same permutation σ_α^β, i.e.

$$\pi_\alpha^\beta(T) = \Big(P_{\sigma_\alpha^\beta(1)} \cdot e_\alpha^\beta, \ldots, P_{\sigma_\alpha^\beta(Q)} \cdot e_\alpha^\beta\Big) \quad \text{and} \quad \pi_\alpha^\beta(S) = \Big(R_{\sigma_\alpha^\beta(1)} \cdot e_\alpha^\beta, \ldots, R_{\sigma_\alpha^\beta(Q)} \cdot e_\alpha^\beta\Big).$$

It is enough to choose $4\,\delta = \min_{\alpha,\beta}\left\{|P_i \cdot e_\alpha^\beta - P_j \cdot e_\alpha^\beta| \,:\, P_i \cdot e_\alpha^\beta \neq P_j \cdot e_\alpha^\beta\right\}$. Indeed, let us assume that $R_i \cdot e_\alpha^\beta \leq R_j \cdot e_\alpha^\beta$. Then, two cases occur:

(a) $R_j \cdot e_\alpha^\beta - R_i \cdot e_\alpha^\beta \geq 2\delta$,

(b) $R_j \cdot e_\alpha^\beta - R_i \cdot e_\alpha^\beta < 2\delta$.

In case (a), since $S \in B_\delta(T)$, we deduce that $P_i \cdot e_\alpha^\beta \leq R_i \cdot e_\alpha^\beta + \delta \leq R_j \cdot e_\alpha^\beta - \delta \leq P_j \cdot e_\alpha^\beta$. In case (b), instead, we infer that $|P_j \cdot e_\alpha^\beta - P_i \cdot e_\alpha^\beta| \leq R_j \cdot e_\alpha^\beta + \delta - R_i \cdot e_\alpha^\beta - \delta < 4\,\delta$, which, in turn, by the choice of δ, leads to $P_j \cdot e_\alpha^\beta = P_i \cdot e_\alpha^\beta$. Hence, in both cases we have $P_i \cdot e_\alpha^\beta \leq P_j \cdot e_\alpha^\beta$, which means that $P_i \cdot e_\alpha^\beta$ can be ordered in increasing way by the same permutation σ_α^β.

Therefore, exploiting the fact that the vectors $\pi_\alpha^\beta(T)$ and $\pi_\alpha^\beta(S)$ are ordered by the same permutation σ_α^β, we have that, for T and S as above, it holds

$$\begin{aligned}
|\boldsymbol{\xi}_{BW}(T) - \boldsymbol{\xi}_{BW}(S)|^2 &= h^{-1} \sum_{\alpha=1}^h \sum_{\beta=1}^n |\pi_\alpha^\beta(T) - \pi_\alpha^\beta(S)|^2 \\
&= h^{-1} \sum_{\alpha=1}^h \sum_{\beta=1}^n \sum_{i=1}^Q |P_{\sigma_\alpha^\beta(i)} \cdot e_\alpha^\beta - R_{\sigma_\alpha^\beta(i)} \cdot e_\alpha^\beta|^2 \\
&= h^{-1} \sum_{\alpha=1}^h \sum_{i=1}^Q |P_i - R_i|^2 \\
&= h^{-1} \sum_{\alpha=1}^h \mathcal{G}(T,S)^2 = \mathcal{G}(T,S)^2.
\end{aligned}$$

This concludes the proof of the corollary.

2.2. Properties of Q-valued Sobolev functions

In this section we prove some of the basic properties of Sobolev Q-functions which will be used in the proofs of the regularity theorems. It is clear that, using $\boldsymbol{\xi}$, one can identify measurable, Lipschitz and Hölder Q-valued functions f with the corresponding maps $\boldsymbol{\xi} \circ f$ into $\mathbb{R}^N$, which are, respectively, measurable, Lipschitz, Hölder functions taking values in $\mathcal{Q}$ a.e. We now show that the same holds for the Sobolev classes of Definition 0.5.

THEOREM 2.4. *Let $\boldsymbol{\xi}$ be the map of Theorem 2.1. Then, a Q-valued function f belongs to the Sobolev space $W^{1,p}(\Omega, \mathcal{A}_Q)$ according to Definition 0.5 if and only if $\boldsymbol{\xi} \circ f$ belongs to $W^{1,p}(\Omega, \mathbb{R}^N)$. Moreover, there exists a constant $C = C(n, Q)$ such that*

$$|D(\boldsymbol{\xi} \circ f)| \le |Df| \le C\, |D(\boldsymbol{\xi} \circ f)|.$$

PROOF. Let f be a Q-valued function such that $g = \boldsymbol{\xi} \circ f \in W^{1,p}(\Omega, \mathbb{R}^N)$. Note that the map $\Upsilon_T : \mathcal{Q} \ni y \mapsto \mathcal{G}(\boldsymbol{\xi}^{-1}(y), T)$ is Lipschitz, with a Lipschitz constant C that can be bounded independently of $T \in \mathcal{A}_Q$. Therefore, $\mathcal{G}(f, T) = \Upsilon_T \circ g$ is a Sobolev function and $|\partial_j\,(\Upsilon_T \circ g)| \le C|\partial_j g|$ for every $T \in \mathcal{A}_Q$. So, f fulfills the requirements (i) and (ii) of Definition 0.5, with $\varphi_j = C\,|\partial_j g|$, from which, in particular, $|Df| \le C\,|D(\boldsymbol{\xi} \circ f)|$.

Vice versa, assume that f is in $W^{1,p}(\Omega, \mathcal{A}_Q)$ and let φ_j be as in Definition 0.5. Choose a countable dense subset $\{T_i\}_{i \in N}$ of $\mathcal{A}_Q$, and recall that any Lipschitz real-valued function Φ on $\mathcal{A}_Q$ can be written as

$$\Phi(\cdot) = \sup_{i \in \mathbb{N}} \big\{\Phi(T_i) - \mathrm{Lip}\,(\Phi)\ \mathcal{G}(\cdot, T_i)\big\}.$$

This implies that $\partial_j\,(\Phi \circ f) \in L^p$ with $|\partial_j\,(\Phi \circ f)| \le \mathrm{Lip}(\Phi)\,\varphi_j$. Therefore, since Ω is bounded, $\Phi \circ f \in W^{1,p}(\Omega)$. Being $\boldsymbol{\xi}$ a Lipschitz map with $\mathrm{Lip}(\boldsymbol{\xi}) \le 1$, we conclude that $\boldsymbol{\xi} \circ f \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $|D(\boldsymbol{\xi} \circ f)| \le |Df|$. $\square$

We now use the theorem above to transfer in a straightforward way several classical properties of Sobolev spaces to the framework of Q-valued mappings. In particular, in the subsequent subsections we deal with Lusin type approximations, trace theorems, Sobolev and Poincaré inequalities, and Campanato–Morrey estimates. Finally Subsection 2.2.5 contains a useful technical lemma estimating the energy of interpolating functions on spherical shells.

2.2.1. Lipschitz approximation and approximate differentiability. We start with the Lipschitz approximation property for Q-valued Sobolev functions.

PROPOSITION 2.5 (Lipschitz approximation). *Let f be in $W^{1,p}(\Omega, \mathcal{A}_Q)$. For every $\lambda > 0$, there exists a Lipschitz Q-function f_λ such that* $\mathrm{Lip}\,(f_\lambda) \le \lambda$ *and*

$$\big|\big\{x \in \Omega\,:\, f(x) \neq f_\lambda(x)\big\}\big| \le \frac{C}{\lambda^p} \int_\Omega \big(|Df|^p + \mathcal{G}(f, Q\,[\![0]\!])^p\big), \tag{2.9}$$

where the constant C depends only on Q, m and Ω.

PROOF. Consider $\boldsymbol{\xi} \circ f$: by the Lusin-type approximation theorem for classical Sobolev functions (see, for instance, [**AF88**] or 6.6.3 of [**EG92**]), there exists a Lipschitz function $h_\lambda : \Omega \to \mathbb{R}^N$ such that $|\{x \in \Omega\,:\, \boldsymbol{\xi} \circ f(x) \neq h_\lambda(x)\}| \le$

$(C/\lambda^p)\,\|\boldsymbol{\xi}\circ f\|^p_{W^{1,p}}$. Clearly, the function $f_\lambda = \boldsymbol{\xi}^{-1}\circ\boldsymbol{\rho}\circ h_\lambda$ has the desired property. □

A direct corollary of the Lipschitz approximation and of Theorem 1.13 is that any Sobolev Q-valued map is approximately differentiable almost everywhere.

Definition 2.6 (Approximate Differentiability). A Q-valued function f is approximately differentiable in x_0 if there exists a measurable subset $\tilde\Omega\subset\Omega$ containing x_0 such that $\tilde\Omega$ has density 1 at x_0 and $f|_{\tilde\Omega}$ is differentiable at x_0.

Corollary 2.7. *Any $f\in W^{1,p}(\Omega,\mathcal{A}_Q)$ is approximately differentiable a.e.*

The approximate differential of f at x_0 can then be defined as $D(f|_{\tilde\Omega})$ because it is independent of the set $\tilde\Omega$. With a slight abuse of notation, we will denote it by Df, as the classical differential. Similarly, we can define the approximate directional derivatives. Moreover, for these quantities we use the notation of Section 1.3, that is

$$Df=\sum_i [\![Df_i]\!] \quad\text{and}\quad \partial_\nu f=\sum_i [\![\partial_\nu f_i]\!]\,,$$

with the same convention as in Remark 1.11, i.e. the first-order approximation is given by $T_{x_0}f=\sum_i [\![f_i(x_0)+Df_i(x_0)\cdot(x-x_0)]\!]$.

Proof of Corollary 2.7. For every $k\in\mathbb{N}$, choose a Lipschitz function f_k such that $\Omega\setminus\Omega_k := \{f\neq f_k\}$ has measure smaller than k^{-p}. By Rademacher's Theorem 1.13, f_k is differentiable a.e. on Ω. Thus, f is approximately differentiable at a.e. point of Ω_k. Since $|\Omega\setminus\cup_k\Omega_k|=0$, this completes the proof. □

Finally, observe that the chain-rule formulas of Proposition 1.12 have an obvious extension to approximate differentiable functions.

Proposition 2.8. *Let $f:\Omega\to\mathcal{A}_Q(\mathbb{R}^n)$ be approximate differentiable at x_0. If Ψ and F are as in Proposition* 1.12, *then* (1.16) *and* (1.17) *holds. Moreover,* (1.15) *holds when Φ is a diffeomorphism.*

Proof. The proof follows trivially from Proposition 1.12 and Definition 2.6. □

2.2.2. Trace properties. Next, we show that the trace of a Sobolev Q-function as defined in Definition 0.7 corresponds to the classical trace for $\boldsymbol{\xi}\circ f$. First we introduce the definition of weak convergence for Q-valued functions.

Definition 2.9 (Weak convergence). Let $f_k, f\in W^{1,p}(\Omega,\mathcal{A}_Q)$. We say that f_k converges weakly to f for $k\to\infty$, (and we write $f_k\rightharpoonup f$) in $W^{1,p}(\Omega,\mathcal{A}_Q)$, if

(i) $\int\mathcal{G}(f_k,f)^p\to 0$, for $k\to\infty$;
(ii) there exists a constant C such that $\int|Df_k|^p\le C<\infty$ for every k.

Proposition 2.10 (Trace of Sobolev Q-functions). *Let $f\in W^{1,p}(\Omega,\mathcal{A}_Q)$. Then, there is a unique function $g\in L^p(\partial\Omega,\mathcal{A}_Q)$ such that $f|_{\partial\Omega}=g$ in the sense of Definition* 0.7. *Moreover, $f|_{\partial\Omega}=g$ if and only if $\boldsymbol{\xi}\circ f|_{\partial\Omega}=\boldsymbol{\xi}\circ g$ in the usual sense, and the set of mappings*

$$W^{1,2}_g(\Omega,\mathcal{A}_Q):=\left\{f\in W^{1,2}(\Omega,\mathcal{A}_Q)\,:\,f|_{\partial\Omega}=g\right\} \tag{2.10}$$

is sequentially weakly closed in $W^{1,p}$.

PROOF. For what concerns the existence, let $g = \boldsymbol{\xi}^{-1}(\boldsymbol{\xi}\circ f|_{\partial\Omega})$. Since $\boldsymbol{\xi}\circ f|_{\partial\Omega} = \boldsymbol{\xi}\circ g$, for every Lipschitz real-valued map Φ on $\mathcal{Q}$, we clearly have $\Phi\circ\boldsymbol{\xi}\circ f|_{\partial\Omega} = \Phi\circ\boldsymbol{\xi}\circ g$. Since $\Upsilon_T(\cdot) := \mathcal{G}(\xi^{-1}(\cdot), T)$ is a Lipschitz map on $\mathcal{Q}$ for every $T\in\mathcal{A}_Q$, we conclude that $f|_{\partial\Omega} = g$ in the sense of Definition 0.7.

The uniqueness is an easy consequence of the following observation: if h and g are maps in $L^p(\partial\Omega, \mathcal{A}_Q)$ such that $\mathcal{G}(h(x), T) = \mathcal{G}(g(x), T)$ for $\mathcal{H}^{n-1}$-a.e. x and for every $T\in\mathcal{A}_Q$, then $h = g$. Indeed, fixed a countable dense subset $\{T_i\}_{i\in\mathbb{N}}$ of $\mathcal{A}_Q$, we have

$$\mathcal{G}\big(h(x), g(x)\big) = \sup_i \big|\mathcal{G}(h(x), T_i) - \mathcal{G}(g(x), T_i)\big| = 0 \quad \mathcal{H}^{n-1}\text{-a.e.}$$

The last statement of the proposition follows easily and the proof is left to the reader. □

2.2.3. Sobolev and Poincaré inequalities. As usual, for $p < m$ we set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{m}$.

PROPOSITION 2.11 (Sobolev Embeddings). *The following embeddings hold:*

(*i*) *if $p < m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$ for every $q\in[1, p^*]$, and the inclusion is compact when $q < p^*$;*

(*ii*) *if $p = m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$, for every $q\in[1, +\infty)$, with compact inclusion;*

(*iii*) *if $p > m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega, \mathcal{A}_Q)$, for $\alpha = 1 - \frac{m}{p}$, with compact inclusion if $\alpha < 1 - \frac{m}{p}$.*

PROOF. Since f is a L^q (resp. Hölder) Q-function if and only if $\boldsymbol{\xi}\circ f$ is L^q (resp. Hölder), the proposition follows trivially from Theorem 2.4 and the Sobolev embeddings for $\boldsymbol{\xi}\circ f$ (see, for example, [**Ada75**] or [**Zie89**]). □

PROPOSITION 2.12 (Poincaré inequality). *Let M be a connected bounded Lipschitz open set of an m-dimensional Riemannian manifold and let $p < m$. There exists a constant $C = C(p, m, n, Q, M)$ with the following property: for every $f\in W^{1,p}(M, \mathcal{A}_Q)$, there exists a point $\overline{f}\in\mathcal{A}_Q$ such that*

$$\left(\int_M \mathcal{G}(f, \overline{f})^{p^*}\right)^{\frac{1}{p^*}} \leq C\left(\int_M |Df|^p\right)^{\frac{1}{p}}. \tag{2.11}$$

REMARK 2.13. Note that the point $\overline{f}$ in the Poincaré inequality is not uniquely determined. Nevertheless, in analogy with the classical setting, we call it a *mean* for f.

PROOF. Set $h := \boldsymbol{\xi}\circ f : M \to \mathcal{Q}\subset\mathbb{R}^N$. By Theorem 2.4, $h\in W^{1,p}(M, \mathbb{R}^N)$. Recalling the classical Poincaré inequality (see, for instance, [**Ada75**] or [**Zie89**]), there exists a constant $C = C(p, m, M)$ such that, if $\overline{h} = ⨍_M h$, then

$$\left(\int_M \left|h(x) - \overline{h}\right|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C\left(\int_M |Dh|^p\right)^{\frac{1}{p}}. \tag{2.12}$$

Let now $v\in\mathcal{Q}$ be such that $\left|\overline{h} - v\right| = \operatorname{dist}\left(\overline{h}, \mathcal{Q}\right)$ (v exists because $\mathcal{Q}$ is closed). Then, since h takes values in $\mathcal{Q}$ almost everywhere, by (2.12) we infer

$$\left(\int_M \left|\overline{h} - v\right|^{p^*} dx\right)^{\frac{1}{p^*}} \leq \left(\int_M \left|\overline{h} - h(x)\right|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C\left(\int_M |Dh|^p\right)^{\frac{1}{p}}. \tag{2.13}$$

Therefore, using (2.12) and (2.13), we end up with

$$\|h - v\|_{L^{p^*}} \leq \left\|h - \overline{h}\right\|_{L^{p^*}} + \left\|\overline{h} - v\right\|_{L^{p^*}} \leq 2\,C\,\|Dh\|_{L^p}\,.$$

Hence, it is immediate to verify, using the biLipschitz continuity of $\boldsymbol{\xi}$, that (2.11) is satisfied with $\overline{f} = \boldsymbol{\xi}^{-1}(v)$ and a constant $C(p, m, n, Q, M)$. □

2.2.4. Campanato–Morrey estimates. We prove next the Campanato–Morrey estimates for Q-functions, a crucial tool in the proof of Theorem 0.9.

PROPOSITION 2.14. *Let $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ and $\alpha \in (0,1]$ be such that*

$$\int_{B_r(y)} |Df|^2 \leq A\, r^{m-2+2\alpha} \quad \text{for every } y \in B_1 \text{ and a.e. } r \in]0, 1 - |y|[.$$

Then, for every $0 < \delta < 1$, there is a constant $C = C(m, n, Q, \delta)$ with

$$\sup_{x,y \in \overline{B_\delta}} \frac{\mathcal{G}(f(x), f(y))}{|x-y|^\alpha} =: [f]_{C^{0,\alpha}(\overline{B_\delta})} \leq C\sqrt{A}. \tag{2.14}$$

PROOF. Consider $\boldsymbol{\xi} \circ f$: as shown in Theorem 2.4, there exists a constant C depending on $\mathrm{Lip}(\boldsymbol{\xi})$ and $\mathrm{Lip}(\boldsymbol{\xi}^{-1})$ such that

$$\int_{B_r(y)} |D(\boldsymbol{\xi} \circ f)(x)|^2 dx \leq C\, A\, r^{m-2+2\alpha}$$

Hence, the usual Campanato–Morrey estimates (see, for example, 3.2 in [**HL97**]) provide the existence of a constant $C = C(m, \alpha, \delta)$ such that

$$|\boldsymbol{\xi} \circ f(x) - \boldsymbol{\xi} \circ f(y)| \leq C\sqrt{A}\,|x-y|^\alpha \quad \text{for every } x, y \in \overline{B_\delta}.$$

Thus, composing with $\boldsymbol{\xi}^{-1}$, we conclude the desired estimate (2.14). □

2.2.5. A technical Lemma. This last subsection contains a technical lemma which estimates the Dirichlet energy of an interpolation between two functions defined on concentric spheres. The lemma is particularly useful to construct competitors for Dir-minimizing maps.

LEMMA 2.15 (Interpolation Lemma). *There is a constant $C = C(m, n, Q)$ with the following property. Let $r > 0$, $g \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$ and $f \in W^{1,2}(\partial B_{r(1-\varepsilon)}, \mathcal{A}_Q)$. Then, there exists $h \in W^{1,2}(B_r \setminus B_{r(1-\varepsilon)}, \mathcal{A}_Q)$ such that $h|_{\partial B_r} = g$, $h|_{\partial B_{r(1-\varepsilon)}} = f$ and*

$$\begin{aligned} \mathrm{Dir}(h, B_r \setminus B_{r(1-\varepsilon)}) \leq C\,\varepsilon\, r \left[\mathrm{Dir}(g, \partial B_r) + \mathrm{Dir}(f, \partial B_{r(1-\varepsilon)})\right] + \\ + \frac{C}{\varepsilon\, r} \int_{\partial B_r} \mathcal{G}\left(g(x), f\left((1-\varepsilon)\,x\right)\right)^2 dx. \end{aligned} \tag{2.15}$$

PROOF. By a scaling argument, it is enough to prove the lemma for $r = 1$. As usual, we consider $\psi = \boldsymbol{\xi} \circ g$ and $\varphi = \boldsymbol{\xi} \circ f$. For $x \in \partial B_1$ and $t \in [1-\varepsilon, 1]$, we define

$$\Phi(t\,x) = \frac{(t - 1 + \varepsilon)\,\psi(x) + (1-t)\,\varphi\left((1-\varepsilon)\,x\right)}{\varepsilon},$$

and $\overline{\Phi} = \boldsymbol{\rho} \circ \Phi$. It is straightforward to verify that $\overline{\Phi}$ belongs to $W^{1,2}(B_1 \setminus B_{1-\varepsilon}, \mathcal{Q})$. Moreover, the Lipschitz continuity of $\boldsymbol{\rho}$ and an easy computation yield the following estimate,

$$\begin{aligned}\int_{B_1\setminus B_{1-\varepsilon}} \left|D\,\overline{\Phi}\right|^2 &\leq C\int_{B_1\setminus B_{1-\varepsilon}} |D\,\Phi|^2\\ &\leq C\int_{1-\varepsilon}^1\int_{\partial B_1}\left(|\partial_\tau\varphi(x)|^2+|\partial_\tau\psi(x)|^2+\left|\frac{\psi(x)-\varphi\left((1-\varepsilon)x\right)}{\varepsilon}\right|^2\right)dx\,dt\\ &= C\,\varepsilon\left\{\mathrm{Dir}(\psi,\partial B_1)+\mathrm{Dir}(\varphi,\partial B_{1-\varepsilon})\right\}+\\ &\quad + C\,\varepsilon^{-1}\int_{\partial B_1}|\psi(x)-\varphi\left((1-\varepsilon)x\right)|^2\,dx,\end{aligned}$$

where ∂_τ denotes the tangential derivative. Consider, finally, $h=\boldsymbol{\xi}^{-1}\circ\overline{\Phi}$: (2.15) follows easily from the biLipschitz continuity of $\boldsymbol{\xi}$. □

The following is a straightforward corollary.

COROLLARY 2.16. *There exists a constant $C=C(m,n,Q)$ with the following property. For every $g\in W^{1,2}(\partial B_1,\mathcal{A}_Q)$, there is $h\in W^{1,2}(B_1,\mathcal{A}_Q)$ with $h|_{\partial B_1}=g$ and*

$$\mathrm{Dir}(h,B_1)\leq C\,\mathrm{Dir}(g,\partial B_1)+C\int_{\partial B_1}\mathcal{G}(g,Q\,[\![0]\!])^2.$$

2.3. Existence of Dir-minimizing Q-valued functions

In this section we prove Theorem 0.8. We first remark that Almgren's definition of Dirichlet energy differs from ours. More precisely, using our notations, Almgren's definition of the Dirichlet energy is simply

$$\int_\Omega \sum_{\substack{i=1,\ldots,Q\\ j=1,\ldots,m}} |\partial_j f_i(x)|^2\,dx, \tag{2.16}$$

where $\partial_j f_i$ are the approximate partial derivatives of Definition 2.6, which exist almost everywhere thanks to Corollary 2.7. Moreover, (2.16) makes sense because the integrand does not depend upon the particular selection chosen for f. Before proving Theorem 0.8 we will show that our Dirichlet energy coincides with Almgren's.

PROPOSITION 2.17 (Equivalence of the definitions). *For every $f\in W^{1,2}(\Omega,\mathcal{A}_Q)$ and every $j=1,\ldots,m$, we have*

$$|\partial_j f|^2=\sum_i|\partial_j f_i|^2\quad a.e. \tag{2.17}$$

Therefore the Dirichlet energy $\mathrm{Dir}(f,\Omega)$ *of Definition* 0.6 *coincides with* (2.16).

REMARK 2.18. Fix a point x_0 of approximate differentiability for f and consider $T_{x_0}(x)=\sum[\![f_i(x_0)+Df_i(x_0)\cdot(x-x_0)]\!]$ its first order approximation at x_0. Note that the integrand in (2.16) coincides with $\sum_i|Df_i(x_0)|^2$ (where $|L|$ denotes the Hilbert-Schmidt norm of the matrix L) and it is independent of the orthonormal coordinate system chosen for $\mathbb{R}^m$. Thus, Proposition 2.17 (and its obvious counterpart when the domain is a Riemannian manifold) implies that $\mathrm{Dir}(f,\Omega)$ is as well independent of this choice.

REMARK 2.19. In the sequel, we will often use the following notation: given a Q-point $T \in \mathcal{A}_Q(\mathbb{R}^n)$, $T = \sum_i [\![P_i]\!]$, we set

$$|T|^2 := \mathcal{G}(T, Q\,[\![0]\!])^2 = \sum_i |P_i|^2.$$

In the same fashion, for $f : \Omega \to \mathcal{A}_Q$, we define the function $|f| : \Omega \to \mathbb{R}$ by setting $|f|(x) = |f(x)|$. Then, Proposition 2.17 asserts that, since we understand Df and $\partial_j f$ as maps into, respectively, $\mathcal{A}_Q(\mathbb{R}^{n\times m})$ and $\mathcal{A}_Q(\mathbb{R}^n)$, this notation is consistent with the definitions of $|Df|$ and $|\partial_j f|$ given in (0.3) and (0.2).

The Dirichlet energy of a function $f \in W^{1,2}$ can be recovered, moreover, as the energy of the composition $\boldsymbol{\xi}_{BW} \circ f$, where $\boldsymbol{\xi}_{BW}$ is the biLipschitz embedding in Corollary 2.2 (compare with Theorem 2.4).

PROPOSITION 2.20. *For every $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$, it holds $|Df| = |D(\boldsymbol{\xi}_{BW} \circ f)|$ a.e. In particular,* $\mathrm{Dir}(f, \Omega) = \int_\Omega |D(\boldsymbol{\xi}_{BW} \circ f)|^2$.

Although this proposition gives a great intuition about the energy of Q-valued functions, as already pointed out, we will not use it in the rest of the paper, the reason being that, the theory is in fact independent of the biLipschitz embedding.

2.3.1. Proof the equivalence of the definitions.

PROOF OF PROPOSITION 2.17. We recall the definition of $|\partial_j f|$ and $|Df|$ given in (0.2) and (0.3): chosen a countable dense set $\{T_l\}_{l\in\mathbb{N}} \subset \mathcal{A}_Q$, we define

$$|\partial_j f| = \sup_{l\in\mathbb{N}} |\partial_j \mathcal{G}(f, T_l)| \quad \text{and} \quad |Df|^2 := \sum_{j=1}^m |\partial_j f|^2\,.$$

By Proposition 2.5, we can consider a sequence $g^k = \sum_{i=1}^Q [\![g_i^k]\!]$ of Lipschitz functions with the property that $|\{g^k \neq f\}| \leq 1/k$. Note that $|\partial_j f| = |\partial_j g^k|$ and $\sum_i |\partial_j g_i^k|^2 = \sum_i |\partial_j f_i|^2$ almost everywhere on $\{g^k = f\}$. Thus, it suffices to prove the proposition for each Lipschitz function g^k.

Therefore, we assume from now on that f is Lipschitz. Note next that on the set $E_l = \{x \in \Omega : f(x) = T_l\}$ both $|\partial_j f|$ and $\sum_i |\partial_j f_i|^2$ vanish a.e. Hence, it suffices to show (2.17) on any point x_0 where f and all $\mathcal{G}(f, T_l)$ are differentiable and $f(x_0) \notin \{T_l\}_{l\in\mathbb{N}}$.

Fix such a point, which, without loss of generality, we can assume to be the origin, $x_0 = 0$. Let $T_0 f$ be the first oder approximation of f at 0. Since $\mathcal{G}(\cdot, T_l)$ is a Lipschitz function, we have $\mathcal{G}(f(y), T_l) = \mathcal{G}(T_0 f(y), T_l) + o(|y|)$. Therefore, $g(y) := \mathcal{G}(T_0 f(y), T_l)$ is differentiable at 0 and $\partial_j g(0) = \partial_j \mathcal{G}(f, T_l)(0)$.

We assume, without loss of generality, that $\mathcal{G}(f(0), T_l)^2 = \sum_i |f_i(0) - P_i|^2$, where $T_l = \sum_i [\![P_i]\!]$. Next, we consider the function

$$h(y) := \sqrt{\sum_i |f_i(0) + Df_i(0)\cdot y - P_i|^2}.$$

Then, $g \leq h$. Since $h(0) = g(0)$, we conclude that $h - g$ has a minimum at 0. Recall that both h and g are differentiable at 0 and $h(0) = g(0)$. Thus, we conclude $\nabla h(0) = \nabla g(0)$, which in turn yields the identity

$$\partial_j\, \mathcal{G}(f, T_l)(0) = \partial_j g(0) = \partial_j h(0) = \sum_i \frac{(f_i(0) - P_i)\cdot \partial_j f_i(0)}{\sqrt{\sum_i |f_i(0) - P_i|^2}}. \tag{2.18}$$

Using the Cauchy-Schwarz inequality and (2.18), we deduce that

$$|\partial_j f|(0)^2 = \sup_{l\in\mathbb{N}} |\partial_j \mathcal{G}(f,T_l)(0)|^2 \leq \sum_i |\partial_j f_i(0)|^2 . \tag{2.19}$$

If the right hand side of (2.19) vanishes, then we clearly have equality. Otherwise, let $Q_i = f_i(0) + \lambda\, \partial_j f_i(0)$, where λ is a small constant to be chosen later, and consider $T = \sum_i [\![Q_i]\!]$. Since $\{T_l\}$ is a dense subset of $\mathcal{A}_Q$, for every $\varepsilon > 0$ we can find a point $T_l = \sum_i [\![P_i]\!]$ such that

$$P_i = f_i(0) + \lambda\, \partial_j f_i(0) + \lambda\, R_i, \quad \text{with } |R_i| \leq \varepsilon \text{ for every } i.$$

Now we choose λ and ε small enough to ensure that $\mathcal{G}(f(0),T_l)^2 = \sum_i |f_i(0) - P_i|^2$ (indeed, recall that, if $f_i(0) = f_k(0)$, then $\partial_j f_i(0) = \partial_j f_k(0)$). So, we can repeat the computation above and deduce that

$$\partial_j\, \mathcal{G}(f,T_l)(0) = \sum_i \frac{(f_i(0) - P_i)\cdot \partial_j f_i(0)}{\sqrt{\sum_i |f_i(0) - P_i|^2}} = \sum_i \frac{(\partial_j f_i(0) + R_i)\cdot \partial_j f_i(0)}{\sqrt{\sum_i |\partial_j f_i(0) + R_i|^2}}.$$

Hence,

$$|\partial_j f|(0) \geq \sum_i \frac{(\partial_j f_i(0))^2 + \varepsilon |\partial_j f_i(0)|}{\sqrt{\sum_i (|\partial_j f_i(0)| + \varepsilon)^2}}.$$

Letting $\varepsilon \to 0$, we obtain the inequality $|\partial_j f|(0) \geq \sum_j (\partial_j f_i(0))^2$. □

Proof of Proposition 2.20. As for Proposition 2.17, it is enough to show the proposition for a Lipschitz function f. We prove that the functions $|Df|$ and $|D(\boldsymbol{\xi}_{BW} \circ f)|$ coincide on each point of differentiability of f.

Let x_0 be such a point and let $T_{x_0} f(x) = \sum_i [\![f_i(x_0) + Df_i(x_0)\cdot(x - x_0)]\!]$ be the first order expansion of f in x_0. Since $\mathcal{G}(f(x), T_{x_0} f(x)) = o(|x - x_0|)$ and $\mathrm{Lip}(\boldsymbol{\xi}_{BW}) = 1$, it is enough to prove that $|Df|(x_0) = |D(\boldsymbol{\xi}_{BW} \circ T_{x_0} f)(x_0)|$.

Using the fact that $Df_i(x_0) = Df_j(x_0)$ when $f_i(x_0) = f_j(x_0)$, it follows easily that, for every x with $|x - x_0|$ small enough,

$$\mathcal{G}(T_{x_0} f(x), f(x_0))^2 = \sum_i |Df_i(x_0)\cdot(x - x_0)|^2.$$

Hence, since $\boldsymbol{\xi}_{BW}$ is an isometry in a neighborhood of each point, for $|x - x_0|$ small enough, we infer that

$$|\boldsymbol{\xi}_{BW}(T_{x_0} f(x)) - \boldsymbol{\xi}_{BW}(f(x_0))|^2 = \sum_i |Df_i(x_0)\cdot(x - x_0)|^2. \tag{2.20}$$

For $x = t\,\mathrm{e}_j + x_0$ in (2.20), where the e_j's are the canonical basis in $\mathbb{R}^m$, taking the limit as t goes to zero, we obtain that

$$|\partial_j(\boldsymbol{\xi}_{BW} \circ T_{x_0} f)(x_0)|^2 = \sum_i |\partial_j f_i|^2(x_0).$$

Summing in j and using Proposition 2.17, we conclude that $|Df|(x_0) = |D(\boldsymbol{\xi}_{BW} \circ T_{x_0} f)(x_0)|$, which concludes the proof. □

2.3.2. Proof of Theorem 0.8. Let $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$ be given. Thanks to Propositions 2.10 and 2.11, it suffices to verify the sequential weak lower semicontinuity of the Dirichlet energy. To this aim, let $f_k \rightharpoonup f$ in $W^{1,2}(\Omega, \mathcal{A}_Q)$: we want to show that

$$\mathrm{Dir}(f, \Omega) \leq \liminf_{k\to\infty} \mathrm{Dir}(f_k, \Omega). \tag{2.21}$$

Let $\{T_l\}_{l\in\mathbb{N}}$ be a dense subset of $\mathcal{A}_Q$ and recall that $|\partial_j f|^2 = \sup_l \big(\partial_j \mathcal{G}(f, T_l)\big)^2$. Thus, if we set

$$h_{j,N} = \max_{l\in\{1,\ldots,N\}} \big(\partial_j \mathcal{G}(f, T_l)\big)^2,$$

we conclude that $h_{j,N} \uparrow |\partial_j f|^2$. Next, for every N, denote by $\mathcal{P}_N$ the collections $P = \{E_l\}_{l=1}^N$ of N disjoint measurable subsets of Ω. Clearly, it holds

$$h_{j,N} = \sup_{P\in\mathcal{P}} \sum_{E_l\in P} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \mathbf{1}_{E_l}.$$

By the Monotone Convergence Theorem, we conclude

$$\mathrm{Dir}(f, \Omega) = \sum_{j=1}^m \sup_N \int h_{j,N}^2 = \sum_{j=1}^m \sup_N \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2.$$

Fix now a partition $\{F_1, \ldots, F_N\}$ such that, for a given $\varepsilon > 0$,

$$\sum_l \int_{F_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \geq \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 - \varepsilon.$$

Then, we can find compact sets $\{K_1, \ldots, K_N\}$ with $K_l \subset F_l$ and

$$\sum_l \int_{K_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \geq \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 - 2\varepsilon.$$

Since the K_l's are disjoint compact sets, we can find disjoint open sets $U_l \supset K_l$. So, denote by $\mathcal{O}_N$ the collections of N pairwise disjoint open sets of Ω. We conclude

$$\mathrm{Dir}(f, \Omega) = \sum_{j=1}^m \sup_N \int h_{j,N}^2 = \sum_{j=1}^m \sup_N \sup_{P\in\mathcal{O}_N} \sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2. \tag{2.22}$$

Note that, since $\mathcal{G}(f_k, T_l) \to \mathcal{G}(f, T_l)$ strongly in $L^2(\Omega)$, then $\partial_j \mathcal{G}(f_k, T_l) \rightharpoonup \partial_j \mathcal{G}(f, T_l)$ in $L^2(U)$ for every open $U \subset \Omega$. Hence, for every N and every $P \in \mathcal{O}_N$, we have

$$\sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \leq \liminf_{k\to+\infty} \sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f_k, T_l)\big)^2 \leq \liminf_{k\to\infty} \int_\Omega |\partial_j f_k|^2.$$

Taking the supremum in $\mathcal{O}_N$ and in N, and then summing in j, in view of (2.22), we achieve (2.21).

CHAPTER 3

Regularity theory

This chapter is devoted to the proofs of the two Regularity Theorems 0.9 and 0.11. In Section 3.1 we derive some Euler-Lagrange conditions for Dir-minimizers, whereas in Section 3.2 we prove a maximum principle for Q-valued functions. Using these two results, we prove Theorem 0.9 in Section 3.3. Then, in Section 3.4 we introduce Almgren's frequency function and prove his fundamental estimate. The frequency function is the main tool for the blow-up analysis of Section 3.5, which gives useful information on the rescalings of Dir-minimizing Q-functions. Finally, in Section 3.6 we combine this analysis with a version of Federer's reduction argument to prove Theorem 0.11.

3.1. First variations

There are two natural types of variations that can be used to perturb Dir-minimizing Q-valued functions. The first ones, which we call inner variations, are generated by right compositions with diffeomorphisms of the domain. The second, which we call outer variations, correspond to "left compositions" as defined in Subsection 1.3.1. More precisely, let f be a Dir-minimizing Q-valued map.

(IV) Given $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, for ε sufficiently small, $x \mapsto \Phi_\varepsilon(x) = x + \varepsilon\varphi(x)$ is a diffeomorphism of Ω which leaves $\partial\Omega$ fixed. Therefore,

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega |D(f \circ \Phi_\varepsilon)|^2. \tag{3.1}$$

(OV) Given $\psi \in C^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that $\mathrm{supp}\,(\psi) \subset \Omega' \times \mathbb{R}^n$ for some $\Omega' \subset\subset \Omega$, we set $\Psi_\varepsilon(x) = \sum_i [\![f_i(x) + \varepsilon\psi(x, f_i(x))]\!]$ and derive

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega |D\,\Psi_\varepsilon|^2. \tag{3.2}$$

The identities (3.1) and (3.2) lead to the following proposition.

Proposition 3.1 (First variations). *For every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, we have*

$$2\int \sum_i \langle Df_i : Df_i \cdot D\varphi\rangle - \int |Df|^2 \,\mathrm{div}\,\varphi = 0. \tag{3.3}$$

For every $\psi \in C^\infty(\Omega_x \times \mathbb{R}^n_u, \mathbb{R}^n)$ such that

$$\mathrm{supp}\,(\psi) \subset \Omega' \times \mathbb{R}^n \quad \textit{for some } \Omega' \subset\subset \Omega,$$

and

$$|D_u\psi| \le C < \infty \quad \textit{and} \quad |\psi| + |D_x\psi| \le C\,(1 + |u|)\,, \tag{3.4}$$

we have

$$\int \sum_i \big\langle Df_i(x) : D_x\psi(x, f_i(x))\big\rangle\, dx + \int \sum_i \big\langle Df_i(x) : D_u\psi(x, f_i(x))\cdot Df_i(x)\big\rangle\, dx = 0. \tag{3.5}$$

Testing (3.3) and (3.5) with suitable φ and ψ, we get two key identities. In what follows, ν will always denote the outer unit normal on the boundary ∂B of a given ball.

PROPOSITION 3.2. *Let $x \in \Omega$. Then, for a.e. $0 < r < dist(x, \partial\Omega)$, we have*

$$(m-2)\int_{B_r(x)} |Df|^2 = r\int_{\partial B_r(x)} |Df|^2 - 2\,r\int_{\partial B_r(x)} \sum_i |\partial_\nu f_i|^2, \tag{3.6}$$

$$\int_{B_r(x)} |Df|^2 = \int_{\partial B_r(x)} \sum_i \langle \partial_\nu f_i, f_i\rangle. \tag{3.7}$$

REMARK 3.3. The identities (3.6) and (3.7) are classical facts for $\mathbb{R}^n$-valued harmonic maps f, which can be derived from the Laplace equation $\Delta f = 0$.

3.1.1. Proof of Proposition 3.1. We apply formula (1.15) of Proposition 2.8 to compute

$$D(f\circ\Phi_\varepsilon)(x) = \sum_i \llbracket Df_i(x+\varepsilon\varphi(x)) + \varepsilon[Df_i(x+\varepsilon\varphi(x))]\cdot D\varphi(x)\rrbracket\,. \tag{3.8}$$

For ε sufficiently small, Φ_ε is a diffeomorphism. We denote by Φ_ε^{-1} its inverse. Then, inserting (3.8) in (3.3), changing variables in the integral ($x = \Phi_\varepsilon^{-1}(y)$) and differentiating in ε, we get

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega \sum_i |Df_i(y) + \varepsilon Df_i\cdot D\varphi(\Phi_\varepsilon^{-1}(y))|^2 \,\det\big(D\Phi_\varepsilon^{-1}(y)\big)\, dy \\ &= 2\int\sum_i \big\langle Df_i(y) : Df_i(y)\cdot D\varphi(y)\big\rangle\, dy - \int \sum_i |Df_i(y)|^2 \operatorname{div}\varphi(y)\, dy. \end{aligned}$$

This shows (3.3). As for (3.5), using (1.16) and then differentiating in ε, the proof is straightforward (the hypotheses in (3.4) ensure the summability of the various integrands involved in the computation).

3.1.2. Proof of Proposition 3.2. Without loss of generality, we assume $x = 0$. We test (3.3) with a function φ of the form $\varphi(x) = \phi(|x|)\,x$, where ϕ is a function in $C^\infty([0,\infty))$, with $\phi \equiv 0$ on $[r,\infty)$, $r < \operatorname{dist}(0,\partial\Omega)$, and $\phi \equiv 1$ in a neighborhood of 0. Then,

$$D\varphi(x) = \phi(|x|)\,\mathrm{Id} + \phi'(|x|)\,x\otimes\frac{x}{|x|} \quad\text{and}\quad \operatorname{div}\varphi(x) = m\,\phi(|x|) + |x|\,\phi'(|x|), \tag{3.9}$$

where Id denotes the $m\times m$ identity matrix. Note that

$$\partial_\nu f_i(x) = Df_i(x)\cdot\frac{x}{|x|}.$$

Then, inserting (3.9) into (3.3), we get

$$
\begin{aligned}
0 = 2 \int |Df(x)|^2\, \phi(|x|)\, dx + 2 \int \sum_{i=1}^{Q} |\partial_\nu f_i(x)|^2\, \phi'(|x|)\, |x|\, dx \\
- m \int |Df(x)|^2\, \phi(|x|)\, dx - \int |Df(x)|^2\, \phi'(|x|)\, |x|\, dx.
\end{aligned}
$$

By a standard approximation procedure, it is easy to see that we can test with

$$
\phi(t) = \phi_n(t) := \begin{cases} 1 & \text{for } t \le r - 1/n, \\ n\,(r-t) & \text{for } r - 1/n \le t \le r. \end{cases} \tag{3.10}
$$

With this choice we get

$$
\begin{aligned}
0 = (2-m) \int |Df(x)|^2\, \phi_n(|x|)\, dx - \frac{2}{n} \int_{B_r \setminus B_{r-1/n}} \sum_{i=1}^{Q} |\partial_\nu f_i(x)|^2\, |x|\, dx \\
+ \frac{1}{n} \int_{B_r \setminus B_{r-1/n}} |Df(x)|^2\, |x|\, dx.
\end{aligned}
$$

Let $n \uparrow \infty$. Then, the first integral converges towards $(2-m) \int_{B_r} |Df|^2$. As for the second and third integral, for a.e. r, they converge, respectively, to

$$
-r \int_{\partial B_r} \sum_{i=1}^{Q} |\partial_\nu f_i|^2 \quad \text{and} \quad r \int_{\partial B_r} |Df|^2.
$$

Thus, we conclude (3.6).

Similarly, test (3.5) with $\psi(x,u) = \phi(|x|)\, u$. Then,

$$
D_u \psi(x,u) = \phi(|x|)\, \mathrm{Id} \quad \text{and} \quad D_x \psi(x,u) = \phi'(|x|)\, u \otimes \frac{x}{|x|}\,. \tag{3.11}
$$

Inserting (3.11) into (3.5) and differentiating in ε, we get

$$
0 = \int |Df(x)|^2\, \phi(|x|)\, dx + \int \sum_{i=1}^{Q} \langle f_i(x), \partial_\nu f_i(x) \rangle\, \phi'(|x|)\, dx.
$$

Therefore, choosing ϕ as in (3.10), we can argue as above and, for $n \uparrow \infty$, we conclude (3.7).

3.2. A maximum principle for Q-valued functions

The two propositions of this section play a key role in the proof of the Hölder regularity for Dir-minimizing Q-functions when the domain has dimension strictly larger than two. Before stating them, we introduce two important functions on $\mathcal{A}_Q(\mathbb{R}^n)$.

Definition 3.4 (Diameter and separation). Let $T = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$. The *diameter* and the *separation* of T are defined, respectively, as

$$
d(T) := \max_{i,j} |P_i - P_j| \quad \text{and} \quad s(T) := \min \big\{ |P_i - P_j| : P_i \neq P_j \big\},
$$

with the convention that $s(T) = +\infty$ if $T = Q\,[\![P]\!]$.

The following proposition is an elementary extension of the usual maximum principle for harmonic functions.

Proposition 3.5 (Maximum Principle). *Let $f : \Omega \to \mathcal{A}_Q$ be* Dir*-minimizing, $T \in \mathcal{A}_Q$ and $r < s(T)/4$. Then, $\mathcal{G}(f(x), T) \leq r$ for $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$ implies that $\mathcal{G}(f, T) \leq r$ almost everywhere on Ω.*

The next proposition allows to decompose Dir-minimizing functions and, hence, to argue inductively on the number of values. Its proof is based on Proposition 3.5 and a simple combinatorial lemma.

Proposition 3.6 (Decomposition for Dir-minimizers). *There exists a positive constant $\alpha(Q) > 0$ with the following property. If $f : \Omega \to \mathcal{A}_Q$ is* Dir*-minimizing and there exists $T \in \mathcal{A}_Q$ such that $\mathcal{G}(f(x), T) \leq \alpha(Q)\, d(T)$ for $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$, then there exists a decomposition of $f = [\![g]\!] + [\![h]\!]$ into two simpler* Dir*-minimizing functions.*

3.2.1. Proof of Proposition 3.5. The proposition follows from the next lemma.

Lemma 3.7. *Let T and r be as in Proposition 3.5. Then, there exists a retraction $\vartheta : \mathcal{A}_Q \to \overline{B_r(T)}$ such that*

(*i*) *$\mathcal{G}(\vartheta(S_1), \vartheta(S_2)) < \mathcal{G}(S_1, S_2)$ if $S_1 \notin \overline{B_r(T)}$,*
(*ii*) *$\vartheta(S) = S$ for every $S \in \overline{B_r(T)}$.*

We assume the lemma for the moment and argue by contradiction for Proposition 3.5. We assume, therefore, the existence of a Dir-minimizing f with the following properties:

(*a*) $f(x) \in \overline{B_r(T)}$ for a.e. $x \in \partial\Omega$;
(*b*) $f(x) \notin \overline{B_r(T)}$ for every $x \in E \subset \Omega$, where E is a set of positive measure.

Therefore, there exist $\varepsilon > 0$ and a set E' with positive measure such that $f(x) \notin B_{r+\varepsilon}(T)$ for every $x \in E'$. By (*ii*) of Lemma 3.7 and (*a*), $\vartheta \circ f$ has the same trace as f. Moreover, by (*i*) of Lemma 3.7, $|D(\vartheta \circ f)| \leq |Df|$ a.e. and, by (*i*) and (*b*), $|D(\vartheta \circ f)| < |Df|$ a.e. on E'. This implies $\mathrm{Dir}(\vartheta \circ f, \Omega) < \mathrm{Dir}(f, \Omega)$, contradicting the minimizing property of f.

Proof of Lemma 3.7. First of all, we write

$$T = \sum_{j=1}^{J} k_j\, [\![Q_j]\!]\,,$$

where $|Q_j - Q_i| > 4\,r$ for every $i \neq j$.

If $\mathcal{G}(S, T) < 2r$, then $S = \sum_{j=1}^{J} [\![S_j]\!]$ with $S_j \in B_{2r}(k_j\, [\![Q_j]\!]) \subset \mathcal{A}_{k_j}$. If, in addition, $\mathcal{G}(S, T) \geq r$, then we set

$$S_j = \sum_{l=1}^{k_j} [\![S_{l,j}]\!]\,,$$

and we define

$$\vartheta(S) = \sum_{j=1}^{J} \sum_{l=1}^{k_j} \left[\!\!\left[\frac{2r - \mathcal{G}(T, S)}{\mathcal{G}(T, S)} (S_{l,j} - Q_j) + Q_j \right]\!\!\right].$$

We then extend ϑ to $\mathcal{A}_Q$ by setting

$$\vartheta(S) = \begin{cases} T & \text{if } S \notin B_{2r}(T), \\ S & \text{if } S \in B_r(T). \end{cases}$$

It is immediate to verify that ϑ is continuous and has all the required properties. $\square$

3.2.2. Proof of Proposition 3.6. The key idea is simple. If the separation of T were not too small, we could apply directly Proposition 3.5. When the separation of T is small, we can find a point S which is not too far from T and whose separation is sufficiently large. Roughly speaking, it suffices to "collapse" the points of the support of T which are too close.

LEMMA 3.8. *For every $0 < \varepsilon < 1$, we set $\beta(\varepsilon, Q) = (\varepsilon/3)^{3^Q}$. Then, for every $T \in \mathcal{A}_Q$ with $s(T) < \infty$, there exists a point $S \in \mathcal{A}_Q$ such that*

$$\beta(\varepsilon, Q)\, d(T) \le s(S) < \infty, \tag{3.12}$$

$$\mathcal{G}(S,T) \le \varepsilon\, s(S). \tag{3.13}$$

Assuming Lemma 3.8, we conclude the proof of Proposition 3.6. Set $\varepsilon = 1/8$ and $\alpha(Q) = \varepsilon\, \beta(\varepsilon, Q) = 24^{-3^Q}/8$. From Lemma 3.8, we deduce the existence of an S satisfying (3.12) and (3.13). Then, there exists $\delta > 0$ such that, for almost every $x \in \partial\Omega$,

$$\mathcal{G}(f(x), S) \le \mathcal{G}(f(x), T) + \mathcal{G}(T, S) \overset{(3.13)}{\le} \alpha(Q)\, d(T) + \frac{s(S)}{8} - \delta \overset{(3.12)}{\le} \frac{s(S)}{4} - \delta.$$

So, we may apply Proposition 3.5 and infer that $\mathcal{G}(f(x), S) \le \frac{s(S)}{4} - \delta$ for almost every x in Ω. The decomposition of f in simpler Dir-minimizing functions is now a simple consequence of the definitions. More precisely, if $S = \sum_{j=1}^{J} k_j\, [\![Q_j]\!] \in \mathcal{A}_Q$, with the Q_j's all different, then $f(x) = \sum_{j=1}^{J} [\![f_j(x)]\!]$, where the f_j's are Dir-minimizing k_j-valued functions with values in the balls $B_{\frac{s(S)}{4}-\delta}(k_j\, [\![Q_j]\!])$.

PROOF OF LEMMA 3.8. For $Q \le 2$, we have $d(T) \le s(T)$ and it suffices to choose $S = T$. We now prove the general case by induction. Let $Q \ge 3$ and assume the lemma holds for $Q-1$. Let $T = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$. Two cases can occur:

(a) either $s(T) \ge (\varepsilon/3)^{3^Q}\, d(T)$;
(b) or $s(T) < (\varepsilon/3)^{3^Q}\, d(T)$.

In case (a), since the separation of T is sufficiently large, the point T itself, i.e. $S = T$, fulfills (3.13) and (3.12). In the other case, since the points P_i are not all equal ($s(T) < \infty$), we can take P_1 and P_2 realizing the separation of T, i.e.

$$|P_1 - P_2| = s(T) \le \left(\frac{\varepsilon}{3}\right)^{3^Q} d(T). \tag{3.14}$$

Moreover, since $Q \ge 3$, we may also assume that, suppressing P_1, we do not reduce the diameter, i.e. that

$$d(T) = d(\tilde{T}), \quad \text{where} \quad \tilde{T} = \sum_{i=2}^{Q} [\![P_i]\!]\,. \tag{3.15}$$

For $\tilde{T}$, we are now in the position to use the inductive hypothesis (with $\varepsilon/3$ in place of ε). Hence, there exists $\tilde{S} = \sum_{j=1}^{Q-1} [\![Q_j]\!]$ such that

$$\left(\frac{\varepsilon}{9}\right)^{3^{Q-1}} d\left(\tilde{T}\right) \le s\left(\tilde{S}\right) \quad \text{and} \quad \mathcal{G}\left(\tilde{S}, \tilde{T}\right) \le \frac{\varepsilon}{3} s\left(\tilde{S}\right). \tag{3.16}$$

Without loss of generality, we can assume that

$$|Q_1 - P_2| \le \mathcal{G}(\tilde{S}, \tilde{T}). \tag{3.17}$$

Therefore, $S = [\![Q_1]\!] + [\![\tilde{S}]\!] \in \mathcal{A}_Q$ satisfies (3.12) and (3.13). Indeed, since $s(S) = s(\tilde{S})$, we infer

$$\left(\frac{\varepsilon}{3}\right)^{3^Q} d(T) \stackrel{(3.15)}{\le} \frac{\varepsilon}{3}\left(\frac{\varepsilon}{9}\right)^{3^{Q-1}} d\left(\tilde{T}\right) \stackrel{(3.16)}{\le} \frac{\varepsilon}{3} s\left(\tilde{S}\right) = \frac{\varepsilon}{3} s(S), \tag{3.18}$$

and

$$\begin{aligned}\mathcal{G}(S,T) &\le \mathcal{G}(\tilde{S},\tilde{T}) + |Q_1 - P_1| \le \mathcal{G}(\tilde{S},\tilde{T}) + |Q_1 - P_2| + |P_2 - P_1| \\ &\stackrel{(3.14),\,(3.17)}{\le} 2\,\mathcal{G}(\tilde{S},\tilde{T}) + \left(\frac{\varepsilon}{3}\right)^{3^Q} d(T) \stackrel{(3.16),\,(3.18)}{\le} \frac{2\,\varepsilon}{3} s(S) + \frac{\varepsilon}{3} s(S) = \varepsilon\, s(S).\end{aligned}$$

□

3.3. Hölder regularity

Now we pass to prove the Hölder continuity of Dir-minimizing Q-valued functions. Theorem 0.9 is indeed a simple consequence of the following theorem.

THEOREM 3.9. *There exist constants $\alpha = \alpha(m, Q) \in]0, 1[$ (with $\alpha = \frac{1}{Q}$ when $m = 2$) and $C = C(m, n, Q, \delta)$ with the following property. If $f : B_1 \to \mathcal{A}_Q$ is* Dir*-minimizing, then*

$$[f]_{C^{0,\alpha}(\overline{B_\delta})} = \sup_{x,y \in \overline{B_\delta}} \frac{\mathcal{G}(f(x), f(y))}{|x-y|^\alpha} \le C\, \mathrm{Dir}(f, \Omega)^{\frac{1}{2}} \quad \textit{for every } 0 < \delta < 1.$$

The proof of Theorem 3.9 consists of two parts: the first is stated in the following proposition which gives the crucial estimate; the second is a standard application of the Campanato–Morrey estimates (see Section 2.2, Proposition 2.14).

PROPOSITION 3.10. *Let $f \in W^{1,2}(B_r, \mathcal{A}_Q)$ be* Dir*-minimizing and suppose that*

$$g = f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_Q).$$

Then, we have that

$$\mathrm{Dir}(f, B_r) \le C(m)\, r\, \mathrm{Dir}(g, \partial B_r), \tag{3.19}$$

where $C(2) = Q$ and $C(m) < (m-2)^{-1}$.

The minimizing property of f enters heavily in the proof of this last proposition, where the estimate is achieved by exhibiting a suitable competitor. This is easier in dimension 2 because we can use Proposition 1.5 for g. In higher dimension the argument is more complicated and relies on Proposition 3.6 to argue by induction on Q. Now, assuming Proposition 3.10, we proceed with the proof of Theorem 3.9.

3.3.1. Proof of Theorem 3.9. Set

$$\gamma(m) := \begin{cases} 2\,Q^{-1} & \text{for } m = 2, \\ C(m)^{-1} - m + 2 & \text{for } m > 2, \end{cases}$$

where $C(m)$ is the constant in (3.19). We want to prove that

$$\int_{B_r} |Df|^2 \leq r^{m-2+\gamma} \int_{B_1} |Df|^2 \quad \text{for every } 0 < r \leq 1. \tag{3.20}$$

Define $h(r) = \int_{B_r} |Df|^2$. Note that h is absolutely continuous and that

$$h'(r) = \int_{\partial B_r} |Df|^2 \geq \mathrm{Dir}(f, \partial B_r) \quad \text{for a.e. } r, \tag{3.21}$$

where, according to Definitions 0.5 and 0.6, $\mathrm{Dir}(f, \partial B_r)$ is given by

$$\mathrm{Dir}(f, \partial B_r) = \int_{\partial B_r} |\partial_\tau f|^2,$$

with $|\partial_\tau f|^2 = |Df|^2 - \sum_{i=1}^{Q} |\partial_\nu f_i|^2$. Here ∂_τ and ∂_ν denote, respectively, the tangential and the normal derivatives. We remark further that (3.21) can be improved for $m = 2$. Indeed, in this case the outer variation formula (3.6), gives an equipartition of the Dirichlet energy in the radial and tangential parts, yielding

$$h'(r) = \int_{\partial B_r} |Df|^2 = \frac{\mathrm{Dir}(f, \partial B_r)}{2}. \tag{3.22}$$

Therefore, (3.21) (resp. (3.22) when $m = 2$) and (3.19) imply

$$(m - 2 + \gamma)\, h(r) \leq r\, h'(r). \tag{3.23}$$

Integrating this differential inequality, we obtain (3.20):

$$\int_{B_r} |Df|^2 = h(r) \leq r^{m-2+\gamma}\, h(1) = r^{m-2+\gamma} \int_{B_1} |Df|^2.$$

Now we can use the Campanato–Morrey estimates for Q-valued functions given in Proposition 2.14 in order to conclude the Hölder continuity of f with exponent $\alpha = \frac{\gamma}{2}$.

3.3.2. Proof of Proposition 3.10: the planar case. It is enough to prove (3.19) for $r = 1$, because the general case follows from an easy scaling argument. We first prove the following simple lemma.

Remark 3.11. In this subsection we introduce a complex notation which will be also useful later. We identify the plane $\mathbb{R}^2$ with $\mathbb{C}$ and therefore we regard the unit disk as

$$\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\} = \{r\, e^{i\theta} \,:\, 0 \leq r < 1,\, \theta \in \mathbb{R}\}$$

and the unit circle as

$$\mathbb{S}^1 = \partial\,\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| = 1\} = \{e^{i\theta} \,:\, \theta \in \mathbb{R}\}.$$

Lemma 3.12. *Let $\zeta \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ and consider the Q-valued function f defined by*

$$f(x) = \sum_{z^Q = x} [\![\zeta(z)]\!]\,.$$

Then, the function f belongs to $W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ and

$$\operatorname{Dir}(f, \mathbb{D}) = \int_{\mathbb{D}} |D\zeta|^2 . \tag{3.24}$$

Moreover, if $\zeta|_{\mathbb{S}^1} \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^n)$, then $f|_{\mathbb{S}^1} \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_Q)$ and

$$\operatorname{Dir}(f|_{\mathbb{S}^1}, \mathbb{S}^1) = \frac{1}{Q} \int_{\mathbb{S}^1} |\partial_\tau \zeta|^2 . \tag{3.25}$$

PROOF. Define the following subsets of the unit disk,

$$\mathcal{D}_j = \left\{ r\, e^{i\theta} : 0 < r < 1,\ (j-1)\, 2\pi/Q < \theta < j\, 2\pi/Q \right\}$$

and

$$\mathcal{C} = \left\{ r\, e^{i\theta} : 0 < r < 1,\ \theta \neq 0 \right\},$$

and let $\varphi_j : \mathcal{C} \to \mathcal{D}_j$ be determinations of the Q^{th}-root, i.e.

$$\varphi_j \left(re^{i\theta} \right) = r^{\frac{1}{Q}}\, e^{i \left(\frac{\theta}{Q} + (j-1)\, \frac{2\pi}{Q} \right)}.$$

It is easily recognized that $f|_{\mathcal{C}} = \sum_j [\![\zeta \circ \varphi_j]\!]$. So, by the invariance of the Dirichlet energy under conformal mappings, one deduces that $f \in W^{1,2}(\mathcal{C}, \mathcal{A}_Q)$ and

$$\operatorname{Dir}(f, \mathcal{C}) = \sum_{i=1}^{Q} \operatorname{Dir}(\zeta \circ \varphi_i, \mathcal{C}) = \int_{\mathbb{D}} |D\zeta|^2 . \tag{3.26}$$

From the above argument and from (3.26), it is straightforward to infer that f belongs to $W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ and (3.24) holds. Finally, (3.25) is a simple computation left to the reader. □

We now prove Proposition 3.10. Let $g = \sum_{j=1}^{J} [\![g_j]\!]$ be a decomposition into irreducible k_j-functions as in Proposition 1.5. Consider, moreover, the $W^{1,2}$ functions $\gamma_j : \mathbb{S}^1 \to \mathbb{R}^n$ "unrolling" the g_j as in Proposition 1.5 (*ii*):

$$g_j(x) = \sum_{z^{k_j} = x} [\![\gamma_j(z)]\!] .$$

We take the harmonic extension ζ_l of γ_l in $\mathbb{D}$, and consider the k_l-valued functions f_l obtained "rolling" back the ζ_l: $f_l(x) = \sum_{z^{k_l} = x} [\![\zeta_l(z)]\!]$. The Q-function $\tilde{f} = \sum_{l=1}^{J} [\![f_l]\!]$ is an admissible competitor for f, since $\tilde{f}|_{\mathbb{S}^1} = f|_{\mathbb{S}^1}$. By a simple computation on planar harmonic functions, it is easy to see that

$$\int_{\mathbb{D}} |D\zeta_l|^2 \leq \int_{\mathbb{S}^1} |\partial_\tau \gamma_l|^2 . \tag{3.27}$$

Hence, from (3.24), (3.25) and (3.27), we easily conclude (3.19):

$$\begin{aligned}\operatorname{Dir}(f, \mathbb{D}) \leq \operatorname{Dir}\left(\tilde{f}, \mathbb{D}\right) = \sum_{l=1}^{J} \operatorname{Dir}(f_l, \mathbb{D}) &\stackrel{(3.24)}{=} \sum_{l=1}^{J} \int_{\mathbb{D}} |D\zeta_l|^2 \\ &\stackrel{(3.27)}{\leq} \sum_{l=1}^{J} \int_{\mathbb{S}^1} |\partial_\tau \gamma_l|^2 \stackrel{(3.25)}{=} \sum_{l=1}^{J} k_l \operatorname{Dir}(g_l, \mathbb{S}^1) \leq Q \operatorname{Dir}(g, \mathbb{S}^1).\end{aligned}$$

3.3.3. Proof of Proposition 3.10: the case $m \geq 3$. To understand the strategy of the proof, fix a Dir-minimizing f and consider the "radial" competitor $h(x) = f(x/|x|)$. An easy computation shows the inequality $\mathrm{Dir}(h, B_1) \leq (m-2)^{-1}\mathrm{Dir}(f, \partial B_1)$. In order to find a better competitor, set $\tilde{f}(x) = \sum_i [\![\varphi(|x|) f_i(x/|x|)]\!]$. With a slight abuse of notation, we will denote this function by $\varphi(|x|)f(x/|x|)$. We consider moreover functions φ which are 1 for $t = 1$ and smaller than 1 for $t < 1$. These competitors are, however, good only if $f|_{\partial B_1}$ is not too far from $Q\,[\![0]\!]$.

Of course, we can use competitors of the form

$$\sum_i \left[\!\!\left[v + \varphi(|x|)\left(f_i\left(\frac{x}{|x|}\right) - v\right)\right]\!\!\right], \tag{3.28}$$

which are still suitable if, roughly speaking,

(C) on ∂B_1, $f(x)$ is not too far from $Q\,[\![v]\!]$, i.e. from a point of multiplicity Q.

A rough strategy of the proof could then be the following. We approximate $f|_{\partial B_1}$ with a $\tilde{f} = [\![f_1]\!] + \ldots + [\![f_J]\!]$ decomposed into simpler $W^{1,2}$ functions f_j each of which satisfies (C). We interpolate on a corona $B_1 \setminus B_{1-\delta}$ between f and $\tilde{f}$, and we then use the competitors of the form (3.28) to extend $\tilde{f}$ to $B_{1-\delta}$. In fact, we shall use a variant of this idea, arguing by induction on Q.

Without loss of generality, we assume that

$$\mathrm{Dir}(g, \partial B_1) = 1. \tag{3.29}$$

Moreover, we recall the notation $|T|$ and $|f|$ introduced in Remark 1.11 and fix the following one for the translations:

$$\text{if } v \in \mathbb{R}^n,\ \text{then } \tau_v(T) := \sum_i [\![T_i - v]\!]\,,\ \text{for every } T = \sum_i [\![T_i]\!] \in \mathcal{A}_Q.$$

Step 1. Radial competitors. Let $\overline{g} = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$ be a mean for g, so that the Poincaré inequality in Proposition 2.12 holds, and assume that the diameter of $\overline{g}$ (see Definition 3.4) is smaller than a constant $M > 0$,

$$d(\overline{g}) \leq M. \tag{3.30}$$

Let $P = Q^{-1}\sum_{i=1}^{Q} P_i$ be the center of mass of $\overline{g}$ and consider $\tilde{f} = \tau_P \circ f$ and $h = \tau_P \circ g$. It is clear that $h = \tilde{f}|_{\partial B_1}$ and that $\overline{h} = \tau_P(\overline{g})$ is a mean for h. Moreover, by (3.30),

$$\left|\overline{h}\right|^2 = \sum_i |P_i - P|^2 \leq Q\,M^2.$$

So, using the Poincaré inequality, we get

$$\int_{\partial B_1} |h|^2 \leq 2\int_{\partial B_1} \mathcal{G}\left(h, \overline{h}\right)^2 + 2\int_{\partial B_1} \left|\overline{h}\right|^2 \leq C\,\mathrm{Dir}(g, \partial B_1) + C\,M^2 \overset{(3.29)}{\leq} C_M, \tag{3.31}$$

where C_M is a constant depending on M.

We consider the Q-function $\hat{f}(x) := \varphi(|x|)\,h\left(\frac{x}{|x|}\right)$, where φ is a $W^{1,2}([0,1])$ function with $\varphi(1) = 1$. From (3.31) and the chain-rule in Proposition 1.12, one can infer the following estimate:

$$\int_{B_1}\left|D\hat{f}\right|^2 = \left(\int_{\partial B_1}|h|^2\right)\int_0^1 \varphi'(r)^2\, r^{m-1}dr + \left(\int_{\partial B_1}|Dh|^2\right)\int_0^1 \varphi(r)^2\, r^{m-3}dr$$
$$\leq \int_0^1 \left(\varphi(r)^2\, r^{m-3} + C_M \varphi'(r)^2\, r^{m-1}\right)dr =: I(\varphi).$$

Since $\tau_{-P}(\hat{f})$ is a suitable competitor for f, one deduces that

$$\mathrm{Dir}(f, B_1) \leq \inf_{\substack{\varphi\in W^{1,2}([0,1])\\ \varphi(1)=1}} I(\varphi).$$

We notice that $I(1) = \frac{1}{m-2}$, as pointed out at the beginning of the section. On the other hand, $\varphi \equiv 1$ cannot be a minimum for I because it does not satisfy the corresponding Euler–Lagrange equation. So, there exists a constant $\gamma_M > 0$ such that

$$\mathrm{Dir}(f, B_1) \leq \inf_{\substack{\varphi\in W^{1,2}([0,1])\\ \varphi(1)=1}} I(\varphi) = \frac{1}{m-2} - 2\,\gamma_M. \tag{3.32}$$

In passing, we note that, when $Q = 1$, $d(T) = 0$ and hence this argument proves the first induction step of the proposition (which, however, can be proved in several other ways).

Step 2. Splitting procedure: the inductive step. Let Q be fixed and assume that the proposition holds for every $Q^* < Q$. Assume, moreover, that the diameter of $\overline{g}$ is bigger than a constant $M > 0$, which will be chosen later:

$$d(\overline{g}) > M$$

Under these hypotheses, we want to construct a suitable competitor for f. As pointed out at the beginning of the proof, the strategy is to decompose f in suitable pieces in order to apply the inductive hypothesis. To this aim:

(*a*) let $S = \sum_{j=1}^{J} k_j\, [\![Q_j]\!] \in \mathcal{A}_Q$ be given by Lemma 3.8 applied to $\varepsilon = \frac{1}{16}$ and $T = \overline{g}$, i.e. S such that

$$\beta\, M \leq \beta\, d(\overline{g}) < s(S) = \min_{i\neq j} |Q_i - Q_j|, \tag{3.33}$$

$$\mathcal{G}(S, \overline{g}) < \frac{s(S)}{16}, \tag{3.34}$$

where $\beta = \beta(1/16, Q)$ is the constant of Lemma 3.8;

(*b*) let $\vartheta : \mathcal{A}_Q \to B_{s(S)/8}(S)$ be given by Lemma 3.7 applied to $T = S$ and $r = \frac{s(S)}{8}$.

We define $h \in W^{1,2}(\partial B_{1-\eta})$ by $h\left((1-\eta)x\right) = \vartheta\left(g(x)\right)$, where $\eta > 0$ is a parameter to be fixed later, and take $\hat{h}$ a Dir-minimizing Q-function on $B_{1-\eta}$ with trace h. Then, we consider the following competitor,

$$\tilde{f} = \begin{cases} \hat{h} & \text{on} \quad B_{1-\eta} \\ \text{interpolation between } \hat{h} \text{ and } g \text{ as in Lemma 2.15,} \end{cases}$$

and we pass to estimate its Dirichlet energy.

By Proposition 3.6, since $\hat{h}$ has values in $\overline{B_{s(S)/8}(S)}$, $\hat{h}$ can be decomposed into two Dir-minimizing K and L-valued functions, with $K, L < Q$. So, by inductive hypothesis, there exists a positive constant ζ such that

$$\mathrm{Dir}\left(\hat{h}, B_{1-\eta}\right) \leq \left(\frac{1}{m-2} - \zeta\right)(1-\eta)\,\mathrm{Dir}(h, \partial B_{1-\eta}) \leq \left(\frac{1}{m-2} - \zeta\right)\mathrm{Dir}(g, \partial B_1), \tag{3.35}$$

where the last inequality follows from $\mathrm{Lip}(\vartheta) = 1$.

Therefore, combining (3.35) with Lemma 2.15, we can estimate

$$\mathrm{Dir}\left(\tilde{f}, B_1\right) \leq \left(\frac{1}{m-2} - \zeta + C\eta\right)\mathrm{Dir}(g, \partial B_1) + \frac{C}{\eta}\int_{\partial B_1} \mathcal{G}\big(g, \vartheta(g)\big)^2, \tag{3.36}$$

with $C = C(n, m, Q)$. Note that

$$\mathcal{G}\left(\overline{g}, \vartheta(g(x))\right) \leq \mathcal{G}\left(g(x), \overline{g}\right) \quad \text{for every } x \in \partial B_1,$$

because, by (3.34), $\vartheta(\overline{g}) = \overline{g}$. Hence, if we define

$$E := \left\{x \in \partial B_1 : g(x) \neq \vartheta(g(x))\right\} = \left\{x \in \partial B_1 : g(x) \notin \overline{B_{s(S)/8}(S)}\right\},$$

the last term in (3.36) can be estimated as follows:

$$\begin{aligned}
\int_{\partial B_1} \mathcal{G}\big(g, \vartheta(g)\big)^2 &= \int_E \mathcal{G}\big(g, \vartheta(g)\big)^2 \leq 2\int_E \left[\mathcal{G}\big(g, \overline{g}\big)^2 + \mathcal{G}\big(\overline{g}, \vartheta(g)\big)^2\right] \\
&\leq 4\int_E \mathcal{G}\big(g, \overline{g}\big)^2 dx \leq 4\,\|\mathcal{G}\big(g, \overline{g}\big)^2\|_{L^q}\,|E|^{(q-1)/q} \\
&\leq C\,\mathrm{Dir}(g, \partial B_1)\,|E|^{(q-1)/q} = C\,|E|^{(q-1)/q},
\end{aligned} \tag{3.37}$$

where the exponent q can be chosen to be $(m-1)/(m-3)$ if $m > 3$, otherwise any $q < \infty$ if $m = 3$.

We are left only with the estimate of $|E|$. Note that, for every $x \in E$,

$$\mathcal{G}(g(x), \overline{g}) \geq \mathcal{G}(g(x), S) - \mathcal{G}(\overline{g}, S) \overset{(3.34)}{\geq} \frac{s(S)}{8} - \frac{s(S)}{16} = \frac{s(S)}{16}.$$

So, we deduce that

$$|E| \leq \left|\left\{\mathcal{G}(g, \overline{g}) \geq \frac{s(S)}{16}\right\}\right| \leq \frac{C}{s(S)^2}\int_{\partial B_1} \mathcal{G}(g, \overline{g})^2 \overset{(3.33)}{\leq} \frac{C}{M^2}\mathrm{Dir}(g, \partial B_1). \tag{3.38}$$

Hence, collecting the bounds (3.35), (3.37) and (3.38), we conclude that

$$\mathrm{Dir}\big(\tilde{f}, B_1\big) \leq \left(\frac{1}{m-2} - \zeta + C\eta + \frac{C}{\eta M^\nu}\right), \tag{3.39}$$

where $C = C(n, m, Q)$ and $\nu = \nu(m)$.

Step 3. Conclusion. We are now ready to conclude. First of all, note that ζ is a fixed positive constant given by the inductive assumption that the proposition holds for $Q^* < Q$. We then choose η so that $C\eta < \zeta/2$ and M so large that $C/(\eta M^\nu) < \zeta/4$, where C is the constant in (3.39). Therefore, the constants M, γ_M and η depend only on n,m and Q. With this choice, Step 2 shows that

$$\mathrm{Dir}(f, B_1) \leq \mathrm{Dir}\big(\tilde{f}, B_1\big) \overset{(3.39)}{\leq} \left(\frac{1}{m-2} - \frac{\zeta}{4}\right)\mathrm{Dir}(g, \partial B_1), \quad \text{if } d(\overline{g}) > M;$$

whereas Step 1 implies

$$\mathrm{Dir}(f, B_1) \overset{(3.32)}{\leq} \left(\frac{1}{m-2} - 2\,\gamma_M\right) \mathrm{Dir}(g, \partial B_1), \quad \text{if } d(\overline{g}) \leq M.$$

This concludes the proof.

3.4. Frequency function

We next introduce Almgren's frequency function and prove his celebrated estimate.

DEFINITION 3.13 (The frequency function). Let f be a Dir-minimizing function, $x \in \Omega$ and $0 < r < \mathrm{dist}(x, \partial\Omega)$. We define the functions

$$D_{x,f}(r) = \int_{B_r(x)} |Df|^2, \quad H_{x,f}(r) = \int_{\partial B_r} |f|^2 \quad \text{and} \quad I_{x,f}(r) = \frac{r D_{x,f}(r)}{H_{x,f}(r)}. \tag{3.40}$$

$I_{x,f}$ is called the *frequency function*.

When x and f are clear from the context, we will often use the shorthand notation $D(r)$, $H(r)$ and $I(r)$.

REMARK 3.14. Note that, by Theorem 3.9, $|f|^2$ is a continuous function. Therefore, $H_{x,f}(r)$ is a well-defined quantity for every r. Moreover, if $H_{x,f}(r) = 0$, then, by minimality, $f|_{B_r(x)} \equiv 0$. So, except for this case, $I_{x,f}(r)$ is always well defined.

THEOREM 3.15. *Let f be* Dir*-minimizing and $x \in \Omega$. Either there exists ϱ such that $f|_{B_\varrho(x)} \equiv 0$ or $I_{x,f}(r)$ is an absolutely continuous nondecreasing positive function on $]0, dist(x, \partial\Omega)[$.*

A simple corollary of Theorem 3.15 is the existence of the limit

$$I_{x,f}(0) = \lim_{r \to 0} I_{x,f}(r),$$

when the frequency function is defined for every r. The same computations as those in Theorem 3.15 yield the following two corollaries.

COROLLARY 3.16. *Let f be* Dir*-minimizing in B_ϱ. Then, $I_{0,f}(r) \equiv \alpha$ if and only if f is α-homogeneous, i.e.*

$$f(y) = |y|^\alpha f\left(\frac{y\,\varrho}{|y|}\right). \tag{3.41}$$

REMARK 3.17. In (3.41), with a slight abuse of notation, we use the following convention (already adopted in Subsection 3.3.3). If β is a scalar function and $f = \sum_i [\![f_i]\!]$ a Q-valued function, we denote by βf the function $\sum_i [\![\beta\, f_i]\!]$.

COROLLARY 3.18. *Let f be* Dir*-minimizing in B_ϱ. Let $0 < r < t \leq \varrho$ and suppose that $I_{0,f}(r) = I(r)$ is defined for every r (i.e. $H(r) \neq 0$ for every r). Then, the following estimates hold:*

(i) *for almost every $r \leq s \leq t$,*

$$\left.\frac{d}{d\tau}\right|_{\tau=s} \left[\ln\left(\frac{H(\tau)}{\tau^{m-1}}\right)\right] = \frac{2\,I(r)}{r} \tag{3.42}$$

and

$$\left(\frac{r}{t}\right)^{2I(t)} \frac{H(t)}{t^{m-1}} \leq \frac{H(r)}{r^{m-1}} \leq \left(\frac{r}{t}\right)^{2I(r)} \frac{H(t)}{t^{m-1}}; \tag{3.43}$$

(ii) if $I(t) > 0$, *then*

$$\frac{I(r)}{I(t)}\left(\frac{r}{t}\right)^{2I(t)}\frac{D(t)}{t^{m-2}} \le \frac{D(r)}{r^{m-2}} \le \left(\frac{r}{t}\right)^{2I(r)}\frac{D(t)}{t^{m-2}}. \tag{3.44}$$

3.4.1. Proof of Theorem 3.15. We assume, without loss of generality, that $x = 0$. D is an absolutely continuous function and

$$D'(r) = \int_{\partial B_r} |Df|^2 \quad \text{for a.e. } r. \tag{3.45}$$

As for $H(r)$, note that $|f|$ is the composition of f with a Lipschitz function, and therefore belongs to $W^{1,2}$. It follows that $|f|^2 \in W^{1,1}$ and hence that $H \in W^{1,1}$.

In order to compute H', note that the distributional derivative of $|f|^2$ coincides with the approximate differential a.e. Therefore, Proposition 2.8 justifies (for a.e. r) the following computation:

$$\begin{aligned}
H'(r) &= \frac{d}{dr}\int_{\partial B_1} r^{m-1}|f(ry)|^2 dy \\
&= (m-1)r^{m-2}\int_{\partial B_1}|f(ry)|^2 dy + \int_{\partial B_1} r^{m-1}\frac{\partial}{\partial r}|f(ry)|^2\,dy \\
&= \frac{m-1}{r}\int_{\partial B_r}|f|^2 + 2\int_{\partial B_r}\sum_i \langle \partial_\nu f_i, f_i\rangle.
\end{aligned}$$

Using (3.6), we then conclude

$$H'(r) = \frac{m-1}{r}H(r) + 2D(r). \tag{3.46}$$

Note, in passing, that, since H and D are continuous, $H \in C^1$ and (3.46) holds pointwise.

If $H(r) = 0$ for some r, then, as already remarked, $f|_{B_r} \equiv 0$. In the opposite case, we conclude that $I \in C \cap W^{1,1}_{loc}$. To show that I is nondecreasing, it suffices to compute its derivative a.e. and prove that it is nonnegative. Using (3.45) and (3.46), we infer that

$$\begin{aligned}
I'(r) &= \frac{D(r)}{H(r)} + \frac{r\,D'(r)}{H(r)} - r\,D(r)\frac{H'(r)}{H(r)^2} \\
&= \frac{D(r)}{H(r)} + \frac{r\,D'(r)}{H(r)} - (m-1)\frac{D(r)}{H(r)} - 2r\frac{D(r)^2}{H(r)^2} \\
&= \frac{(2-m)D(r) + r\,D'(r)}{H(r)} - 2\,r\,\frac{D(r)^2}{H(r)^2} \quad \text{for a.e. } r. \qquad (3.47)
\end{aligned}$$

Recalling (3.6) and (3.7) and using the Cauchy–Schwartz inequality, from (3.47) we conclude that, for almost every r,

(3.48)

$$I'(r) = \frac{r}{H(r)^2}\left\{\int_{\partial B_r(x)}|\partial_\nu f|^2 \cdot \int_{\partial B_r(x)}|f|^2 - \left(\int_{\partial B_r(x)}\sum_i\langle\partial_\nu f_i, f_i\rangle\right)^2\right\} \ge 0.$$

3.4.2. Proof of Corollary 3.16. Let f be a Dir-minimizing Q-valued function. Then, $I(r) \equiv \alpha$ if and only if equality occurs in (3.48) for almost every r, i.e. if and only if there exist constants λ_r such that

$$f_i(y) = \lambda_r \, \partial_\nu f_i(y), \text{ for almost every } r \text{ and a.e. } y \text{ with } |y| = r. \tag{3.49}$$

Recalling (3.7) and using (3.49), we infer that, for such r,

$$\alpha = I(r) = \frac{r\,D(r)}{H(r)} = \frac{r\int_{\partial B_r}\sum_i \langle \partial_\nu f_i, f_i\rangle}{\int_{\partial B_r}\sum_i |f_i|^2} \overset{(3.49)}{=} \frac{r\lambda_r \int_{\partial B_r}\sum_i |f_i|^2}{\int_{\partial B_r}\sum_i |f_i|^2} = r\lambda_r.$$

So, summarizing, $I(r) \equiv \alpha$ if and only if

$$f_i(y) = \frac{\alpha}{|y|}\,\partial_\nu f_i(y) \quad \text{for almost every } y. \tag{3.50}$$

Let us assume that (3.41) holds. Then, (3.50) is clearly satisfied and, hence, $I(r) \equiv \alpha$. On the other hand, assuming that the frequency is constant, we now prove (3.41). To this aim, let $\sigma_y = \{r\,y : 0 \le r \le \varrho\}$ be the radius passing through $y \in \partial B_1$. Note that, for almost every y, $f|_{\sigma_y} \in W^{1,2}_{loc}$; so, for those y, recalling the $W^{1,2}$-selection in Proposition 1.2, we can write $f|_{\sigma_y} = \sum_i [\![f_i|_{\sigma_y}]\!]$, where $f_i|_{\sigma_y} : [0,\varrho] \to \mathbb{R}^n$ are $W^{1,2}_{loc}$ functions. By (3.50), we infer that $f_i|_{\sigma_y}$ solves the ordinary differential equation

$$(f_i|_{\sigma_y})'(r) = \frac{\alpha}{r}\, f_i|_{\sigma_y}(r), \quad \text{for a.e. } r.$$

Hence, for a.e. $y \in \partial B_1$ and for every $r \in (0,\varrho]$, $f_i|_{\sigma_y}(r) = r^\alpha\, f\,(y)$, thus concluding (3.41).

3.4.3. Proof of Corollary 3.18. The proof is a straightforward consequence of equation (3.46). Indeed, (3.46) implies, for almost every s,

$$\frac{d}{d\tau}\Big|_{\tau=s}\left(\frac{H(\tau)}{\tau^{m-1}}\right) = \frac{H'(s)}{s^{m-1}} - \frac{(m-1)\,H(s)}{s^m} \overset{(3.46)}{=} \frac{2\,D(s)}{s^{m-1}},$$

which, in turn, gives (3.42). Integrating (3.42) and using the monotonicity of I, one obtains (3.43). Finally, (3.44) follows from (3.43), using the identity $I(r) = \frac{r\,D(r)}{H(r)}$.

3.5. Blow-up of Dir-minimizing Q-valued functions

Let f be a Q-function and assume $f(y) = Q\,[\![0]\!]$ and $\mathrm{Dir}(f, B_\varrho(y)) > 0$ for every ϱ. We define the blow-ups of f at y in the following way,

$$f_{y,\varrho}(x) = \frac{\varrho^{\frac{m-2}{2}}\, f(\varrho\,x + y)}{\sqrt{\mathrm{Dir}(f, B_\varrho(y))}}. \tag{3.51}$$

The main result of this section is the convergence of blow-ups of Dir-minimizing functions to homogeneous Dir-minimizing functions, which we call *tangent functions.*

To simplify the notation, we will not display the subscript y in $f_{y,\rho}$ when y is the origin.

THEOREM 3.19. *Let $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ be* Dir-*minimizing. Assume $f(0) = Q\,[\![0]\!]$ and $\mathrm{Dir}(f, B_\varrho) > 0$ for every $\varrho \le 1$. Then, for any sequence $\{f_{\varrho_k}\}$ with $\rho_k \downarrow 0$, a subsequence, not relabeled, converges locally uniformly to a function $g : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ with the following properties:*

(*a*) $\mathrm{Dir}(g, B_1) = 1$ *and $g|_\Omega$ is* Dir-*minimizing for any bounded Ω;*

(*b*) $g(x) = |x|^\alpha \, g\left(\frac{x}{|x|}\right)$, *where* $\alpha = I_{0,f}(0) > 0$ *is the frequency of* f *at* 0.

Theorem 3.19 is a direct consequence of the estimate on the frequency function and of the following convergence result for Dir-minimizing functions.

PROPOSITION 3.20. *Let* $f_k \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *be* Dir*-minimizing* Q*-functions weakly converging to* f*. Then, for every open* $\Omega' \subset\subset \Omega$, $f|_{\Omega'}$ *is* Dir*-minimizing and it holds moreover that* $\mathrm{Dir}(f, \Omega') = \lim_k \mathrm{Dir}(f_k, \Omega')$.

REMARK 3.21. In fact, a suitable modification of our proof shows that the property of being Dir-minimizing holds on Ω. However, we never need this stronger property in the sequel.

Assuming Proposition 3.20, we prove Theorem 3.19.

PROOF OF THEOREM 3.19. We show later $I_{0,f}(0) > 0$. We consider any ball B_N of radius N centered at 0. It follows from estimate (3.44) and $I_{0,f}(0) > 0$ that $\mathrm{Dir}(f_\varrho, B_N)$ is uniformly bounded in ϱ. Hence, the functions f_ϱ are all Dir-minimizing and Theorem 3.9 implies that the f_{ϱ_k}'s are locally equi-Hölder continuous. Since $f_\varrho(0) = Q\,[\![0]\!]$, the f_ϱ's are also locally uniformly bounded and the Ascoli–Arzelà theorem yields a subsequence (not relabeled) converging uniformly on compact subsets of $\mathbb{R}^m$ to a continuous Q-valued function g. This implies easily the weak convergence (as defined in Definition 2.9), so we can apply Proposition 3.20 and conclude (*a*) (note that $\mathrm{Dir}(f_\varrho, B_1) = 1$ for every ϱ). Observe next that, for every $r > 0$,

$$I_{0,g}(r) = \frac{r\,\mathrm{Dir}(g, B_r)}{\int_{\partial B_r} |g|^2} = \lim_{\varrho\to 0} \frac{r\,\mathrm{Dir}(f_\varrho, B_r)}{\int_{\partial B_r} |f_\varrho|^2} = \lim_{\varrho\to 0} \frac{\varrho\, r\,\mathrm{Dir}(f, B_{\varrho\, r})}{\int_{\partial B_{\varrho\, r}} |f|^2} = I_{0,f}(0). \tag{3.52}$$

So, (*b*) follows from Corollary 3.16, once we have shown that $I_{0,f}(0) > 0$. Since $f(0) = Q\,[\![0]\!]$, $H(r) \leq C r D(r)$, for some constant C. Indeed, assume w.l.o.g. $r = 1$. If $|x| = 1$, then $|f(x)|^2 \leq 2\mathcal{G}(f(x/2), f(0))^2 + 2\mathcal{G}(f(x), f(x/2))^2 \leq CD(1) + 2\int_{1/2}^1 |Df(\tau x)|^2 d\tau$ (the last step follows from Theorem 3.9). Integrating the inequality in $x \in \partial B_1$ we conclude $H(1) \leq CD(1)$. □

PROOF OF PROPOSITION 3.20. We consider the case of $\Omega = B_1$: the general case is a routine modification of the arguments (and, besides, we never need it in the sequel). Since the f_k's are Dir-minimizing and, hence, locally Hölder equi-continuous, and since the f_k's converge strongly in L^2 to f, they actually converge to f uniformly on compact sets. Set $D_r = \liminf_k \mathrm{Dir}(f_k, B_r)$ and assume by contradiction that $f|_{B_r}$ is not Dir-minimizing or $\mathrm{Dir}(f, B_r) < D_r$ for some $r < 1$. Under this assumption, we can find $r_0 > 0$ such that, for every $r \geq r_0$, there exist a $g \in W^{1,2}(B_r, \mathcal{A}_Q)$ with

$$g|_{\partial B_r} = f|_{\partial B_r} \quad \text{and} \quad \gamma_r := D_r - \mathrm{Dir}(g, B_r) > 0. \tag{3.53}$$

Fatou's Lemma implies that $\liminf_k \mathrm{Dir}(f_k, \partial B_r)$ is finite for almost every r,

$$\int_0^1 \liminf_{k\to+\infty} \mathrm{Dir}(f_k, \partial B_r)\, dr \leq \liminf_{k\to+\infty} \int_0^1 \mathrm{Dir}(f_k, \partial B_r)\, dr \leq C < +\infty.$$

Passing, if necessary, to a subsequence, we can fix a radius $r \geq r_0$ such that

$$\mathrm{Dir}(f, \partial B_r) \leq \lim_{k\to+\infty} \mathrm{Dir}(f_k, \partial B_r) \leq M < +\infty. \tag{3.54}$$

We now show that (3.53) contradicts the minimality of f_k in B_r for large n. Let, indeed, $0<\delta<r/2$ to be fixed later and consider the functions $\tilde{f}_k$ on B_r defined by

$$\tilde{f}_k(x)=\begin{cases} g\left(\frac{r\,x}{r-\delta}\right) & \text{for } \; x\in B_{r-\delta},\\ h_k(x) & \text{for } \; x\in B_r\setminus B_{r-\delta},\end{cases}$$

where the h_k's are the interpolations provided by Lemma 2.15 between $f_k\in W^{1,2}(\partial B_r,\mathcal{A}_Q)$ and $g\left(\frac{r\,x}{r-\delta}\right)\in W^{1,2}(B_{r-\delta},\mathcal{A}_Q)$. We claim that, for large k, the functions $\tilde{f}_k$ have smaller Dirichlet energy than f_k, thus contrasting the minimizing property of f_k, and concluding the proof. Indeed, recalling the estimate in Lemma 2.15, we have

$$\begin{aligned}\mathrm{Dir}\big(\tilde{f}_k,B_r\big)&\le \mathrm{Dir}\big(\tilde{f}_k,B_{r-\delta}\big)+C\delta\Big[\mathrm{Dir}\big(\tilde{f}_k,\partial B_{r-\delta}\big)+\mathrm{Dir}\big(f_k,\partial B_r\big)\Big]+\frac{C}{\delta}\int_{\partial B_r}\mathcal{G}\big(f_k,\tilde{f}_k\big)^2\\ &\le \mathrm{Dir}(g,B_r)+C\,\delta\,\mathrm{Dir}(g,\partial B_r)+C\,\delta\,\mathrm{Dir}(f_k,\partial B_r)+\frac{C}{\delta}\int_{\partial B_r}\mathcal{G}(f_k,g)^2.\end{aligned}$$

Choose now δ such that $4\,C\,\delta\,(M+1)\le\gamma_r$, where M and γ_r are the constants in (3.54) and (3.53). Using the uniform convergence of f_k to f, we conclude, for k large enough,

$$\begin{aligned}\mathrm{Dir}\big(\tilde{f}_k,B_r\big)&\stackrel{(3.53),\,(3.54)}{\le} D_r-\gamma_r+C\,\delta\,M+C\,\delta\,(M+1)+\frac{C}{\delta}\int_{\partial B_r}\mathcal{G}(f_k,f)^2,\\ &\le D_r-\frac{\gamma_r}{2}+\frac{C}{\delta}\int_{\partial B_r}\mathcal{G}(f_k,f)^2<D_r-\frac{\gamma_r}{4}.\end{aligned}$$

This gives the contradiction. □

3.6. Estimate of the singular set

In this section we estimate the Hausdorff dimension of the singular set of Dir-minimizing Q-valued functions as in Theorem 0.11. The main point of the proof is contained in Proposition 3.22, estimating the size of the set of singular points with multiplicity Q. Theorem 0.11 follows then by an easy induction argument on Q.

Proposition 3.22. *Let Ω be connected and $f\in W^{1,2}(\Omega,\mathcal{A}_Q(\mathbb{R}^n))$ be* Dir*-minimizing. Then, either $f=Q\,[\![\zeta]\!]$ with $\zeta:\Omega\to\mathbb{R}^n$ harmonic in Ω, or the set*

$$\Sigma_{Q,f}=\{x\in\Omega\,:\,f(x)=Q\,[\![y]\!]\,,\ \textit{for some } y\in\mathbb{R}^n\}$$

(which is relatively closed in Ω) has Hausdorff dimension at most $m-2$ and it is locally finite for $m=2$.

We will make a frequent use of the function $\sigma:\Omega\to\mathbb{N}$ given by the formula

$$\sigma(x)=\mathrm{card}(\mathrm{supp}\,f(x)). \tag{3.55}$$

Note that σ is lower semicontinuous because f is continuous. This implies, in turn, that $\Sigma_{Q,f}$ is closed.

3.6.1. Preparatory Lemmas. We first state and prove two lemmas which will be used in the proof of Proposition 3.22. The first reduces Proposition 3.22 to the case where all points of multiplicity Q are of the form $Q[\![0]\!]$. In order to state it, we introduce the map $\boldsymbol{\eta}:\mathcal{A}_Q(\mathbb{R}^n)\to\mathbb{R}^n$ which takes each measure $T=\sum_i [\![P_i]\!]$ to its center of mass,

$$\boldsymbol{\eta}(T)=\frac{\sum_i P_i}{Q}.$$

LEMMA 3.23. *Let $f:\Omega\to\mathcal{A}_Q(\mathbb{R}^n)$ be* Dir*-minimizing. Then,*

(a) *the function $\boldsymbol{\eta}\circ f:\Omega\to\mathbb{R}^n$ is harmonic;*
(b) *for every $\zeta:\Omega\to\mathbb{R}^n$ harmonic, $g:=\sum_i [\![f_i+\zeta]\!]$ is as well* Dir*-minimizing.*

PROOF. The proof of (a) follows from plugging $\psi(x,u)=\zeta(x)\in C_c^\infty(\Omega,\mathbb{R}^n)$ in the variations formula (3.5) of Proposition 3.1. Indeed, from the chain-rule (1.17), one infers easily that $Q\,D(\boldsymbol{\eta}\circ f)=\sum_i Df_i$ and hence, from (3.5) we get $\int\langle D(\boldsymbol{\eta}\circ f):D\zeta\rangle=0$. The arbitrariness of $\zeta\in C_c^\infty(\Omega,\mathbb{R}^n)$ gives (a).

To show (b), let h be any Q-valued function with $h|_{\partial\Omega}=f|_{\partial\Omega}$: we need to verify that, if $\tilde h:=\sum_i [\![h_i+\zeta]\!]$, then $\mathrm{Dir}(g,\Omega)\le\mathrm{Dir}(\tilde h,\Omega)$. From Almgren's form of the Dirichlet energy (see (2.16)), we get

$$\begin{aligned}
\mathrm{Dir}(g,\Omega)&=\int_\Omega\sum_{i,j}|\partial_j g_i|^2=\int_\Omega\sum_{i,j}\left\{|\partial_j f_i|^2+|\partial_j\zeta|^2+2\,\partial_j f_i\,\partial_j\zeta\right\}\\
&\overset{\text{min. of }f}{\le}\int_\Omega\sum_{i,j}\left\{|\partial_j h_i|^2+|\partial_j\zeta|^2\right\}+2Q\int_\Omega D(\boldsymbol{\eta}\circ f)\cdot D\zeta\\
&=\mathrm{Dir}(\tilde h,\Omega)+2Q\int_\Omega\left\{D(\boldsymbol{\eta}\circ f)-D(\boldsymbol{\eta}\circ h)\right\}\cdot D\zeta. \qquad (3.56)
\end{aligned}$$

Since $\boldsymbol{\eta}\circ f$ and $\boldsymbol{\eta}\circ h$ have the same trace on $\partial\Omega$ and ζ is harmonic, the last integral in (3.56) vanishes. □

The second lemma characterizes the blow-ups of homogeneous functions and is the starting point of the reduction argument used in the proof of Proposition 3.22.

LEMMA 3.24 (Cylindrical blow-up). *Let $g:B_1\to\mathcal{A}_Q(\mathbb{R}^n)$ be an α-homogeneous and* Dir*-minimizing function with $\mathrm{Dir}(g,B_1)>0$ and set $\beta=I_{z,g}(0)$. Suppose, moreover, that $g(z)=Q[\![0]\!]$ for $z=e_1/2$. Then, the tangent functions h to g at z are β-homogeneous with $\mathrm{Dir}(h,B_1)=1$ and satisfy:*

(a) *$h(s\,e_1)=Q[\![0]\!]$ for every $s\in\mathbb{R}$;*
(b) *$h(x_1,x_2,\ldots,x_m)=\hat h(x_2,\ldots,x_m)$, where $\hat h:\mathbb{R}^{m-1}\to\mathcal{A}_Q(\mathbb{R}^n)$ is* Dir*-minimizing on any bounded open subset of $\mathbb{R}^{m-1}$.*

PROOF. The first part of the proof follows from Theorem 3.19, while (a) is straightforward. We need only to verify (b). To simplify notations, we pose $x'=(0,x_2,\ldots,x_m)$: we show that $h(x')=h(s\,e_1+x')$ for every s and x'. This is an easy consequence of the homogeneity of both g and h. Recall that h is the local uniform limit of g_{z,ϱ_k} for some $\rho_k\downarrow 0$ and set $C_k:=\mathrm{Dir}(g,B_{\varrho_k}(z))^{-1/2}$, $\beta=I_{z,g}(0)$

and $\lambda_k := \frac{1}{1-2\varrho_k\, s}$, where $z = e_1/2$. Hence, we have

$$h(s\,e_1 + x') \overset{\text{hom. of } h}{=} \lim_{k\uparrow\infty} C_k\, \frac{g_{z,\varrho_k}\,(s\lambda_k\, e_1 + \lambda_k x')}{\lambda_k^{\beta}} = \lim_{k\uparrow\infty} C_k\, \frac{g\,(\lambda_k\, z + \lambda_k\, \varrho_k\, x')}{\lambda_k^{\beta}}$$
$$\overset{\text{hom. of } g}{=} \lim_{\varrho\to 0} C_k\, \frac{{\lambda_k}^{\alpha}\, g_{z,\varrho_k}\,(x')}{\lambda_k^{\beta}} = h(x'),$$

where we used $\lambda_k z + \lambda_k\, \varrho_k\, x' = z + s\lambda_k\, \varrho_k\, e_1 + \lambda_k\, \varrho_k\, x'$ and $\lim_{k\uparrow\infty} \lambda_k = 1$.

The minimizing property of $\hat{h}$ is a consequence of the Dir-minimality of h. It suffices to show it on every ball $B \subset \mathbb{R}^{m-1}$ for which $\hat{h}|_{\partial B} \in W^{1,2}$. To fix ideas, assume B to be centered at 0 and to have radius R. Assume the existence of a competitor $\tilde{h} \in W^{1,2}(B)$ such that $\mathrm{Dir}(\tilde{h}, B) \leq D(\hat{h}, B) - \gamma$ and $\tilde{h}|_{\partial B} = \hat{h}|_{\partial B}$. We now construct a competitor h' for h on a cylinder $C_L = [-L, L] \times B_R$. First of all we define

$$h'(x_1, x_2, \ldots, x_n) = \tilde{h}(x_2, \ldots, x_n) \quad \text{for } |x_1| \leq L - 1.$$

It remains to "fill in" the two cylinders $C_L^1 =]L-1, L[\times B_R$ and $C_L^2 =]-L, -(L-1)[\times B_R$. Let us consider the first cylinder. We need to define h' in C_L^1 in such a way that $h' = h$ on the lateral surface $]L-1, L[\times \partial B_R$ and on the upper face $\{L\} \times B_R$ and $h' = \tilde{h}$ on the lower face $\{L-1\} \times B_R$. Now, since the cylinder C_L^1 is biLipschitz to a unit ball, recalling Corollary 2.16, this can be done with a $W^{1,2}$ map.

Denote by u and v the upper and lower "filling" maps in the case $L = 1$ By the x_1-invariance of our construction, the maps

$$u_L(x_1, \ldots, x_m) := u(x_1 - L, \ldots, x_m) \quad \text{and} \qquad v_L(x_1, \ldots, x_m) = u(x_1 + L, \ldots, x_m)$$

can be taken as filling maps for any $L \geq 1$. Therefore, we can estimate

$$\mathrm{Dir}(h', C_L) - D(h, C_L) \leq \left(\mathrm{Dir}\left(h', C_L^1 \cup C_L^2\right) - \mathrm{Dir}\left(h, C_L^1 \cup C_L^2\right)\right) - 2\,(L-1)\,\gamma$$
$$=: \Lambda - 2\,(L-1)\,\gamma,$$

where Λ is a constant independent of L. Therefore, for a sufficiently large L, we have $D(h', C_L) < D(h, C_L)$ contradicting the minimality of h in C_L. □

3.6.2. Proof of Proposition 3.22. With the help of these two lemmas we conclude the proof of Proposition 3.22. First of all we notice that, by Lemma 3.23, it suffices to consider Dir-minimizing function f such that $\boldsymbol{\eta} \circ f \equiv 0$. Under this assumption, it follows that $\Sigma_{Q,f} = \{x : f(x) = Q\,[\![0]\!]\}$. Now we divide the proof into two parts, being the case $m = 2$ slightly different from the others.

The planar case $m = 2$. We prove that, except for the case where all sheets collapse, $\Sigma_{Q,f}$ consists of isolated points. Without loss of generality, let $0 \in \Sigma_{Q,f}$ and assume the existence of $r_0 > 0$ such that $\mathrm{Dir}(f, B_r) > 0$ for every $r \leq r_0$ (note that, when we are not in this case, then $f \equiv Q\,[\![0]\!]$ in a neighborhood of 0). Suppose by contradiction that 0 is not an isolated point in $\Sigma_{Q,f}$, i.e. there exist $x_k \to 0$ such that $f(x_k) = Q\,[\![0]\!]$. By Theorem 3.19, the blow-ups $f_{|x_k|}$ converge uniformly, up to a subsequence, to some homogeneous Dir-minimizing function g, with $\mathrm{Dir}(g, B_1) = 1$ and $\boldsymbol{\eta} \circ g \equiv 0$. Moreover, since $f(x_k)$ are Q-multiplicity points, we deduce that there exists $w \in \mathbb{S}^1$ such that $g(w) = Q\,[\![0]\!]$. Up to rotations, we can assume that $w = e_1$. Considering the blowup of g in the point $e_1/2$, by Lemma 3.24, we find a new tangent function h with the property that $h(0, x_2) = \hat{h}(x_2)$ for some function $\hat{h} : \mathbb{R} \to \mathcal{A}_Q$ which is Dir-minimizing on every interval. Moreover, since $\mathrm{Dir}(h, B_1) = 1$, clearly $\mathrm{Dir}\big(\hat{h}, I\big) > 0$, where $I = [-1, 1]$. Note also that $\boldsymbol{\eta} \circ \hat{h} \equiv 0$ and $\hat{h}(0) = Q\,[\![0]\!]$. From the 1-d selection criterion in Proposition 1.5, this is clearly a contradiction. Indeed, by a simple comparison argument, it is easily seen that every Dir-minimizing 1-d function $\hat{h}$ is an affine function of the form $\hat{h}(x) = \sum_i [\![L_i(x)]\!]$ with the property that either $L_i(x) \neq L_j(x)$ for every x or $L_i(x) = L_j(x)$ for every x. Since $\hat{h}(0) = Q\,[\![0]\!]$, we would conclude that $\hat{h} = Q\,[\![L]\!]$ for some linear L. On the other hand, by $\boldsymbol{\eta} \circ \hat{h} \equiv 0$ we would conclude $L = 0$, contradicting $\mathrm{Dir}(\hat{h}, I) > 0$.

We conclude that, if $x \in \Sigma_{Q,f}$, either x is isolated, or $U \subset \Sigma_{Q,f}$ for some neighborhood of x. Since Ω is connected, we conclude that, either $\Sigma_{Q,f}$ consists of isolated points, or $\Sigma_{Q,f} = \Omega$.

The case $m \geq 3$. In this case we use the so-called Federer's reduction argument (following closely the exposition in Appendix A of [**Sim83**]). We denote by $\mathcal{H}^t$ the Hausdorff t-dimensional measure and by $\mathcal{H}^t_\infty$ the Hausdorff pre-measure defined by

$$\mathcal{H}^t_\infty(A) = \inf\left\{\sum_{k\in\mathbb{N}} \mathrm{diam}(E_k)^t \;:\; A \subset \cup_{k\in\mathbb{N}} E_k\right\}. \tag{3.57}$$

We use this simple property of the Hausdorff pre-measures $\mathcal{H}^t_\infty$: if K_l are compact sets converging to K in the sense of Hausdorff, then

$$\limsup_{l\to+\infty} \mathcal{H}^t_\infty(K_l) \leq \mathcal{H}^t_\infty(K). \tag{3.58}$$

To prove (3.58), note first that the infimum on (3.57) can be taken over open coverings. Next, given an open covering of K, use its compactness to find a finite subcovering and the convergence of K_l to conclude that it covers K_l for l large enough (see the proof of Theorem A.4 in [**Sim83**] for more details).

Step 1. *Let $t > 0$. If $\mathcal{H}^t_\infty\left(\partial\Sigma_{Q,f} \cap \Omega\right) > 0$, then there exists a function $g \in W^{1,2}(B_1, \mathcal{A}_Q)$ with the following properties:*

(a_1) *g is a homogeneous* Dir*-minimizing function with* $\mathrm{Dir}(g, B_1) = 1$*;*
(b_1) *$\boldsymbol{\eta} \circ g \equiv 0$;*
(c_1) *$\mathcal{H}^t_\infty\left(\Sigma_{Q,g}\right) > 0$.*

We note that $\mathcal{H}^t_\infty$-almost every point $x \in \Sigma_{Q,f}$ is a point of positive t density (see Theorem 3.6 in [**Sim83**]), i.e.

$$\limsup_{r\to 0} \frac{\mathcal{H}^t_\infty\left(\partial\Sigma_{Q,f} \cap \Omega \cap B_r(x)\right)}{r^t} > 0.$$

So, since $\mathcal{H}^t_\infty\left(\partial\Sigma_{Q,f} \cap \Omega\right) > 0$, from Theorem 3.19 we conclude the existence of a point $x \in \Sigma_{Q,f}$ and a sequence of radii $\varrho_k \to 0$ such that the blow-ups $f_{x,2\varrho_k}$ converge uniformly to a function g satisfying (a_1) and (b_1), and

$$\limsup_{k\to+\infty} \frac{\mathcal{H}^t_\infty\left(\Sigma_{Q,f} \cap B_{\varrho_k}(x)\right)}{{\varrho_k}^t} > 0. \tag{3.59}$$

From the uniform convergence of $f_{x,2\varrho_k}$ to g, we deduce easily that, up to subsequence, the compact sets $K_k = \overline{B_{\frac{1}{2}}} \cap \Sigma_{Q,f_{x,2\varrho_k}}$ converge in the sense of Hausdorff to a compact set $K \subseteq \Sigma_{Q,g}$. So, from the semicontinuity property (3.58), we infer

(c_1),

$$\mathcal{H}^t_\infty(\Sigma_{Q,g}) \geq \mathcal{H}^t_\infty(K) \geq \limsup_{k\to+\infty} \mathcal{H}^t_\infty(K_k) \geq \limsup_{k\to+\infty} \mathcal{H}^t_\infty(B_{\frac{1}{2}} \cap \Sigma_{Q,f_{x,2\varrho_k}})$$
$$= \limsup_{k\to+\infty} \frac{\mathcal{H}^t_\infty(\Sigma_{Q,f} \cap B_{\varrho_k}(x))}{{\varrho_k}^t} \overset{(3.59)}{>} 0.$$

Step 2. Let $t > 0$ and g satisfying (a_1)-(c_1) of Step 1. Suppose, moreover, that there exists $1 \leq l \leq m-2$, with $l-1 < t$, such that

$$g(x) = \hat{g}(x_l, \ldots, x_m). \tag{3.60}$$

Then, there exists a function $h \in W^{1,2}(B_1, \mathcal{A}_Q)$ with the following properties:

- (a_2) *h is a homogeneous* Dir*-minimizing function with* $\mathrm{Dir}(h, B_1) = 1$*;*
- (b_2) $\boldsymbol{\eta} \circ h \equiv 0$*;*
- (c_2) $\mathcal{H}^t_\infty(\Sigma_{Q,h}) > 0$*;*
- (d_2) $h(x) = \hat{h}(x_{l+1}, \ldots, x_m)$*.*

We notice that $\mathcal{H}^t_\infty\left(\mathbb{R}^{l-1} \times \{0\}\right) = 0$, being $t > l-1$. So, since $\mathcal{H}^t_\infty(\Sigma_{Q,g}) > 0$, we can find a point $0 \neq x = (0, \ldots, 0, x_l, \ldots, x_m) \in \Sigma_{Q,g}$ of positive density for $\mathcal{H}^t_\infty \llcorner \Sigma_{Q,g}$. By the same argument of Step 1, we can blow-up at x obtaining a function h with properties (a_2), (b_2) and (c_2). Moreover, using Lemma 3.24, one immediately infers (d_2).

Step 3. Conclusion: Federer's reduction argument.

Let now $t > m-2$ and suppose $\mathcal{H}^t(\partial\Sigma_{Q,f} \cap \Omega) > 0$. Then, up to rotations, we may apply Step 1 once and Step 2 repeatedly until we end up with a Dir-minimizing function h with properties (a_2)-(c_2) and depending only on two variables, $h(x) = \hat{h}(x_1, x_2)$. This implies that $\hat{h}$ is a planar Q-valued Dir-minimizing function such that $\boldsymbol{\eta} \circ \hat{h} \equiv 0$, $\mathrm{Dir}(\hat{h}, B_1) = 1$ and $\mathcal{H}^{t-m+2}\left(\Sigma_{Q,\hat{h}}\right) > 0$. As shown in the proof of the planar case, this is impossible, since $t - m + 2 > 0$ and the singularities are at most countable. So, we deduce that $\mathcal{H}^t(\partial\Sigma_{Q,f} \cap \Omega) = 0$ and hence either $\Sigma_{Q,f} = \partial\Sigma_{Q,f} \cap \Omega$ or $\Sigma_{Q,f} = \Omega$, thus concluding the proof.

3.6.3. Proof of Theorem 0.11. Let σ be as in (3.55). It is then clear that, if x is a regular point, then σ is continuous at x.

On the other hand, let x be a point of continuity of σ and write $f(x) = \sum_{j=1}^J k_j [\![P_j]\!]$, where $P_i \neq P_j$ for $i \neq j$. Since the target of σ is discrete, it turns out that $\sigma \equiv J$ in a neighborhood U of x. Hence, by the continuity of f, in a neighborhood $V \subset U$ of x, there is a continuous decomposition $f = \sum_{j=1}^J \{f_j\}$ in k_j-valued functions, with the property that $f_j(y) \neq f_i(y)$ for every $y \in V$ and $f_j = k_j [\![g_j]\!]$ for each j. Moreover, it is easy to check that each g_j must necessarily be a harmonic function, so that x is a regular point for f. Therefore, we conclude

$$\Sigma_f = \{x : \sigma \text{ is discontinuous at } x\}. \tag{3.61}$$

The continuity of f implies easily the lower semicontinuity of σ, which in turn shows, through (3.61), that Σ is relatively closed.

In order to estimate the Hausdorff dimension of Σ_f, we argue by induction on the number of values. For $Q = 1$ there is nothing to prove, since Dir-minimizing $\mathbb{R}^n$-valued functions are classical harmonic functions. Next, we assume that the theorem holds for every Q^*-valued functions, with $Q^* < Q$, and prove it for Q-valued functions. If $f = Q [\![\zeta]\!]$ with ζ harmonic, then $\Sigma_f = \emptyset$ and the proposition is proved. If this is not the case, we consider first $\Sigma_{Q,f}$ the set of points of multiplicity Q: it is a subset of Σ_f and we know from Proposition 3.22 that it is a closed subset of Ω with Hausdorff dimension at most $m-2$ and at most countable if $m=2$. Then, we consider the open set $\Omega' = \Omega \setminus \Sigma_{Q,f}$. Thanks to the continuity of f, we can find countable open balls B_k such that $\Omega' = \cup_k B_k$ and $f|_{B_k}$ can be decomposed as the sum of two multiple-valued Dir-minimizing functions:

$$f|_{B_k} = [\![f_{k,Q_1}]\!] + [\![f_{k,Q_2}]\!], \quad \text{with } Q_1 < Q,\, Q_2 < Q,$$

and

$$\mathrm{supp}\,(f_{k,Q_1}(x)) \cap \mathrm{supp}\,(f_{k,Q_2}(x)) = \emptyset \quad \text{for every } \; x \in B_k.$$

Clearly, it follows from this last condition that

$$\Sigma_f \cap B_k = \Sigma_{f_{k,Q_1}} \cup \Sigma_{f_{k,Q_2}}.$$

Moreover, f_{k,Q_1} and f_{k,Q_2} are both Dir-minimizing and, by inductive hypothesis, $\Sigma_{f_{k,Q_1}}$ and $\Sigma_{f_{k,Q_2}}$ are closed subsets of B_k with Hausdorff dimension at most $m-2$. We conclude that

$$\Sigma_f = \Sigma_{Q,f} \cup \bigcup_{k\in\mathbb{N}} \left(\Sigma_{f_{k,Q_1}} \cup \Sigma_{f_{k,Q_2}}\right)$$

has Hausdorff dimension at most $m-2$ and it is at most countable if $m=2$.

CHAPTER 4

Intrinsic theory

In this chapter we develop more systematically the metric theory of Q-valued Sobolev functions. The aim is to provide a second proof of all the propositions and lemmas in Section 2.2, independent of Almgren's embedding and retraction $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. Some of the properties proved in this section are actually true for Sobolev spaces taking values in fairly general metric targets, whereas some others do depend on the specific structure of $\mathcal{A}_Q(\mathbb{R}^n)$.

4.1. Metric Sobolev spaces

To our knowledge, metric space-valued Sobolev-type spaces were considered for the first time by Ambrosio in [**Amb90**] (in the particular case of BV mappings). The same issue was then considered later by several other authors in connection with different problems in geometry and analysis (see for instance [**GS92**], [**KS93**], [**Ser94**], [**Jos97**], [**JZ00**], [**CL01**] and [**HKST01a**]). The definition adopted here differs slightly from that of Ambrosio (see Definition 0.5) and was proposed later, for general exponents, by Reshetnyak (see [**Res97**] and [**Res04**]). In fact, it turns out that the two points of view are equivalent, as witnessed by the following proposition.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. A Q-valued function f belongs to $W^{1,p}(\Omega, \mathcal{A}_Q)$ if and only if there exists a function $\psi \in L^p(\Omega, \mathbb{R}^+)$ such that, for every Lipschitz function $\phi : \mathcal{A}_Q \to \mathbb{R}$, the following two conclusions hold:*

(a) *$\phi \circ f \in W^{1,p}(\Omega)$;*

(b) *$|D\,(\phi \circ f)\,(x)| \leq Lip(\phi)\,\psi(x)$ for almost every $x \in \Omega$.*

This fact was already remarked by Reshetnyak. The proof relies on the observation that Lipschitz maps with constant less than 1 can be written as suprema of translated distances. This idea, already used in [**Amb90**], underlies in a certain sense the embedding of separable metric spaces in ℓ^∞, a fact exploited first in the pioneering work [**Gro83**] by Gromov (see also the works [**AK00a**], [**AK00b**] and [**HKST01b**], where this idea has been used in various situations).

Proof. Since the distance function from a point is a Lipschitz map, with Lipschitz constant 1, one implication is trivial. To prove the opposite, consider a Sobolev Q-valued function f: we claim that (a) and (b) hold with $\psi = \left(\sum_j \varphi_j^2\right)^{1/2}$, where the φ_j's are the functions in Definition 0.5. Indeed, take a Lipschitz function $\phi \in \mathrm{Lip}(\mathcal{A}_Q)$. By treating separately the positive and the negative part of the function, we can assume, without loss of generality, that $\phi \geq 0$. If $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{A}_Q$ is a dense subset and $L = \mathrm{Lip}(\varphi)$, it is a well known fact that $\phi(T) = \inf_i \left\{\phi(T_i) + L\,\mathcal{G}(T_i, T)\right\}$. Therefore,

$$\phi \circ f = \inf_i \left\{\phi(T_i) + L\,\mathcal{G}(T_i, f)\right\} =: \inf_i g_i. \tag{4.1}$$

Since $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$, each $g_i \in W^{1,p}(\Omega)$ and the inequality $|D(\phi\circ f)| \le \sup_i |Dg_i|$ holds a.e. On the other hand, $|Dg_i| = L\,|D\mathcal{G}(f,T_i)| \le L\sqrt{\sum_j \varphi_j^2}$ a.e. This completes the proof. $\square$

In the remaining sections of this chapter, we first prove the existence of $|\partial_j f|$ (as defined in the Introduction) and prove the explicit formula (0.2). Then, we introduce a metric on $W^{1,p}(\Omega, \mathcal{A}_Q)$, making it a complete metric space. This part of the theory is in fact valid under fairly general assumptions on the target space: the interested reader will find suitable analogs in the aforementioned papers.

4.1.1. Representation formulas for $|\partial_j f|$.

PROPOSITION 4.2. *For every Sobolev Q-valued function $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$, there exist $g_j \in L^p$, for $j = 1, \ldots, m$, with the following two properties:*

(i) *$|\partial_j \mathcal{G}(f,T)| \le g_j$ a.e. for every $T \in \mathcal{A}_Q$;*
(ii) *if $\varphi_j \in L^p$ is such that $|\partial_j \mathcal{G}(f,T)| \le \varphi_j$ for all $T \in \mathcal{A}_Q$, then $g_j \le \varphi_j$ a.e.*

These functions are unique and will be denoted by $|\partial_j f|$. Moreover, chosen a countable dense subset $\{T_i\}_{i\in\mathbb{N}}$ of $\mathcal{A}_Q$, they satisfy the equality (0.2).

PROOF. The uniqueness of the functions g_j is an obvious corollary of their property (ii). It is enough to prove that $g_j = |\partial_j f|$ as defined in (0.2) satisfies (i), because it obviously satisfies (ii). Let $T \in \mathcal{A}_Q$ and $\{T_{i_k}\} \subseteq \{T_i\}$ be such that $T_{i_k} \to T$. Then, $\mathcal{G}(f, T_{i_k}) \to \mathcal{G}(f,T)$ in L^p and, hence, for every $\psi \in C_c^\infty(\Omega)$,

$$\left|\int \partial_j \mathcal{G}(f,T)\,\psi\right| = \lim_{i_k\to+\infty}\left|\int \mathcal{G}(f,T_{i_k})\,\partial_j\psi\right| = \lim_{i_k\to+\infty}\left|\int \partial_j\mathcal{G}(f,T_{i_k})\,\psi\right| \le \int g_j\,|\psi|. \tag{4.2}$$

Since (4.2) holds for every ψ, we conclude $|\partial_j \mathcal{G}(f,T)| \le g_j$ a.e. $\square$

4.1.2. A metric on $W^{1,p}(\Omega, \mathcal{A}_Q)$.

Given f and $g \in W^{1,p}(\Omega, \mathcal{A}_Q)$, define

$$d_{W^{1,p}}(f,g) = \|\mathcal{G}(f,g)\|_{L^p} + \sum_{j=1}^m \left\|\sup_i \left|\partial_j \mathcal{G}(f,T_i) - \partial_j \mathcal{G}(g,T_i)\right|\right\|_{L^p}. \tag{4.3}$$

PROPOSITION 4.3. $\left(W^{1,p}(\Omega, \mathcal{A}_Q), d_{W^{1,p}}\right)$ *is a complete metric space and*

$$d_{W^{1,p}}(f_k, f) \to 0 \quad \Rightarrow \quad |Df_k| \stackrel{L^p}{\to} |Df|. \tag{4.4}$$

PROOF. The proof that $d_{W^{1,p}}$ is a metric is a simple computation left to the reader; we prove its completeness. Let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence for $d_{W^{1,p}}$. Then, it is a Cauchy sequence in $L^p(\Omega, \mathcal{A}_Q)$. There exists, therefore, a function $f \in L^p(\Omega, \mathcal{A}_Q)$ such that $f_k \to f$ in L^p. We claim that f belongs to $W^{1,p}(\Omega, \mathcal{A}_Q)$ and $d_{W^{1,p}}(f_k, f) \to 0$. Since $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$ if and only if $d_{W^{1,p}}(f, 0) < \infty$, it is clear that we need only to prove that $d_{W^{1,p}}(f_k, f) \to 0$. This is a consequence of the following simple observation:

$$\begin{aligned}\left\|\sup_i \left|\partial_j \mathcal{G}(f,T_i) - \partial_j \mathcal{G}(f_k,T_i)\right|\right\|_{L^p}^p &= \sup_{P\in\mathcal{P}} \sum_{E_s\in P} \|\partial_j \mathcal{G}(f,T_s) - \partial_j \mathcal{G}(f_k,T_s)\|_{L^p(E_s)}^p \\ &\le \lim_{l\to+\infty} d_{W^{1,p}}(f_l, f_k)^p,\end{aligned} \tag{4.5}$$

where $\mathcal{P}$ is the family of finite measurable partitions of Ω. Indeed, by (4.5),

$$\lim_{k\to+\infty} d_{W^{1,p}}(f_k, f) \overset{(4.5)}{\leq} \lim_{k\to+\infty}\left[\|\mathcal{G}(f, f_k)\|_{L^p} + m \lim_{l\to+\infty} d_{W^{1,p}}(f_l, f_k)\right] = 0.$$

We now come to (4.4). Assume $d_{W^{1,p}}(f_k, f) \to 0$ and observe that

$$\begin{aligned}\big||\partial_j f_k| - |\partial_j f_l|\big| &= \left|\sup_i |\partial_j \mathcal{G}(f_k, T_i)| - \sup_i |\partial_j \mathcal{G}(f_k, T_i)|\right| \\ &\leq \sup_i |\partial_j \mathcal{G}(f_k, T_i) - \partial_j \mathcal{G}(f_k, T_i)|\,.\end{aligned}$$

Hence, one can infer $\big\||\partial_j f_k| - |\partial_j f_l|\,\big\|_{L^p} \leq d_{W^{1,p}}(f_k, f_l)$. This implies that $|Df_k|$ is a Cauchy sequence, from which the conclusion follows easily. □

4.2. Metric proofs of the main theorems I

We start now with the metric proofs of the results in Section 2.2.

4.2.1. Lipschitz approximation. In this subsection we prove a strengthened version of Proposition 2.5. The proof uses, in the metric framework, a standard truncation technique and the Lipschitz extension Theorem 1.7 (see, for instance, 6.6.3 in [**EG92**]). This last ingredient is a feature of $\mathcal{A}_Q(\mathbb{R}^n)$ and, in general, the problem of whether or not general Sobolev mappings can be approximated with Lipschitz ones is a very subtle issue already when the target is a smooth Riemannian manifold (see for instance [**SU82**], [**Bet91**], [**HL03**] and [**HR07**]). The truncation technique is, instead, valid in a much more general setting, see for instance [**HKST01b**].

PROPOSITION 4.4 (Lipschitz approximation). *There exists a constant $C = C(m, \Omega, Q)$ with the following property. For every $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$ and every $\lambda > 0$, there exists a Q-function f_λ such that* $\mathrm{Lip}\,(f_\lambda) \leq C\,\lambda$,

$$|E_\lambda| = \big|\{x \in \Omega \,:\, f(x) \neq f_\lambda(x)\}\big| \leq \frac{C\||Df|\|_{L^p}^p}{\lambda^p} \tag{4.6}$$

and $d_{W^{1,p}}(f, f_\lambda) \leq C d_{W^{1,p}}(f, Q[\![0]\!])$. Moreover, $d_{W^{1,p}}(f, f_\lambda) = o(1)$ and $|E_\lambda| = o(\lambda^{-p})$.

PROOF. We consider the case $1 \leq p < \infty$ ($p = \infty$ is immediate) and we set

$$\Omega_\lambda = \{x \in \Omega \,:\, M(|Df|) \leq \lambda\},$$

where M is the Maximal Function Operator (see [**Ste93**] for the definition). By rescaling, we can assume $\||Df|\|_{L^p} = 1$. As a consequence, we can also assume $\lambda \geq C(m, \Omega, Q)$, where $C(m, \Omega, Q)$ will be chosen later.

Notice that, for every $T \in \mathcal{A}_Q$ and every $j \in \{1, \ldots, m\}$,

$$M\big(\,|\partial_j \mathcal{G}(f, T)|\,\big) \leq M(|Df|) \leq \lambda \quad \text{in } \Omega_\lambda.$$

By standard calculation (see, for example, 6.6.3 in [**EG92**]), we deduce that, for every T, $\mathcal{G}(f, T)$ is $(C\,\lambda)$-Lipschitz in Ω_λ, with $C = C(m)$. Therefore,

$$\big|\mathcal{G}(f(x), T) - \mathcal{G}(f(y), T)\big| \leq C\,\lambda\,|x - y| \quad \forall\, x, y \in \Omega_\lambda \text{ and } \forall\, T \in \mathcal{A}_Q. \tag{4.7}$$

From (4.7), we get a Lipschitz estimate for $f|_{\Omega_\lambda}$ by setting $T = f(x)$. We can therefore use Theorem 1.7 to extend $f|_{\Omega_\lambda}$ to a Lipschitz function f_λ with $\mathrm{Lip}(f_\lambda) \leq C\lambda$.

The standard weak $(p-p)$ estimate for maximal functions (see [**Ste93**]) yields

$$|\Omega \setminus \Omega_\lambda| \leq \frac{C}{\lambda^p} \int_{\Omega\setminus\Omega_{\lambda/2}} |Df|^p \leq \frac{C}{\lambda^p}\, o(1), \tag{4.8}$$

which implies (4.6) and $|E_\lambda| = o(\lambda^{-p})$. Observe also that, from (4.8), it follows that

$$\int_{\Omega\setminus\Omega_\lambda} |Df_\lambda|^p \leq C \int_{\Omega\setminus\Omega_{\lambda/2}} |Df|^p. \tag{4.9}$$

It remains to prove $d_{W^{1,p}}(f, f_\lambda) \leq C d_{W^{1,p}}(f, Q\llbracket 0 \rrbracket)$ and $d_{W^{1,p}}(f_\lambda, f) \to 0$. By (4.9), it suffices to show

$$\|\mathcal{G}(f_\lambda, Q\llbracket 0 \rrbracket)\|_{L^p} \leq C d_{W^{1,p}}(f, Q\llbracket 0 \rrbracket) \quad \text{and} \quad \|\mathcal{G}(f_\lambda, f)\|_{L^p} \to 0\,.$$

We first choose the constant $C(m, \Omega, Q) \leq \lambda$ so to guarantee that $2|\Omega_\lambda| \geq |\Omega|$. Set $g := \mathcal{G}(f, Q\llbracket 0 \rrbracket)$, $g_\lambda := \mathcal{G}(f_\lambda, Q\llbracket 0 \rrbracket)$ and $h = g - g_\lambda$. Let $\bar{h}$ be the average of h over Ω and use the Poincaré inequality and the fact that h vanishes on Ω_λ to conclude that

$$\begin{aligned}\frac{|\Omega|}{2}|\bar{h}|^p \leq |\Omega_\lambda||\bar{h}|^p \leq \int |h - \bar{h}|^p \leq C\|Dh\|_{L^p}^p \\ \leq C\int_{\Omega\setminus\Omega_\lambda} (|Df|^p + |Df_\lambda|^p) \leq C \int_{\Omega\setminus\Omega_{\lambda/2}} |Df|^p.\end{aligned}$$

Therefore,

$$\|h\|_{L^p}^p \leq C \int_{\Omega\setminus\Omega_{\lambda/2}} |Df|^p\,.$$

So, using the triangle inequality, we conclude that

$$\|\mathcal{G}(f_\lambda, Q\llbracket 0 \rrbracket)\|_{L^p} \leq \|\mathcal{G}(f, Q\llbracket 0 \rrbracket)\|_{L^p} + C\||Df|\|_{L^p} \leq C d_{W^{1,p}}(f, Q\llbracket 0 \rrbracket)$$

and

$$\begin{aligned}\|\mathcal{G}(f, f_\lambda)\|)_{L^p} &= \|\mathcal{G}(f, Q\llbracket 0 \rrbracket)\|_{L^p(\Omega\setminus\Omega_\lambda)} + \|h\|_{L^p} \\ &\leq \|\mathcal{G}(f, Q\llbracket 0 \rrbracket)\|_{L^p(\Omega\setminus\Omega_\lambda)} + C\||Df|\|_{L^p(\Omega\setminus\Omega_{\lambda/2})}.\end{aligned} \tag{4.10}$$

Since $|\Omega \setminus \Omega_\lambda| \downarrow 0$, the right hand side of (4.10) converges to 0 as $\lambda \downarrow 0$. □

4.2.2. Trace theory. Next, we show the existence of the trace of a Q-valued Sobolev function as defined in Definition 0.7. Moreover, we prove that the space of functions with given trace $W^{1,p}_g(\Omega, \mathcal{A}_Q)$ defined in (2.10) is closed under weak convergence. A suitable trace theory can be build in a much more general setting (see the aforementioned papers). Here, instead, we prefer to take advantage of Proposition 4.4 to give a fairly short proof.

PROPOSITION 4.5. *Let $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. Then, there exists an unique $g \in L^p(\partial\Omega, \mathcal{A}_Q)$ such that*

$$(\varphi \circ f)|_{\partial\Omega} = \varphi \circ g \qquad \textit{for all} \quad \varphi \in \mathrm{Lip}\,(\mathcal{A}_Q). \tag{4.11}$$

We denote g by $f|_{\partial\Omega}$. Moreover, the following set is closed under weak convergence:

$$W^{1,2}_g(\Omega, \mathcal{A}_Q) := \big\{f \in W^{1,2}(\Omega, \mathcal{A}_Q) \,:\, f|_{\partial\Omega} = g\big\}.$$

PROOF. Consider a sequence of Lipschitz functions f_k with $d_{W^{1,p}}(f_k, f) \to 0$ (whose existence is ensured from Proposition 4.4). We claim that $f_k|_{\partial\Omega}$ is a Cauchy sequence in $L^p(\partial\Omega, \mathcal{A}_Q)$. To see this, notice that, if $\{T_i\}_{i\in\mathbb{N}}$ is a dense subset of $\mathcal{A}_Q$,

$$\mathcal{G}(f_k, f_l) = \sup_i |\mathcal{G}(f_k, T_i) - \mathcal{G}(f_l, T_i)| .$$

Moreover, recalling the classical estimate for the trace of a real-valued Sobolev functions, $\|f|_{\partial\Omega}\|_{L^p} \leq C \|f\|_{W^{1,p}}$, we conclude that

$$\begin{aligned}
\|\mathcal{G}(f_k, f_l)\|^p_{L^p(\partial\Omega)} &\leq C \int_\Omega \mathcal{G}(f_k, f_l)^p + \sum_j \int_\Omega |\partial_j \mathcal{G}(f_k, f_l)|^p \\
&\leq C \int_\Omega \mathcal{G}(f_k, f_l)^p + \sum_j \int_\Omega \sup_i |\partial_j \mathcal{G}(f_k, T_i) - \partial_j \mathcal{G}(f_l, T_i)|^p \\
&\leq C\, d_{W^{1,p}}(f_k, f_l)^p, \qquad (4.12)
\end{aligned}$$

(where we used the identity $|\partial_j (\sup_i g_i)| \leq \sup_i |\partial_j g_i|$, which holds true if there exists an $h \in L^p(\Omega)$ with $|g_i|, |Dg_i| \leq h \in L^p(\Omega)$).

Let, therefore, g be the L^p-limit of f_k. For every $\varphi \in \mathrm{Lip}(\mathcal{A}_Q)$, we clearly have that $(\varphi \circ f_k)|_{\partial\Omega} \to \varphi \circ g$ in L^p. But, since $\varphi \circ f_k \to \varphi \circ f$ in $W^{1,p}(\Omega)$, the limit of $(\varphi \circ f_k)|_{\partial\Omega}$ is exactly $(\varphi \circ f)|_{\partial\Omega}$. This shows (4.11). We now come to the uniqueness. Assume that g and $\hat{g}$ satisfy (4.11). Then, $\mathcal{G}(g, T_i) = \mathcal{G}(\hat{g}, T_i)$ almost everywhere on $\partial\Omega$ and for every i. This implies

$$\mathcal{G}(g, \hat{g}) = \sup_i |\mathcal{G}(g, T_i) - \mathcal{G}(\hat{g}, T_i)| = 0 \quad \text{a.e. on } \Omega,$$

i.e. $g = \hat{g}$ a.e.

Finally, as for the last assertion of the proposition, note that $f_k \rightharpoonup f$ in the sense of Definition 2.9 if and only if $\varphi \circ f_k \rightharpoonup \varphi \circ f$ for any Lipschitz function φ. Therefore, the proof that the set $W^{1,2}_g$ is closed is a direct consequence of the corresponding fact for classical Sobolev spaces of real-valued functions. □

4.2.3. Sobolev embeddings. The following proposition is an obvious consequence of the definition and holds under much more general assumptions.

PROPOSITION 4.6 (Sobolev Embeddings). *The following embeddings hold:*

(*i*) *if* $p < m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$ *for every* $q \in [1, p^*]$, *where* $p^* = \frac{m\,p}{m-p}$, *and the inclusion is compact when* $q < p^*$;

(*ii*) *if* $p = m$, *then* $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$, *for every* $q \in [1, +\infty)$, *with compact inclusion.*

REMARK 4.7. In Proposition 2.11 we have also shown that

(*iii*) if $p > m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega, \mathcal{A}_Q)$, for $\alpha = 1 - \frac{m}{p}$, with compact inclusion for $\alpha < 1 - \frac{m}{p}$.

It is not difficult to give an intrinsic proof of it. However, in the regularity theory of Chapters 3 and 5, (iii) is used only in the case $m = 1$, which has already been shown in Proposition 1.2.

PROOF. Recall that $f \in L^p(\Omega, \mathcal{A}_Q)$ if and only if $\mathcal{G}(f, T) \in L^p(\Omega)$ for some (and, hence, any) T. So, the inclusions in (*i*) and (*ii*) are a trivial corollary of the usual Sobolev embeddings for real-valued functions, which in fact yields the inequality

$$\|\mathcal{G}(f, Q[\![0]\!])\|_{L^q(\Omega)} \leq C(n, \Omega, Q) d_{W^{1,p}}(f, Q[\![0]\!]) . \tag{4.13}$$

As for the compactness of the embeddings when $q < p^*$, consider a sequence $\{f_k\}_{k\in\mathbb{N}}$ of Q-valued Sobolev functions with equibounded $d_{W^{1,p}}$-distance from a point:

$$d_{W^{1,p}}(f_k, Q[\![0]\!]) = \|\mathcal{G}(f_k, Q[\![0]\!])\|_{L^p} + \sum_j \||\partial_j f_k|\|_{L^p} \leq C < +\infty.$$

For every $l \in \mathbb{N}$, let $f_{k,l}$ be the function given by Proposition 4.4 choosing $\lambda = l$.

From the Ascoli–Arzelà Theorem and a diagonal argument, we find a subsequence (not relabeled) f_k such that, for any fixed l, $\{f_{k,l}\}_k$ is a Cauchy sequence in C^0. We now use this to show that f_k is a Cauchy sequence in L^q. Indeed,

$$\|\mathcal{G}(f_k, f_{k'})\|_{L^q} \leq \|\mathcal{G}(f_k, f_{k,l})\|_{L^q} + \|\mathcal{G}(f_{k,l}, f_{k',l})\|_{L^q} + \|\mathcal{G}(f_{k',l}, f_{k'})\|_{L^q} . \tag{4.14}$$

We claim that the first and third terms are bounded by $C\, l^{1/q-1/p^*}$. It suffices to show it for the first term. By Proposition 4.4, there is a constant C such that $d_{W^{1,p}}(f_{k,l}, Q[\![0]\!]) \leq C$ for every k and l. Therefore, we infer

$$\begin{aligned}
\|\mathcal{G}(f_k, f_{k,l})\|_{L^q}^q &\leq C \int_{\{f_k \neq f_{k,l}\}} \left[\mathcal{G}(f_k, Q[\![0]\!])^q + \mathcal{G}(f_{k,l}, Q[\![0]\!])^q\right] \\
&\leq \left(\|\mathcal{G}(f_k, [\![0]\!])\|_{L^{p^*}}^q + \|\mathcal{G}(f_{k,l}, [\![0]\!])\|_{L^{p^*}}^q\right) |\{f_k \neq f_{k,l}\}|^{1-q/p^*} \\
&\leq C l^{q/p^*-1},
\end{aligned}$$

where in the last line we have used (4.13) (in the critical case p^*) and the Hölder inequality.

Let ε be a given positive number. Then we can choose l such that the first and third term in (4.14) are both less than $\varepsilon/3$, independently of k. On the other hand, since $\{f_{k,l}\}_k$ is a Cauchy sequence in C^0, there is an N such that $\|\mathcal{G}(f_{k,l}, f_{k',l})\|_{L^q} \leq \varepsilon/3$ for every $k, k' > N$. Clearly, for $k, k' > N$, we then have $\|\mathcal{G}(f_k, f_{k'})\| \leq \varepsilon$. This shows that $\{f_k\}$ is a Cauchy sequence in L^q and hence completes the proof of (i). The compact inclusion in (ii) is analogous. □

4.2.4. Campanato–Morrey estimate. We conclude this section by giving another proof of the Campanato–Morrey estimate in Proposition 2.14.

PROPOSITION 4.8. *Let $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ and $\alpha \in (0,1]$ be such that $\forall \delta \in (0,1)$,*

$$\int_{B_r(x)} |Df|^2 \leq A_\delta r^{m-2+2\alpha} \quad \textit{for every } x \in B_\delta \textit{ and a.e. } r \in]0, 1-|x|].$$

Then, for every $0 < \delta < 1$, there is a constant $C = C(m, n, Q, \delta)$ such that

$$\sup_{x,y\in\overline{B_\delta}} \frac{\mathcal{G}(f(x), f(y))}{|x-y|^\alpha} =: [f]_{C^{0,\alpha}(\overline{B_\delta})} \leq C\sqrt{A_\delta}. \tag{4.15}$$

PROOF. Let $T \in \mathcal{A}_Q$ be given. Then,

$$\int_{B_r} |D\mathcal{G}(f, T)|^2 \leq \int_{B_r} |Df|^2 \leq A\, r^{m-2+2\alpha} \quad \text{for a.e. } r \in]0, 1].$$

By the classical estimate (see 3.2 in [**HL97**]), $\mathcal{G}(f,T)$ is α-Hölder with

$$\sup_{x,y\in\overline{B_\delta}} \frac{|\mathcal{G}(f(x),T)-\mathcal{G}(f(y),T)|}{|x-y|^\alpha} \leq C\sqrt{A},$$

where C is independent of T. This implies easily (4.15). □

4.3. Metric proofs of the main theorems II

We give in this section metric proofs of the two remaining results of Section 2.2: the Poincaré inequality in Proposition 2.12 and the interpolation Lemma 2.15.

4.3.1. Poincaré inequality.

PROPOSITION 4.9 (Poincaré inequality). *Let M be a connected bounded Lipschitz open set of a Riemannian manifold. Then, for every $1 \leq p < m$, there exists a constant $C = C(p,m,n,Q,M)$ with the following property: for every function $f \in W^{1,p}(M,\mathcal{A}_Q)$, there exists a point $\overline{f} \in \mathcal{A}_Q$ such that*

$$\left(\int_M \mathcal{G}(f,\overline{f})^{p^*}\right)^{\frac{1}{p^*}} \leq C\left(\int_M |Df|^p\right)^{\frac{1}{p}}, \tag{4.16}$$

where $p^* = \frac{m\,p}{m-p}$.

A proof of (a variant of) this Poincaré-type inequality appears already, for the case $p=1$ and a compact target, in the work of Ambrosio [**Amb90**]. Here we use, however, a different approach, based on the existence of an isometric embedding of $\mathcal{A}_Q(\mathbb{R}^n)$ into a separable Banach space. We then exploit the linear structure of this larger space to take averages. This idea, which to our knowledge appeared first in [**HKST01b**], works in a much more general framework, but, to keep our presentation easy, we will use all the structural advantages of dealing with the metric space $\mathcal{A}_Q(\mathbb{R}^n)$.

The key ingredients of the proof are the lemmas stated below. The first one is an elementary fact, exploited first by Gromov in the context of metric geometry (see [**Gro83**]) and used later to tackle many problems in analysis and geometry on metric spaces (see [**AK00a**], [**AK00b**] and [**HKST01b**]). The second is an extension of a standard estimate in the theory of Sobolev spaces. Both lemmas will be proved at the end of the subsection.

LEMMA 4.10. *Let (X,d) be a complete separable metric space. Then, there is an isometric embedding $i: X \to B$ into a separable Banach space.*

LEMMA 4.11. *For every $1 \leq p < m$ and $r > 0$, there exists a constant $C = C(p,m,n,Q)$ such that, for every $f \in W^{1,p}(B_r,\mathcal{A}_Q) \cap \mathrm{Lip}\,(B_r,\mathcal{A}_Q)$ and every $z \in B_r$,*

$$\int_{B_r} \mathcal{G}(f(x),f(z))^p dx \leq C\, r^{p+m-1} \int_{B_r} |Df|(x)^p\, |x-z|^{1-m}\, dx. \tag{4.17}$$

PROOF OF PROPOSITION 4.9. *Step* 1. We first assume $M = B_r \subset \mathbb{R}^m$ and f Lipschitz. We regard f as a map taking values in the Banach space B of Lemma 4.10. Since B is a Banach space, we can integrate B-valued functions on Riemannian manifolds using the Bochner integral. Indeed, being f Lipschitz and B a separable Banach space, in our case it is straightforward to check that f is integrable in the sense of Bochner (see [**DU77**]; in fact the theory of the Bochner integral can be applied in much more general situations).

Consider therefore the average of f on M, which we denote by S_f. We will show that

$$\int_{B_r} \|f - S_f\|_B^p \leq C r^p \int_{B_r} |Df|^p. \tag{4.18}$$

First note that, by the usual convexity of the Bochner integral,

$$\|f(x) - S_f\|_B \leq \fint \|f(z) - f(x)\|_B \, dz = \fint \mathcal{G}(f(z), f(x))\, dz.$$

Hence, (4.18) is a direct consequence of Lemma 4.11:

$$\begin{aligned}\int_{B_r} \|f(x) - S_f\|_B^p \, dx &\leq \int_{B_r} \fint_{B_r} \mathcal{G}(f(x), f(z))^p \, dz \, dx \\ &\leq C\, r^{p+m-1} \fint_{B_r} \int_{B_r} |w - z|^{1-m} |Df|(w)^p \, dw \, dz \\ &\leq C\, r^p \int_{B_r} |Df|(w)^p \, dw.\end{aligned}$$

Step 2. Assuming $M = B_r \subset \mathbb{R}^m$ and f Lipschitz, we find a point $\overline{f}$ such that

$$\int_{B_r} \mathcal{G}\left(f, \overline{f}\right)^p \leq C r^p \int_{B_r} |Df|^p. \tag{4.19}$$

Consider, indeed, $\overline{f} \in \mathcal{A}_Q$ a point such that

$$\|S_f - \overline{f}\|_B = \min_{T \in \mathcal{A}_Q} \|S_f - T\|_B. \tag{4.20}$$

Note that $\overline{f}$ exists because $\mathcal{A}_Q$ is locally compact. Then, we have

$$\begin{aligned}\int_{B_r} \mathcal{G}\left(f, \overline{f}\right)^p &\leq C \int_{B_r} \|f - S_f\|_B^p + \int_{B_r} \|S_f - \overline{f}\|_B^p \\ &\overset{(4.18),\,(4.20)}{\leq} C\, r^p \int_{B_r} |Df|^p + C \int_{B_r} \|S_f - f\|_B^p \overset{(4.18)}{\leq} C\, r^p \int_{B_r} |Df|^p.\end{aligned}$$

Step 3. Now we consider the case of a generic $f \in W^{1,p}(B_r, \mathcal{A}_Q)$. From the Lipschitz approximation Theorem 4.4, we find a sequence of Lipschitz functions f_k converging to f, $d_{W^{1,p}}(f_k, f) \to 0$. Fix, now, an index k such that

$$\int_{B_r} \mathcal{G}(f_k, f)^p \leq r^p \int_{B_r} |Df|^p \quad \text{and} \quad \int_{B_r} |Df_k|^p \leq 2 \int_{B_r} |Df|^p, \tag{4.21}$$

and set $\overline{f} = \overline{f_k}$, with the $\overline{f_k}$ found in the previous step. With this choice, we conclude

$$\int_{B_r} \mathcal{G}\left(f, \overline{f}\right)^p \leq C \int_{B_r} \mathcal{G}(f, f_k)^p + \int_{B_r} \mathcal{G}\left(f_k, \overline{f}_k\right)^p \overset{(4.19),\,(4.21)}{\leq} C\, r^p \int_{B_r} |Df|^p. \tag{4.22}$$

Step 4. Using classical Sobolev embeddings, we prove (4.16) in the case of $M = B_r$. Indeed, since $\mathcal{G}(f, \overline{f}) \in W^{1,p}(B_r)$, we conclude

$$\left\|\mathcal{G}(f, \overline{f})\right\|_{L^{p^*}} \leq C \left\|\mathcal{G}(f, \overline{f})\right\|_{W^{1,p}} \overset{(4.22)}{\leq} C \left(\int_{B_r} |Df|^p\right)^{\frac{1}{p}}.$$

Step 5. Finally, we drop the hypothesis of M being a ball. Using the compactness and connectedness of $\overline{M}$, we cover M by finitely many domains $A_1, \ldots, A_N$ biLipschitz to a ball such that $A_k \cap \cup_{i<k} A_i \neq \emptyset$. This reduces the proof of the general statement to that in the case $M = A \cup B$, where A and B are two domains such that $A \cap B \neq \emptyset$ and the Poincaré inequality is valid for both. Under these assumptions, denoting by f_A and f_B two means for f over A and B, we estimate

$$\begin{aligned}\mathcal{G}(f_A, f_B)^{p^*} = \fint_{A\cap B} \mathcal{G}(f_A, f_B)^{p^*} &\leq C \fint_A \mathcal{G}(f_A, f)^{p^*} + C \fint_B \mathcal{G}(f, f_B)^{p^*} \\ &\leq C \left(\int_M |Df|^p \right)^{\frac{p^*}{p}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{A\cup B} \mathcal{G}(f, f_A)^{p^*} &\leq \int_A \mathcal{G}(f, f_A)^{p^*} + \int_B \mathcal{G}(f, f_A)^{p^*} \\ &\leq \int_A \mathcal{G}(f, f_A)^{p^*} + C \int_B \mathcal{G}(f, f_B)^{p^*} + C\, \mathcal{G}(f_A, f_B)^{p^*} |B| \\ &\leq C \left(\int_M |Df|^p \right)^{\frac{p^*}{p}}.\end{aligned}$$

□

Proof of Lemma 4.10. We choose a point $x \in X$ and consider the Banach space $A := \{f \in \mathrm{Lip}(X, \mathbb{R}) : f(x) = 0\}$ with the norm $\|f\|_A = \mathrm{Lip}(f)$. Consider the dual A' and let $i : X \to A'$ be the mapping that to each $y \in X$ associates the element $[y] \in A'$ given by the linear functional $[y](f) = f(y)$. First of all we claim that i is an isometry, which amounts to prove the following identity:

$$d(z, y) = \|[y] - [z]\|_{A'} = \sup_{f(x)=0,\, \mathrm{Lip}(f)\leq 1} |f(y) - f(z)| \quad \forall x, y \in X. \tag{4.23}$$

The inequality $|f(y) - f(z)| \leq d(y, z)$ follows from the fact that $\mathrm{Lip}(f) = 1$. On the other hand, consider the function $f(w) := d(w, y) - d(y, x)$. Then $f(x) = 0$, $\mathrm{Lip}(f) = 1$ and $|f(y) - f(z)| = d(y, z)$.

Next, let C be the subspace generated by finite linear combinations of elements of $i(X)$. Note that C is separable and contains $i(X)$: its closure in A' is the desired separable Banach space B. □

Proof of Lemma 4.11. Fix $z \in B_r$. Clearly the restriction of f to any segment $[x, z]$ is Lipschitz. Using Rademacher, it is easy to justify the following inequality for a.e. x:

$$\mathcal{G}(f(x), f(z)) \leq |x - z| \int_0^1 |Df|(z + t(x - z))\, dt. \tag{4.24}$$

Hence, one has

$$\int_{B_r\cap\partial B_s(z)} \mathcal{G}(f(x),f(z))^p\,dx \overset{(4.24)}{\leq} \int_{B_r\cap\partial B_s(z)}\int_0^1 |x-z|^p\,|Df|(z+t(x-z))^p\,dt\,dx$$

$$\leq s^p\int_0^1\int_{B_r\cap\partial B_{ts}(z)} t^{1-n}|Df|(w)^p\,dw\,dt$$

$$= s^{p+m-1}\int_0^1\int_{B_r\cap\partial B_{ts}(z)} |w-z|^{1-m}|Df|(w)^p\,dw\,dt$$

$$\leq s^{p+m-2}\int_{B_r}|w-z|^{1-m}|Df|(w)^p\,dw. \tag{4.25}$$

Integrating in s the inequality (4.25), we conclude (4.17),

$$\int_{B_r}\mathcal{G}(f(x),f(z))^p\,dx \leq C\,r^{p+m-1}\int_{B_r}|w-z|^{1-m}|Df|(w)^p\,dw.$$

□

4.3.2. Interpolation Lemma. We prove in this section Lemma 2.15 (the statement below is, in fact, slightly simpler: Lemma 2.15 follows however from elementary scaling arguments). In this case, the proof relies in an essential way on the properties of $\mathcal{A}_Q(\mathbb{R}^n)$ and we believe that generalizations are possible only under some structural assumptions on the metric target.

Lemma 4.12 (Interpolation Lemma). *There exists a constant $C = C(m,n,Q)$ with the following property. For any g, $\tilde g \in W^{1,2}(\partial B_1,\mathcal{A}_Q)$, there is $h \in W^{1,2}(B_1\setminus B_{1-\varepsilon},\mathcal{A}_Q)$ such that*

$$h\,(x) = g(x), \qquad h\,((1-\varepsilon)\,x) = \tilde g(x), \quad \textit{for } x\in\partial B_1,$$

and

$$\mathrm{Dir}(h,B_1\setminus B_{1-\varepsilon}) \leq C\left\{\varepsilon\,\mathrm{Dir}(g,\partial B_1)+\varepsilon\,\mathrm{Dir}(\tilde g,\partial B_1)+\varepsilon^{-1}\int_{\partial B_1}\mathcal{G}\,(g,\tilde g)^2\right\}.$$

Proof. For the sake of clarity, we divide the proof into two steps: in the first one we prove the lemma in a simplified geometry (two parallel hyperplanes instead of two concentric spheres); then, we adapt the construction to the case of interest.

Step 1. Interpolation between parallel planes. We let $A = [-1,1]^{m-1}$, $B = A\times[0,\varepsilon]$ and consider two functions g, $\tilde g\in W^{1,2}(A,\mathcal{A}_Q)$. We then want to find a function $h: B\to\mathcal{A}_Q$ such that

$$h(x,0) = g(x) \quad\text{and}\quad h(x,\varepsilon) = \tilde g(x); \tag{4.26}$$

$$\mathrm{Dir}(h,B) \leq C\left(\varepsilon\,\mathrm{Dir}(g,A)+\varepsilon\,\mathrm{Dir}(\tilde g,A)+\varepsilon^{-1}\int_A\mathcal{G}(g,\tilde g)^2\right), \tag{4.27}$$

where the constant C depends only on m, n and Q.

For every $k\in\mathbb{N}_+$, set $A_k = [-1-k^{-1},1+k^{-1}]^{m-1}$, and decompose A_k in the union of $(k+1)^{m-1}$ cubes $\{C_{k,l}\}_{l=1,\ldots,(k+1)^{m-1}}$ with disjoint interiors, side length equal to $2/k$ and faces parallel to the coordinate hyperplanes. We denote by $x_{k,l}$ their centers. Therefore, $C_{k,l} = x_{k,l}+\left[-\frac{1}{k},\frac{1}{k}\right]^{m-1}$. Finally, we subdivide A into the cubes $\{D_{k,l}\}_{l=1,\ldots,k^{m-1}}$ of side $2/k$ and having the points $x_{k,l}$ as vertices, (so $\{D_{k,l}\}$ is the decomposition "dual" to $\{C_{k,l}\}$; see Figure 2).

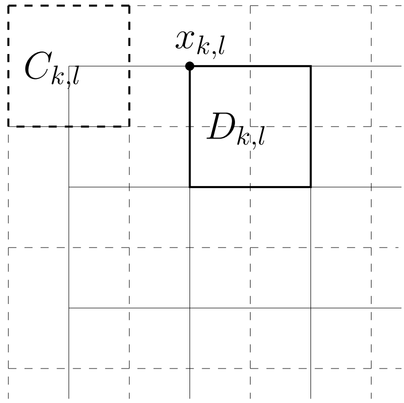

FIGURE 2. The cubes $C_{k,l}$ and $D_{k,l}$.

On each $C_{k,l}$ take a mean $\overline{g}_{k,l}$ of g on $C_{k,l} \cap A$. On A_k we define the piecewise constant functions g_k which takes the constant value $\overline{g}_{k,l}$ on each $C_{k,l}$:

$$g_k \equiv \overline{g}_{k,l} \quad \text{in} \quad C_{k,l}, \quad \text{with} \quad \int_{C_{k,l}\cap A} \mathcal{G}(g, \overline{g}_{k,l})^2 \leq \frac{C}{k^2} \int_{C_{k,l}\cap A} |Dg|^2.$$

In an analogous way, we define $\tilde{g}_k$ from $\tilde{g}$ and denote by $\tilde{g}_{k,l}$ the corresponding averages. Note that $g_k \to g$ and $\tilde{g}_k \to \tilde{g}$ in $L^2(A, \mathcal{A}_Q)$.

We next define a Lipschitz function $f_k : B \to \mathcal{A}_Q$. We set $f_k(x_{k,l}, 0) = \bar{g}_{k,l}$ and $f_k(x_{k,l}, \varepsilon) = \tilde{g}_{k,l}$. We then use Theorem 1.7 to extend f_k on the 1-skeleton of the cubical decomposition given by $D_{k,l} \times [0, \varepsilon]$. We apply inductively Theorem 1.7 to extend f_k to the j-skeletons.

If $V_{k,l}$ and $Z_{k,l}$ denote, respectively, the set of vertices of $D_{k,l} \times \{0\}$ and $D_{k,l} \times \{\varepsilon\}$, we then conclude that

$$\operatorname{Lip}(f_k|_{D_{k,l}\times\{\varepsilon\}}) \leq C \operatorname{Lip}(f_k|_{Z_{k,l}}) \quad \text{and} \quad \operatorname{Lip}(f_k|_{D_{k,l}\times\{0\}}) \leq C \operatorname{Lip}(f_k|_{V_{k,l}}). \tag{4.28}$$

Let $(x_{k,i}, 0)$ and $(x_{k,j}, 0)$ be two adjacent vertices in $V_{k,l}$. Then,

$$\begin{aligned}
\mathcal{G}(f_k(x_{k,i}, 0), f_k(x_{k,j}, 0))^2 = \mathcal{G}(g_k(x_{k,i}), g_k(x_{k,j}))^2 &= \fint_{C_{k,i}\cap C_{k,j}\cap A} \mathcal{G}(g_k(x_{k,i}), g_k(x_{k,j}))^2 \\
&\leq C \fint_{C_{k,i}\cap A} \mathcal{G}(\overline{g}_{k,i}, g)^2 + C \fint_{C_{k,j}\cap A} \mathcal{G}(g, \overline{g}_{k,j})^2 \\
&\leq \frac{C}{k^{m+1}} \int_{C_{k,i}\cup C_{k,j}} |Dg|^2. \qquad (4.29)
\end{aligned}$$

In the same way, if $(x_{k,i}, \varepsilon)$ and $(x_{k,j}, \varepsilon)$ are two adjacent vertices in $Z_{k,l}$, then

$$\mathcal{G}(f_k(x_{k,i}, \varepsilon), f_k(x_{k,j}, \varepsilon))^2 \leq \frac{C}{k^{m+1}} \int_{C_{k,i}\cup C_{k,j}} |D\tilde{g}|^2.$$

Finally, for $(x_{k,i}, 0)$ and $(x_{k,i}, \varepsilon)$, we have

$$\mathcal{G}\big(f_k(x_{k,i}, 0), f_k(x_{k,i}, \varepsilon)\big)^2 = \mathcal{G}(g_{k,i}, \tilde{g}_{k,i})^2 \leq \fint_{C_{k,i}\cap A} \mathcal{G}(g_k, \tilde{g}_k)^2.$$

Hence, if $\{C_{k,\alpha}\}_{\alpha=1,\dots,2^{m-1}}$ are all the cubes intersecting $D_{k,l}$, we conclude that the Lipschitz constant of f_k in $D_{k,l}\times[0,\varepsilon]$ is bounded in the following way:

$$\mathrm{Lip}(f_k|_{D_{k,l}\times[0,\varepsilon]})^2 \le \frac{C}{k^{m-1}}\int_{\cup_\alpha C_{k,\alpha}} \left(|Dg|^2+|D\tilde g|^2+\varepsilon^{-2}\mathcal{G}(g_k,\tilde g_k)^2\right).$$

Observe that each $C_{k,\alpha}$ intersects at most N cubes $D_{k,l}$, for some dimensional constant N. Thus, summing over l, we conclude

$$\mathrm{Dir}(f_k, A\times[0,\varepsilon]) \le C\left(\varepsilon\int_A |Dg|^2+\varepsilon\int_A|D\tilde g|^2+\varepsilon^{-1}\int_A\mathcal{G}(g_k,\tilde g_k)^2\right). \tag{4.30}$$

Next, having fixed $D_{k,l}$, consider one of its vertices, say x'. By (4.28) and (4.29), we conclude

$$\max_{y\in D_{k,l}}\mathcal{G}(f_k(y,0),f_k(x',0))^2 \le \frac{C}{k^{m+1}}\int_{\cup_\alpha C_{k,\alpha}}|Dg|^2.$$

For any $x\in D_{k,l}$, $g_k(x)$ is equal to $f_k(x',0)$ for some vertex $x'\in D_{k,l}$. Thus, we can estimate

$$\int_A \mathcal{G}(f_k(x,0),g_k(x))^2\,dx \le \frac{C}{k^2}\int_A|Dg|^2. \tag{4.31}$$

Recalling that $g_k\to g$ in L^2, we conclude, therefore, that $f_k(\cdot,0)$ converges to g. A similar conclusion can be inferred for $f_k(\cdot,\varepsilon)$.

Finally, from (4.30) and (4.31), we conclude a uniform bound on $\||f_k|\|_{L^2(B)}$. Using the compactness of the embedding $W^{1,2}\subset L^2$, we conclude the existence of a subsequence converging strongly in L^2 to a function $h\in W^{1,2}(B)$. Obviously, h satisfies (4.27). We now want to show that (4.26) holds.

Let $\delta\in]0,\varepsilon[$ and assume that $f_k(\cdot,\delta)\to f(\cdot,\delta)$ in L^2 (which in fact holds for a.e. δ). Then, a standard argument shows that

$$\begin{aligned}\int_A\mathcal{G}(f(x,\delta),g(x))^2\,dx &= \lim_{k\uparrow\infty}\int_A\mathcal{G}(f_k(x,\delta),g_k(x))^2\,dx\\ &\le \limsup_{k\uparrow\infty}\delta\||Df_k|\|^2_{L^2(B)}\le C\delta.\end{aligned}$$

Clearly, this implies that $f(\cdot,0)=g$. An analogous computation shows $f(\cdot,\varepsilon)=\tilde g$.

Step 2. *Interpolation between two spherical shells.* In what follows, we denote by D the closed $(m-1)$-dimensional ball and assume that $\phi_+ : D\to\partial B_1\cap\{x_m\ge 0\}$ is a diffeomorphism. Define $\phi_- : D\to\partial B_1\cap\{x_m\le 0\}$ by simply setting $\phi_-(x)=-\phi_+(x)$. Next, let $\phi: A\to D$ be a biLipschitz homeomorphism, where A is the set in Step 1, and set

$$\varphi_\pm=\phi_\pm\circ\phi,\quad g_{k,\pm}=g\circ\varphi_\pm\quad\text{and}\quad \tilde g_{k,\pm}=\tilde g\circ\varphi_\pm.$$

Consider the Lipschitz approximating functions constructed in Step 1, $f_{k,+}: A\times[0,\varepsilon]\to\mathcal{A}_Q$ interpolating between $g_{k,+}$ and $\tilde g_{k,-}$.

Next, to construct $f_{k,-}$, we use again the cell decomposition of Step 1. We follow the same procedure to attribute the values $f_{k,-}(x_{k,l},0)$ and $f_{k,-}(x_{k,l},\varepsilon)$ on the vertices $x_{k,l}\notin\partial A$. We instead set $f_{k,-}(x_{k,l},0)=f_{k,+}(x_{k,l},0)$ and $f_{k,-}(x_{k,l},\varepsilon)=f_{k,+}(x_{k,l},\varepsilon)$ when $x_{k,l}\in\partial A$. Finally, when using Theorem 1.7 as in Step 1, we take care to set $f_{k,+} = f_{k,-}$ on the skeletons lying in ∂A and we define

$$f_k(x) = \begin{cases} f_{k,+}(\varphi_+^{-1}(x/|x|), 1-|x|) & \text{if } x_m \geq 0 \\ f_{k,-}(\varphi_-^{-1}(x/|x|), 1-|x|) & \text{if } x_m \leq 0\,. \end{cases}$$

Then, f_k is a Lipschitz map. We want to use the estimates of Step 1 in order to conclude the existence of a sequence converging to a function h which satisfies the requirements of the proposition. This is straightforward on $\{x_m \geq 0\}$. On $\{x_m \leq 0\}$ we just have to control the estimates of Step 1 for vertices lying on ∂A. Fix a vertex $x_{k,l} \in \partial A$.

In the procedure of Step 1, $f_{k,-}(x_{k,l}, 0)$ and $f_{k,-}(x_{k,l}, \varepsilon)$ are defined by taking the averages $h_{k,l}$ and $\tilde{h}_{k,l}$ for $g \circ \varphi_-$ and $\tilde{g} \circ \varphi_-$ on the cell $C_{k,l} \cap A$. In the procedure specified above the values of $f_{k,-}(x_{k,l}, 0)$ and $f_{k,-}(x_{k,l}, \varepsilon)$ are given by the averages of $g \circ \varphi_+$ and $\tilde{g} \circ \varphi_+$, which we denote by $g_{k,l}$ and $\tilde{g}_{k,l}$. However, we can estimate the difference in the following way

$$|g_{k,l} - h_{k,l}| \leq \frac{C}{k^{m+2}} \int_{E_{k,l}} |Dg|^2,$$

where $E_{k,l}$ is a suitable cell in ∂B_1 containing $\varphi_+(C_{k,l})$ and $\varphi_-(C_{k,l})$. Since these two cells have a face in common and $\varphi_\pm$ are biLipschitz homeomorphisms, we can estimate the diameter of $E_{k,l}$ with C/k (see Figure 3). Therefore the estimates (4.30) and (4.31) proved in Step 1 hold with (possibly) worse constants. $\square$

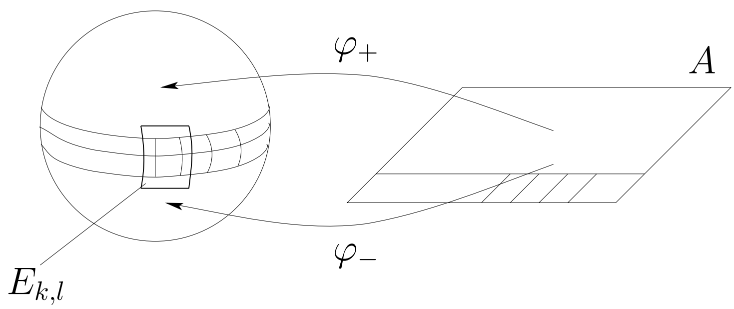

FIGURE 3. The maps $\varphi_\pm$ and the cells $E_{k,l}$.

CHAPTER 5

The improved estimate of the singular set in 2 dimensions

In this final part of the paper we prove Theorem 0.12. The first section gives a more stringent description of 2-d tangent functions to Dir-minimizing functions. The second section uses a comparison argument to show a certain rate of convergence for the frequency function of f. This rate implies the uniqueness of the tangent function. In Section 5.3, we use this uniqueness to get a better description of a Dir-minimizing functions around a singular point: an induction argument on Q yields finally Theorem 0.12.

Throughout the rest of the paper we use the notation introduced in Remark 3.11 and sometimes use (r,θ) in place of $r\,e^{i\theta}$.

5.1. Characterization of 2-d tangent Q-valued functions

In this section we analyze further Dir-minimizing functions $f:\mathbb{D}\to\mathcal{A}_Q(\mathbb{R}^n)$ which are homogeneous, that is

$$f(r,\theta)=r^\alpha\,g(\theta)\quad\text{for some }\alpha>0. \tag{5.1}$$

Recall that, for $T=\sum_i [\![T_i]\!]$ we denote by $\boldsymbol{\eta}(T)$ the center of mass $Q^{-1}\sum_i T_i$.

Proposition 5.1. *Let $f:\mathbb{D}\to\mathcal{A}_Q(\mathbb{R}^n)$ be a nontrivial, α-homogeneous function which is* Dir*-minimizing. Assume in addition that $\boldsymbol{\eta}\circ f=0$. Then,*

(a) *$\alpha=\frac{n^*}{Q^*}\in\mathbb{Q}$, with $\mathrm{MCD}\,(n^*,Q^*)=1$;*
(b) *there exist ($\mathbb{R}$-)linear maps $L_j:\mathbb{C}\to\mathbb{R}^n$ and $k_j\in\mathbb{N}$ such that*

$$f(x)=k_0\,[\![0]\!]+\sum_{j=1}^J k_j\sum_{z^{Q^*}=x}\left[\!\!\left[L_j\cdot z^{n^*}\right]\!\!\right]=:k_0\,[\![0]\!]+\sum_{j=1}^J k_j\,[\![f_j(x)]\!]\,. \tag{5.2}$$

$J\geq 1$ and $k_j\geq 1$ for all $j\geq 1$. If $Q^*>1$ or $k_0>0$, each L_j is injective. If $Q^*=1$, either $J\geq 2$ or $k_0>0$.
(c) *For any $i\neq j$ and any $x\neq 0$, the supports of $f_i(x)$ and $f_j(x)$ are disjoint.*

Proof. Let f be a homogeneous Dir-minimizing Q-valued function. We decompose $g=f|_{\mathbb{S}^1}$ into irreducible $W^{1,2}$ pieces as described in Proposition 1.5. Hence, we can write $g(x)=k_0\,[\![0]\!]+\sum_{j=1}^J k_j\,[\![g_j(x)]\!]$, where

(i) k_0 might vanish, while $k_j>0$ for every $j>0$,
(ii) the g_j's are all distinct, Q_j-valued irreducible $W^{1,2}$ maps such that $g_j(x)\neq Q_j\,[\![0]\!]$ for some $x\in\mathbb{S}^1$.

By the characterization of irreducible pieces, there are $W^{1,2}$ maps $\gamma_j : \mathbb{S}^1 \to \mathbb{R}^n$ such that

$$g_j(x) = \sum_{z^{Q_j}=x} [\![\gamma_j(z)]\!] . \tag{5.3}$$

Recalling (5.1), we extend γ_j to a function β_j on the disk by setting $\beta_j(r,\theta) = r^{\alpha\, Q_j}\gamma_j(\theta)$ and we conclude that

$$f(x) = k_0\,[\![0]\!] + \sum_{j=1}^{J} \sum_{z^{Q_j}=x} [\![\beta_j(z)]\!] =: k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j\,[\![f_j(x)]\!] .$$

It follows that each f_j is an α-homogeneous, Dir-minimizing function which assumes values different from $Q\,[\![0]\!]$ somewhere. By Lemma 3.12, β_j is necessarily a Dir-minimizing $\mathbb{R}^n$-valued function. Since β_j is $(\alpha\, Q_j)$-homogeneous, its coordinates must be homogeneous harmonic polynomials. Moreover, β_j does not vanish identically. Therefore, we conclude that $n_j = \alpha\, Q_j$ is a positive integer. Thus, the components of each β_j are linear combinations of the harmonic functions $(r,\theta) \mapsto r^{n_j}\cos(n_j\theta)$ and $(r,\theta) \mapsto r^{n_j}\sin(n_j\theta)$. It follows that there are (nonzero) $\mathbb{R}$-linear maps $L_j : \mathbb{C} \to \mathbb{R}^n$ such that $\beta_j(z) = L_j \cdot z^{n_j}$.

Next, let n^* and Q^* be the two positive integers determined by $\alpha = n^*/Q^*$ and MCD $(n^*, Q^*) = 1$. Since $n_j/Q_j = \alpha = n^*/Q^*$, we necessarily have $Q_j = m_j Q^*$ for some integer $m_j = \frac{n_j}{n^*} \geq 1$. Hence,

$$g_j(x) = \sum_{z^{m_j Q^*}=x} \left[\!\!\left[L_j \cdot z^{m_j n^*} \right]\!\!\right] .$$

However, if $m_j > 1$, then supp $(g_j) \equiv Q^* \neq Q_j$, so that g_j would not be irreducible. Therefore, $Q_j = Q^*$ for every j.

Next, since $\mathrm{Dir}(f,\mathbb{D}) > 0$, $J \geq 1$. If $Q^* = 1$, $J = 1$ and $k_0 = 0$, then $f = Q\,[\![f_1]\!]$ and f_1 is an $\mathbb{R}^n$-valued function. But then $f_1 = \boldsymbol{\eta} \circ f = 0$, contradicting $\mathrm{Dir}(f,\mathbb{D}) > 0$. Moreover, again using the irreducibility of g_j, for all $x \in \mathbb{S}^1$, the points

$$L_j \cdot z^{n^*} \quad \text{with} \quad z^{Q^*} = x$$

are all distinct. This implies that L_j is injective if $Q^* > 1$. Indeed, assume by contradiction that $L_j \cdot v = 0$ for some $v \neq 0$. Without loss of generality, we can assume that $v = e_1$. Let $x = e^{i\theta/n^*} \in \mathbb{S}^1$, with $\theta/Q^* = \pi/2 - \pi/Q^*$, and let us consider the set

$$R := \{z^{n^*} \in \mathbb{S}^1 : z^{Q^*} = x\} = \{e^{i(\theta+2\pi k)/Q^*}\}.$$

Therefore $w_1 = e^{i\theta/Q^*}$ and $w_2 = e^{i(\theta+2\pi)/Q^*} = e^{i\pi - i\theta/Q^*}$ are two distinct elements of R. However, it is easy to see that $w_1 - w_2 = 2\cos(\theta/Q^*)e_1$. Therefore, $L_j w_1 = L_j w_2$, which is a contradiction. This shows that L_j is injective.

Finally, we argue by contradiction for (c). If (c) were false, up to rotation of the plane and relabeling of the g_i's, we assume that supp $(g_1(0))$ and supp $(g_2(0))$ have a point P in common. We can, then, choose the functions γ_1 and γ_2 of (5.3) so that

$$\gamma_1(0) = \gamma_1(2\pi) = \gamma_2(0) = \gamma_2(2\pi) = P.$$

We then define $\xi : \mathbb{D} \to \mathbb{R}^n$ in the following way:

$$\xi(r,\theta) = \begin{cases} r^{2\,\alpha\, Q^*}\,\gamma_1(2\theta) & \text{if } \theta \in [0,\pi], \\ r^{2\,\alpha\, Q^*}\,\gamma_2(2\theta) & \text{if } \theta \in [\pi, 2\pi]. \end{cases}$$

Then, it is immediate to verify that

$$[\![f_1(x)]\!] + [\![f_2(x)]\!] = \sum_{z^{2Q^*}=x} [\![\xi(z)]\!] \,. \tag{5.4}$$

Therefore, f can be decomposed as

$$f(x) = \sum_{z^{2\,Q^*}=x} [\![\xi(z)]\!] + \left\{ k_0[\![0]\!] + (k_1-1)\,[\![f_1(x)]\!] + (k_2-1)\,[\![f_2(x)]\!] + \sum_{j\geq J} k_j\,[\![f_i(x)]\!] \right\}.$$

It turns out that the map in (5.4) is a Dir-minimizing function, and, hence, that ξ is a $(2\,\alpha\,Q^*)$-homogeneous Dir-minimizing function. Since $2\,\alpha\,Q^* = 2\,n^*$ we conclude the existence of a linear $L : \mathbb{C} \to \mathbb{R}^n$ such that

$$[\![f_1(x)]\!] + [\![f_2(x)]\!] = \sum_{z^{2Q^*}=x} \left[\!\!\left[L\cdot z^{2n^*}\right]\!\!\right] = 2 \sum_{z^{Q^*}=x} \left[\!\!\left[L\cdot z^{n^*}\right]\!\!\right].$$

Hence, for any $x \in \mathbb{S}^1$, the cardinality of the support of $[\![g_1(x)]\!] + [\![g_2(x)]\!]$ is at most Q^*. Since each g_i is irreducible, the cardinality of the support of $[\![g_i(x)]\!]$ is everywhere exactly Q^*. We conclude thus that $g_1(x) = g_2(x)$ for every x, which is a contradiction to assumption (ii) in our decomposition. Arguing analogously we conclude that each L_j is injective when $Q^* = 1$ and $k_0 > 0$. $\square$

5.2. Uniqueness of 2-d tangent functions

The key point of this section is the rate of convergence for the frequency function, as stated in Proposition 5.2. We use here the functions $H_{x,f}$, $D_{x,f}$ and $I_{x,f}$ introduced in Definition 3.13 and drop the subscripts when f is clear from the context and $x = 0$.

PROPOSITION 5.2. *Let $f \in W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ be* Dir*-minimizing, with* $\mathrm{Dir}(f,\mathbb{D}) > 0$ *and set $\alpha = I_{0,f}(0) = I(0)$. Then, there exist constants $\gamma > 0$, $C > 0$, $H_0 > 0$ and $D_0 > 0$ such that, for every $0 < r \leq 1$,*

$$0 \leq I(r) - \alpha \leq C\,r^\gamma, \tag{5.5}$$

$$0 \leq \frac{H(r)}{r^{2\alpha+1}} - H_0 \leq C\,r^\gamma \quad \textit{and} \quad 0 \leq \frac{D(r)}{r^{2\alpha}} - D_0 \leq C\,r^\gamma. \tag{5.6}$$

The proof of this result follows computations similar to those of [**Cha88**]. A simple corollary of (5.5) and (5.6) is the uniqueness of tangent functions.

THEOREM 5.3. *Let $f : \mathbb{D} \to \mathcal{A}_Q(\mathbb{R}^n)$ be a* Dir*-minimizing Q-valued functions, with* $\mathrm{Dir}(f,\mathbb{D}) > 0$ *and $f(0) = Q\,[\![0]\!]$. Then, there exists a unique tangent map g to f at 0 (i.e. the maps $f_{0,\rho}$ defined in* (3.51) *converge locally uniformly to g).*

In the first subsection we prove Theorem 5.3 assuming Proposition 5.2, which will be then proved in the second subsection.

5.2.1. Proof of Theorem 5.3. Set $\alpha = I_{0,f}(0)$ and note that, by Theorem 3.19 and Proposition 5.2, $\alpha = D_0/H_0 > 0$, where D_0 and H_0 are as in (5.6). Without loss of generality, we might assume $D_0 = 1$. So, by (5.6), recalling the definition of blow-up f_ϱ, it follows that

$$f_\varrho(r,\theta) = \varrho^{-\alpha} f(r\,\varrho,\theta)\,(1 + O(\varrho^{\gamma/2})). \tag{5.7}$$

Our goal is to show the existence of a limit function (in the uniform topology) for the blow-up f_ϱ. From (5.7), it is enough to show the existence of a uniform limit for the functions $h_\varrho(r,\theta) = \varrho^\alpha f_\varrho(r\,\varrho,\theta)$. Since $h_\varrho(r,\theta) = r^\alpha h_{r\,\varrho}(1,\theta)$, it suffices to prove the existence of a uniform limit for $h_\varrho|_{\mathbb{S}^1}$. On the other hand, the family of functions $\{h_\varrho\}_{\varrho>0}$ is equi-Hölder (cp. with Theorem 3.19 and (5.6) in Proposition 5.2). Therefore, the existence of an uniform limit is equivalent to the existence of an L^2 limit.

So, we consider $r/2 \le s \le r$ and estimate

$$\begin{aligned}\int_0^{2\pi} \mathcal{G}\left(h_r, h_s\right)^2 &= \int_0^{2\pi} \mathcal{G}\left(\frac{f(r,\theta)}{r^\alpha}, \frac{f(s,\theta)}{s^\alpha}\right)^2 d\theta \le \int_0^{2\pi}\left(\int_s^r \left|\frac{d}{dt}\left(\frac{f(t,\theta)}{t^\alpha}\right)\right| dt\right)^2 d\theta \\ &\le (r-s)\int_0^{2\pi}\int_s^r \left|\frac{d}{dt}\left(\frac{f(t,\theta)}{t^\alpha}\right)\right|^2 dt\,d\theta. \end{aligned}\tag{5.8}$$

This computation can be easily justified because $r \mapsto f(r,\theta)$ is a $W^{1,2}$ function for a.e. θ. Using the chain rule in Proposition 1.12 and the variation formulas (3.6), (3.7) in Proposition 3.2, we estimate (5.8) in the following way:

$$\begin{aligned}\int_0^{2\pi} \mathcal{G}\left(h_r, h_s\right)^2 &\le (r-s)\int_0^{2\pi}\int_s^r \sum_i \left\{\alpha^2\frac{|f_i|^2}{t^{2\alpha+2}} + \frac{|\partial_\nu f_i|^2}{t^{2\alpha}} - 2\,\alpha\,\frac{\langle\partial_\nu f_i, f_i\rangle}{t^{2\alpha+1}}\right\} \\ &\overset{(3.6),\,(3.7)}{=} (r-s)\int_s^r \left\{\alpha^2\frac{H(t)}{t^{2\alpha+3}} + \frac{D'(t)}{2\,t^{2\alpha+1}} - 2\,\alpha\,\frac{D(t)}{t^{2\alpha+2}}\right\} dt \\ &= (r-s)\int_s^r \left\{\frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}\right)' + \alpha^2\frac{H(t)}{2\,t^{2\alpha+3}} - \alpha\,\frac{D(t)}{t^{2\alpha+2}}\right\} dt \\ &= (r-s)\int_s^r \left\{\frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}\right)' + \alpha\,\frac{H(t)}{2\,t^{2\alpha+3}}\left(\alpha - I_{0,f}(t)\right)\right\} dt \\ &\le (r-s)\int_s^r \frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}\right)' dt = (r-s)\int_s^r \frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}} - D_0\right)' dt \end{aligned}\tag{5.9}$$

where the last inequality follows from the monotonicity of the frequency function, which implies, in particular, that $\alpha \le I_{0,f}(t)$ for every t. Integrating by parts the last integral of (5.9), we get

$$\begin{aligned}\int_0^{2\pi} \mathcal{G}\left(h_r, h_s\right)^2 \le (r-s)&\left[\frac{1}{2\,r}\left(\frac{D(r)}{r^{2\alpha}} - D_0\right) - \frac{1}{2\,s}\left(\frac{D(s)}{s^{2\alpha}} - D_0\right)\right] + \\ &+ (r-s)\int_s^r \frac{1}{2t^2}\left(\frac{D(t)}{t^{2\alpha}} - D_0\right).\end{aligned}$$

Recalling that $0 \le D(r)/r^{2\alpha} - D_0 \le C r^\gamma$ and $s = r/2$ we estimate

$$\int_0^{2\pi} \mathcal{G}\left(h_r, h_s\right)^2 \le C\frac{r-s}{s}\,r^\gamma + (r-s)\int_s^r \frac{1}{2t^{2-\gamma}} \le C r^\gamma. \tag{5.10}$$

Let now $s \leq r$ and choose $L \in \mathbb{N}$ such that $r/2^{L+1} < s \leq r/2^L$. Iterating (5.10), we reach

$$\begin{aligned}\|\mathcal{G}(h_r, h_s)\|_{L^2} &\leq \sum_{l=0}^{L-1} \left\|\mathcal{G}\left(h_{r/2^l}, h_{r/2^{l+1}}\right)\right\|_{L^2} + \left\|\mathcal{G}\left(h_{r/2^L}, h_s\right)\right\|_{L^2} \\ &\leq \sum_{l=0}^{L} \frac{r^{\gamma/2}}{\left(2^{\gamma/2}\right)^l} \leq C\, r^{\gamma/2}.\end{aligned}$$

This shows that $h_\varrho|_{\mathbb{S}^1}$ is a Cauchy sequence in L^2 and, hence, concludes the proof.

5.2.2. Proof of Proposition 5.2. The key of the proof is the following estimate:

$$I'(r) \geq \frac{2}{r}\left(\alpha + \gamma - I(r)\right)(I - \alpha)\,. \tag{5.11}$$

We will prove (5.11) in a second step. First we show how to conclude the various statements of the proposition.

Step 1. (5.11)$\Longrightarrow$ *Proposition* 5.2. Since I is monotone nondecreasing (as proved in Theorem 3.15), there exists $r_0 > 0$ such that $\alpha + \gamma - I(r) \geq \gamma/2$ for every $r \leq r_0$. Therefore,

$$I'(r) \geq \frac{\gamma}{r}\left(I(r) - \alpha\right) \quad \forall\, r \leq r_0. \tag{5.12}$$

Integrating the differential inequality (5.12), we get the desired conclusion:

$$I(r) - \alpha \leq \left(\frac{r}{r_0}\right)^\gamma (I(r_0) - \alpha) = C\, r^\gamma.$$

From the computation of H' in (3.46), we deduce easily that

$$\left(\frac{H(r)}{r}\right)' = \frac{2\,D(r)}{r}. \tag{5.13}$$

This implies the following identity:

$$\begin{aligned}\left(\log\frac{H(r)}{r^{2\alpha+1}}\right)' &= \left(\log\frac{H(r)}{r} - \log r^{2\alpha}\right)' \\ &= \frac{r}{H(r)}\left(\frac{H(r)}{r}\right)' - \frac{2\alpha}{r} \overset{(5.13)}{=} \frac{2}{r}\left(I(r) - \alpha\right) \geq 0.\end{aligned} \tag{5.14}$$

So, in particular, we infer the monotonicity of $\log\frac{H(r)}{r^{2\alpha+1}}$ and, hence, of $\frac{H(r)}{r^{2\alpha+1}}$. We can, therefore, integrate (5.14) and use (5.5) in order to achieve that, for $0 < s < r \leq 1$ and for a suitable constant C_γ, the function

$$\log\frac{H(r)}{r^{2\alpha+1}} - C_\gamma\, r^\gamma = \log\left(\frac{H(r)\, e^{-C_\gamma\, r^\gamma}}{r^{2\alpha+1}}\right)$$

is decreasing. So, we conclude the existence of the following limits:

$$\lim_{r\to 0}\frac{H(r)\, e^{-C_\gamma\, r^\gamma}}{r^{2\alpha+1}} = \lim_{r\to 0}\frac{H(r)}{r^{2\alpha+1}} = H_0 > 0,$$

with the bounds, for r small enough,

$$\frac{H(r)}{r^{2\alpha+1}}\left(1 - C\, r^\gamma\right) \leq \frac{H(r)\, e^{-C_\gamma\, r^\gamma}}{r^{2\alpha+1}} \leq H_0 \leq \frac{H(r)}{r^{2\alpha+1}}.$$

This easily concludes the first half of (5.6). The rest of (5.6) follows from the following identity:

$$\frac{D(r)}{r^{2\alpha}} - D_0 = (I(r) - I_0)\,\frac{H(r)}{r^{2\alpha+1}} + I_0\left(\frac{H(r)}{r^{2\alpha+1}} - H_0\right).$$

Indeed, both addendum are positive and bounded by $C\,r^\gamma$.

Step 2. Proof of (5.11). Recalling the computation in (3.47), (5.11) is equivalent to

$$\frac{r\,D'(r)}{H(r)} - \frac{2\,I(r)^2}{r} \geq \frac{2}{r}\big(\alpha+\gamma - I(r)\big)\,(I(r)-\alpha)\,,$$

which, in turn, reduces to

$$(2\alpha+\gamma)\,D(r) \leq \frac{r\,D'(r)}{2} + \frac{\alpha(\alpha+\gamma)\,H(r)}{r}. \tag{5.15}$$

To prove (5.15), we exploit once again the harmonic competitor constructed in the proof of the Hölder regularity for the planar case in Proposition 3.10. Let $r>0$ be a fixed radius and $f(re^{i\theta}) = g(\theta) = \sum_{j=1}^J [\![g_j(\theta)]\!]$ be an irreducible decomposition as in Proposition 1.5. For each irreducible g_j, we find $\gamma_j \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^n)$ and Q_j such that

$$g_j(\theta) = \sum_{i=1}^{Q_j}\left[\!\!\left[\gamma_j\left(\frac{\theta+2\pi i}{Q_j}\right)\right]\!\!\right].$$

We write now the different quantities in (5.15) in terms of the Fourier coefficients of the γ_j's. To this aim, consider the Fourier expansions of the γ_j's,

$$\gamma_j(\theta) = \frac{a_{j,0}}{2} + \sum_{l=1}^{+\infty} r^l\big\{a_{j,l}\cos(l\,\theta) + b_{j,l}\sin(l\,\theta)\big\},$$

and their harmonic extensions

$$\zeta_j(\varrho,\theta) = \frac{a_{j,0}}{2} + \sum_{l=1}^{+\infty} \varrho^l\big\{a_{j,l}\cos(l\,\theta) + b_{j,l}\sin(l\,\theta)\big\}.$$

Recalling Lemma 3.12, we infer the following equalities:

$$D'(r) = 2\sum_j \mathrm{Dir}(g_j, r\,\mathbb{S}^1) = \sum_j \frac{2\,\mathrm{Dir}(\gamma_j, r\,\mathbb{S}^1)}{Q_j} = 2\,\pi\sum_j\sum_l \frac{r^{2l-1}\,l^2}{Q_j}\left(a_{j,l}^2 + b_{j,l}^2\right), \tag{5.16}$$

$$H(r) = \sum_j \int_{r\,\mathbb{S}^1} |g_j|^2 = \sum_j Q_j \int_{r\,\mathbb{S}^1} |\gamma_j|^2 = \pi\sum_j Q_j\left\{\frac{r\,a_{j,0}^2}{2} + \sum_l r^{2l+1}\left(a_{j,l}^2 + b_{j,l}^2\right)\right\}. \tag{5.17}$$

Finally, using the minimality of f,

$$D(r) \leq \sum_j \mathrm{Dir}(\zeta_j, B_r) = \pi\sum_j\sum_l r^{2l}\,l\left(a_{j,l}^2 + b_{j,l}^2\right). \tag{5.18}$$

We deduce from (5.16), (5.17) and (5.18) that, to prove (5.15), it is enough to find a γ such that

$$(2\alpha+\gamma)\,l \leq \frac{l^2}{Q_j} + \alpha\,(\alpha+\gamma)\,Q_j, \quad \text{for every } l\in\mathbb{N} \text{ and every } Q_j,$$

which, in turn, is equivalent to

$$\gamma\, Q_j\,(l-\alpha\, Q_j) \leq (l-\alpha\, Q_j)^2. \tag{5.19}$$

Note that the Q_j's depend on r, the radius we fixed. However, they are always natural numbers less or equal than Q. It is, hence, easy to verify that the following γ satisfies (5.19):

$$\gamma = \min_{1\leq k\leq Q}\left\{\frac{\lfloor \alpha\, k\rfloor + 1 - \alpha\, k}{k}\right\}. \tag{5.20}$$

5.3. The singularities of 2-d Dir-minimizing functions are isolated

We are finally ready to prove Theorem 0.12.

Proof of Theorem 0.12. Our aim is to prove that, if $f:\Omega\to\mathcal{A}_Q$ is Dir-minimizing, then the singular points of f are isolated. The proof is by induction on the number of values Q. The basic step of the induction procedure, $Q=1$, is clearly trivial, since $\Sigma_f=\emptyset$. Now, we assume that the claim is true for any $Q'<Q$ and we will show that it holds for Q as well.

So, we fix $f:\mathbb{R}^2\supset\Omega\to\mathcal{A}_Q$ Dir-minimizing. Since the function $f-Q\,[\![\boldsymbol{\eta}\circ f]\!]$ is still Dir-minimizing and has the same singular set as f (notations as in Lemma 3.23), it is not restrictive to assume $\boldsymbol{\eta}\circ f\equiv 0$.

Next, let $\Sigma_{Q,f}=\{x: f(x)=Q\,[\![0]\!]\}$ and recall that, by the proof of Theorem 0.11, either $\Sigma_{Q,f}=\Omega$ or $\Sigma_{Q,f}$ consists of isolated points. Assuming to be in the latter case, on $\Omega\setminus\Sigma_{Q,f}$, we can locally decompose f as the sum of a Q_1-valued and a Q_2-valued Dir-minimizing function with $Q_1,Q_2<Q$. We can therefore use the inductive hypothesis to conclude that the points of $\Sigma_f\setminus\Sigma_{Q,f}$ are isolated. It remains to show that no $x\in\Sigma_{Q,f}$ is the limit of a sequence of points in $\Sigma_f\setminus\Sigma_{Q,f}$.

Fix $x_0\in\Sigma_{Q,f}$. Without loss of generality, we may assume $x_0=0$. Note that $0\in\Sigma_{Q,f}$ implies $D(r)>0$ for every r such that $B_r\subset\Omega$. Let g be the tangent function to f in 0 . By the characterization in Proposition 5.1, we have

$$g = k_0\,[\![0]\!] + \sum_{j=1}^{J} k_j\,[\![g_j]\!] = \sum_j k_j\,[\![g_j]\!]\,,$$ [1]

where the g_j's are Q^*-valued functions satisfying (a)-(c) of Proposition 5.1 (in particular $\alpha=n^*/Q^*$ is the frequency in 0). So, we are necessarily in one of the following cases:

(i) $\max\{k_0, J-1\}>0$;
(ii) $J=1$, $k_0=0$ and $k_1<Q$.

If case (i) holds, we define

$$d_{i,j} := \min_{x\in\mathbb{S}^1}\,\mathrm{dist}\big(\mathrm{supp}\,(g_i(x)),\mathrm{supp}\,(g_j(x))\big)\quad\text{and}\quad \varepsilon=\min_{i\neq j}\frac{d_{i,j}}{4}. \tag{5.21}$$

By Proposition 5.1(c), we have $\varepsilon>0$. From the uniform convergence of the blow-ups to g, there exists $r_0>0$ such that

$$\mathcal{G}\,(f(x),g(x)) \leq \varepsilon\,|x|^{\alpha}\quad\text{for every}\quad |x|\leq r_0. \tag{5.22}$$

[1] Here we use the convention that the index j runs from 0 to J *only if* $k_0>0$. Otherwise the index runs from 1 to J.

The choice of ε in (5.21) and (5.22) easily implies the existence of f_j, with $j \in \{0, \ldots, J\}$, such that f_0 is a $W^{1,2}$ k_0-valued function, each f_j is a $W^{1,2}$ $(k_j\, Q^*)$-valued function for $j > 0$, and

$$f|_{B_{r_0}} = \sum_{j=0}^{J} [\![f_j]\!] \,. \tag{5.23}$$

It follows that each f_j is a Dir-minimizing function. The sum (5.23) contains at least two terms: so each f_j take less than Q values and we can use our inductive hypothesis to conclude that $\Sigma_f \cap B_{r_0} = \bigcup_j \Sigma_{f_j} \cap B_{r_0}$ consists of isolated points.

If case (ii) holds, then $k\, Q^* = Q$, with $k < Q$, and g is of the form

$$g(x) = \sum_{z^{Q^*}=x} k \left[\!\!\left[L \cdot z^{n^*} \right]\!\!\right],$$

where L is injective. In this case, set

$$d(r) := \min_{z_1^{Q^*}=z_2^{Q^*},\, z_1 \neq z_2,\, |z_i| = r^{1/Q^*}} |L \cdot z_1^{n^*} - L \cdot z_2^{n^*}|.$$

Note that

$$d(r) = c\, r^{\alpha} \quad \text{and} \quad \max_{|x|=r} \operatorname{dist}\big(\operatorname{supp}(f(x)), \operatorname{supp}(g(x))\big) = o(r^{\alpha}).$$

This implies the existence of $r > 0$ and $\zeta \in C(B_r, \mathcal{A}_k(\mathbb{R}^n))$ such that

$$f(x) = \sum_{z^{Q^*}=x} [\![\zeta(z)]\!] \qquad \text{for } |x| < r.$$

Set $\rho = r^{Q^*}$. If $x \neq B_\rho \setminus 0$ and $\sigma < \min\{|x|, \rho - |x|\}$, then obviously $\zeta \in W^{1,2}(B_\sigma(x))$. Thus, $\zeta \in W^{1,2}(B_\rho \setminus B_\sigma)$ for every $\sigma > 0$. On the other hand, after the same computations as in Lemma 3.12, it is easy to show that $\mathrm{Dir}(\zeta, B_\rho \setminus B_\sigma)$ is bounded independently of σ. We conclude that $\zeta \in W^{1,2}(B_\rho \setminus \{0\})$. This implies that $\zeta \in W^{1,2}(B_\rho)$ (see below) and hence we can apply the same arguments of Lemma 3.12 to show that ζ is Dir-minimizing. Therefore, by inductive hypothesis, Σ_ζ consists of isolated points. So, ζ is necessarily regular in a punctured disk $B_\sigma(0) \setminus \{0\}$, which implies the regularity of f in the punctured disk $B_{\sigma^{1/Q^*}} \setminus \{0\}$.

For the reader's convenience, we give a short proof of the claim $\zeta \in W^{1,2}(B_\rho)$. This is in fact a consequence of the identity $W^{1,2}(B_\rho \setminus \{0\}) = W^{1,2}(B_\rho)$ for classical Sobolev spaces, a byproduct of the fact that 2-capacity of a single point in the plain is finite.

Indeed, we claim that, for every $T \in \mathcal{A}_k(\mathbb{R}^n)$, the function $h_T := \mathcal{G}(\zeta, T)$ belongs to $W^{1,2}(B_\rho)$. Fix a test function $\varphi \in C_c^\infty(B_\rho)$ and denote by Λ^i the distributional derivative $\partial_{x_i} h_T$ in $B_\rho \setminus \{0\}$. For every $\sigma \in (0, \rho)$ let $\psi_\sigma \in C_c^\infty(B_\sigma)$ be a cutoff function with the properties:

(i) $0 \leq \psi_\sigma \leq 1$;
(ii) $\|D\psi_\sigma\|_{C^0} \leq C\sigma^{-1}$, where C is a geometric constant independent of σ.

Then,

$$\int h_T\,\partial_{x_i}\varphi = \int h_T\,\partial_{x_i}(\varphi\,\psi_\sigma) + \int h_T\,\partial_{x_i}((1-\psi_\sigma)\,\varphi)$$
$$= \underbrace{\int_{B_\sigma} h_T\,\partial_{x_i}(\varphi\,\psi_\sigma)}_{(I)} - \underbrace{\int \Lambda^i((1-\psi_\sigma)\,\varphi)}_{(II)}.$$

Letting $\sigma \downarrow 0$, (II) converges to $\int \Lambda^i\varphi$. As for (I), we estimate it as follows:

$$|(I)| \le \|\partial_{x_i}(\varphi\,\psi_\sigma)\|_{L^2(B_\sigma)}\,\|h_T\|_{L^2(B_\sigma)}.$$

By the absolute continuity of the integral, $\|h_T\|_{L^2(B_\sigma)} \to 0$ as $\sigma \downarrow 0$. On the other hand, we have the pointwise inequality $|\partial_{x_i}(\varphi\,\psi_\sigma)| \le C(1+\sigma^{-1})$. Therefore, $\|\partial_{x_i}(\varphi\,\psi_\sigma)\|_{L^2(B_\sigma)}$ is bounded independently of σ. This shows that $(I) \downarrow 0$ and hence we conclude the identity $\int h_T\,\partial_{x_i}\varphi = -\int \Lambda^i\varphi$. Thus, Λ is the distributional derivative of h_T in B_ρ. $\square$

REMARK 5.4. Theorem 0.12 is optimal. There are Dir-minimizing functions for which the singular set is not empty. Any holomorphic varieties which can be written as graph of a multi-valued function is Dir-minimizing. For example, the function

$$\mathbb{D} \ni z \mapsto \left[\!\left[z^{\frac{1}{2}} \right]\!\right] + \left[\!\left[-z^{\frac{1}{2}} \right]\!\right] \in \mathcal{A}_2(\mathbb{R}^4),$$

whose graph is the complex variety $\mathscr{V} = \{(z,w) \in \mathbb{C}^2 : |z| < 1,\, w^2 = z\}$, is an example of a Dir-minimizing function with a singular point in the origin. A proof of this result is contained in [**Alm00**]. The question will be addressed also in [**Spa09**].

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